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Steiner radial number resulting from various graph operations

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Abstract

The Steiner n-radial graph of a graph G on p vertices, denoted by $SR_n(G)$, has the vertex set as in G and any $n(2 \leq n \leq p)$ vertices are mutually adjacent in $SR_n(G)$ if and only if they are n-radial in G. When G is disconnected, any n vertices are mutually adjacent in $SR_n(G)$ if not all of them are in the same component. For the edge set of $SR_n(G)$, draw K_n corresponding to each set of n-radial vertices. The Steiner radial number $r_S(G)$ of a graph G is the least positive integer n such that the Steiner n-radial graph of G is complete. In this paper, Steiner radial number has been determined for the line graph of any tree, total graph of any tree, complement of any tree, sum of two non-trivial trees and Mycielskians of some families. For any pair of positive integers $a, b \geq 3$ with $a \leq b$, there exists a graph whose Steiner radial number is a and Steiner radial number of its line graph is b.

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1. Introduction

We consider finite undirected graphs without multiple edges and loops throughout this paper. Let G be a graph on p vertices and S, a set of vertices of G. The subgraph induced by S in G is denoted by $\langle S \rangle$. In [9], the Steiner distance (SD) of S in G denoted by $d_G(S)$, is defined as

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the minimum number of edges in a connected subgraph of G that contains S. Such a subgraph is necessarily a tree and is called a Steiner tree for S in G. Steiner trees play a significant role in various aspects of image processing, such as segmentation, feature extraction, image registration, and compression. They provide efficient methods for connecting and analyzing image features, thereby enhancing the capabilities of automated image analysis systems and improving the accuracy of image-based applications [18, 19]. The Steiner n-eccentricity $e_n(v)$ of a vertex v in a graph G is defined as $e_n(v) = \max\{d_G(S) : S \subseteq V(G)$ with $v \in S$ and $|S| = n\}$. The n-radius $rad_n(G)$ of G is defined as the smallest Stenier n-eccentricity among the vertices of G, and the n-diameter $diam_n(G)$ of G is the largest Steiner n-eccentricity. The concept of SD was further developed in [17, 21].

In [14], KM. Kathiresan et al. introduced the concept of Steiner radial number of a graph G. Any n vertices of a graph G are said to be n-radial to each other if SD between them is equal to the n-radius of the graph G. The Steiner n-radial graph of a graph G, denoted by $SR_n(G)$, has the vertex set as in G and $n(2 \leq n \leq p)$ vertices are mutually adjacent in $SR_n(G)$ if and only if they are n-radial in G. If G is disconnected, any n vertices are mutually adjacent in $SR_n(G)$ if not all of them are in the same component. For the edge set of $SR_n(G)$, draw K_n corresponding to each set of n-radial vertices. When $n = 2$, Steiner n-radial graph of G coincides with radial graph G. For a pair of graphs G and H on p vertices, the least positive integer n such that $SR_n(G) \cong H$, is called the Steiner completion number of G over H. When $H = K_p$, the Steiner completion number of G over $H$is called as Steiner radial number of G. The Steiner radial number $r_S(G)$ of a graph G is the least positive integer n such that the Steiner n-radial graph of G is complete.

The Steiner tree and Steiner radial number are valuable tools in network optimization across various domains, offering efficient solutions for connecting multiple points with minimal distance or cost. Steiner radial graphs are valuable for visualizing complex networks such as social networks, communication networks, or biological networks [7, 23, 24].

The concept of graph operator has found various applications in chemical research [12, 11]. The line graph $L(G)$ of a graph G has vertices corresponding to the edges of G and two vertices are adjacent in $L(G)$ if their corresponding edges of G have a common end vertex [10]. Parameters of line graphs have been applied for the evaluation of structural complexity of molecular graphs and design of novel topological indices [5, 6].

The total graph $T(G)$ of a graph G has vertex set $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent whenever they are neighbours in G [13]. The vertices and edges of a graph are called elements. Two elements of a graph are neighbours if they are either incident or adjacent. Several properties of total graphs are investigated in [1, 2, 3, 4]. A tree is defined as a connected graph that contains no cycles, and various parameters associated with trees have been extensively studied [20, 22]. A vertex of degree zero in G is called an isolated vertex and a vertex of degree one is called a pendant vertex or a leaf. An edge e in a graph G is called a pendant edge if it is incident with a pendant vertex. A vertex of degree $p-1$ is called an universal vertex or full degree vertex. The graph G obtained from $K_{1,m}$ and $K_{1,n}$ by joining their centers by an edge is called a bistar and is denoted by $B(m,n)$. The join $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$ if G_1 and G_2 are two graphs with disjoint vertex sets. A vertex v in G is called peripheral if eccentricity of v is equal to diameter of G.

A power graph G^k (the k^{th} power of a graph G) is the graph whose vertices are those of G and in which two distinct vertices are joined whenever the distance between them in G is at most k [13]. The complement $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if uv is not in $E(G)$ [8]. For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{v'_i : v_i \in V\}$ and edge set $E \cup \{v_i v'_j : v_i v_j \in E\} \cup \{v'_i u : v'_i \in V'\}$ [16]. The following notation can be found in [15]. The set of all connected graphs G for which $r(G) = 1$ and $d(G) = 2$ on p vertices denoted by F_{12}.

Now, we collect some useful results known for Steiner radial number of graphs from [14].

Theorem A [14]. For every tree T with $m(\neq p-1)$ pendant vertices, $r_S(T) = m + 2$.

Theorem B [14]. For any complete bipartite graph K_{p_1,p_2} with $p_1 \leq p_2$ and $p_1 \neq 1$, $r_S(K_{p_1,p_2}) = p_1 + 1$.

Theorem C [14]. If G is disconnected graph of order $p \geq 3$ but not totally disconnected, then $SR_3(G) \cong K_p$.

Theorem D [14]. $r_S(G) = 2$ if and only if G is either complete or totally disconnected.

For graph theoretic terminology, we follow [8].

2. Main Results

In this section, we shall determine the Steiner radial number for the line graph of a graph, the total graph of a graph, the complement of a tree, and the Mycielskians of certain graphs.

In Observation 2.1, we determine the Steiner radial number for the complement of a complete n-partite graph and illustrate the possible Steiner n-radial graphs in Example 2.1.

Observation 2.1. *For any complete n-partite graph other than complete graph, the Steiner radial number of its complement is 3.*

Proof. By Theorem C, the result follows. □

Proposition 2.1. *If $G \in F_{12}$, then $r_S(G) = 3$.*

Proof. Since $G \in F_{12}$, $R(G) \cong G$ and hence $SR_n(G) \cong G$ which is not complete. Therefore, $r_S(G) \geq 3$. SD of any 3-element set having at least one full degree vertex is 2 and SD of any 3-element set in which none of them are adjacent or only 2 vertices are adjacent is 3. Hence for any vertex v of degree $p-1$ in G, $e_3(v) = 2$ and for any vertex v of degree less than $p-1$, $e_3(v)$ is either 2 or 3. Hence 3-radius of G is 2. Hence $SR_3(G) \cong K_p$ and hence $r_S(G) = 3$. □

Example 2.1. *All possible Steiner n-radial graphs of the line graph of $K_{2,4}$ are given below: Let $G = K_{2,4}$, and let its line graph, denoted by $L(G)$, be shown in Figure 1. If we let $n = 2$, $rad_2(L(G)) = 2$ and the sets $S_1 = \{e_{11}, e_{22}\}$, $S_2 = \{e_{11}, e_{23}\}$, $S_3 = \{e_{11}, e_{24}\}$, $S_4 = \{e_{12}, e_{21}\}$, $S_5 = \{e_{12}, e_{23}\}$, $S_6 = \{e_{12}, e_{24}\}$, $S_7 = \{e_{13}, e_{21}\}$, $S_8 = \{e_{13}, e_{22}\}$, $S_9 = \{e_{13}, e_{24}\}$, $S_{10} = \{e_{14}, e_{21}\}$, $S_{11} = \{e_{14}, e_{22}\}$ and $S_{12} = \{e_{14}, e_{23}\}$ are the only sets of 2-radial vertices of $L(G)$. Hence the Steiner 2-radial graph of $L(G)$ is obtained and shown in Figure 2.*

If we take $n = 3$, $rad_3(L(G)) = 3$ and the sets $S_1 = \{e_{11}, e_{12}, e_{23}\}$, $S_2 = \{e_{11}, e_{12}, e_{24}\}$, $S_3 = \{e_{11}, e_{13}, e_{22}\}$, $S_4 = \{e_{11}, e_{13}, e_{24}\}$, $S_5 = \{e_{11}, e_{14}, e_{22}\}$, $S_6 = \{e_{11}, e_{14}, e_{23}\}$, $S_7 = \{e_{11}, e_{22}, e_{23}\}$, $S_8 = \{e_{11}, e_{22}, e_{24}\}$, $S_9 = \{e_{11}, e_{23}, e_{24}\}$, $S_{10} = \{e_{21}, e_{22}, e_{13}\}$, $S_{11} = \{e_{21}, e_{22}, e_{14}\}$, $S_{12} =$

$\{e_{21}, e_{23}, e_{12}\}$, $S_{13} = \{e_{21}, e_{23}, e_{14}\}$, $S_{14} = \{e_{21}, e_{24}, e_{12}\}$, $S_{15} = \{e_{21}, e_{24}, e_{13}\}$, $S_{16} = \{e_{21}, e_{12}, e_{13}\}$, $S_{17} = \{e_{21}, e_{12}, e_{14}\}$ *and* $S_{18} = \{e_{21}, e_{13}, e_{14}\}$ *are the only sets of 3-radial vertices of* $L(G)$. *Hence the Steiner 3-radial graph of* $L(G)$ *is obtained and shown in Figure* 2.

If we let $n = 4$, $rad_4(L(G)) = 4$ *and the sets* $S_1 = \{e_{11}, e_{22}, e_{23}, e_{24}\}$, $S_2 = \{e_{12}, e_{21}, e_{23}, e_{24}\}$, $S_3 = \{e_{13}, e_{21}, e_{22}, e_{24}\}$, $S_4 = \{e_{14}, e_{21}, e_{22}, e_{23}\}$, $S_5 = \{e_{11}, e_{12}, e_{13}, e_{24}\}$, $S_6 = \{e_{11}, e_{13}, e_{14}, e_{22}\}$ *and* $S_7 = \{e_{12}, e_{13}, e_{14}, e_{21}\}$ *are the only sets of 4-radial vertices of* $L(G)$. *Hence the Steiner 4-radial graph of* $L(G)$ *is obtained, which is isomorphic to* $SR_3(L(G))$, *as shown in Figure* 3.

If we take $n = 5$, $rad_5(L(G)) = 4$ *and the sets* $S_1 = \{e_{11}, e_{12}, e_{13}, e_{14}, e_{21}\}$, $S_2 = \{e_{11}, e_{12}, e_{22}, e_{23}, e_{24}\}$, $S_3 = \{e_{13}, e_{14}, e_{22}, e_{23}, e_{24}\}$ *and* $S_4 = \{e_{11}, e_{21}, e_{22}, e_{23}, e_{24}\}$ *are 5-radial vertices of* $L(G)$. *Hence the Steiner 5-radial graph of* $L(G)$ *is obtained and shown in Figure* 3, *which is isomorphic to* K_8.

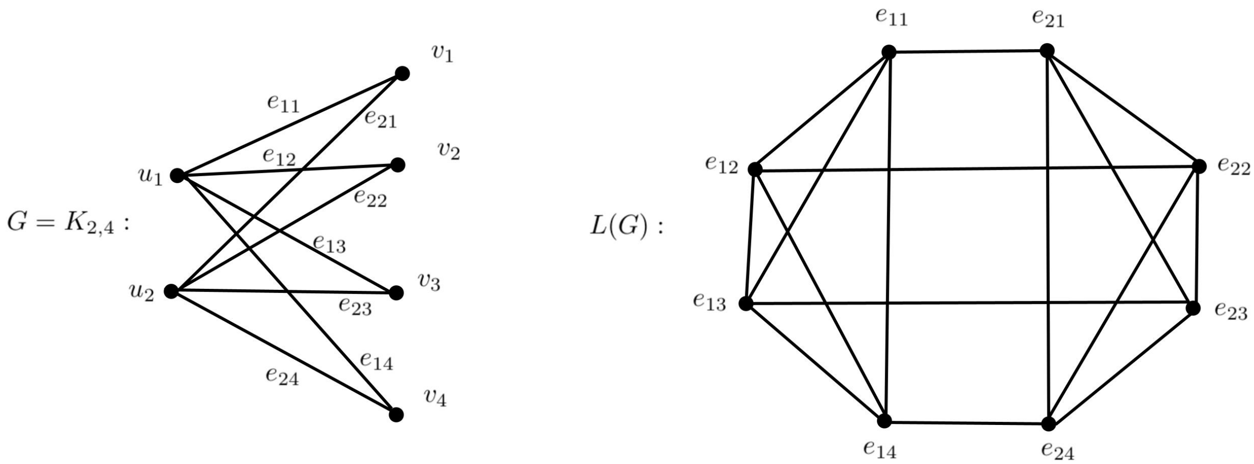

Figure 1. $G = K_{2,4}$ and its Line graph $L(G)$

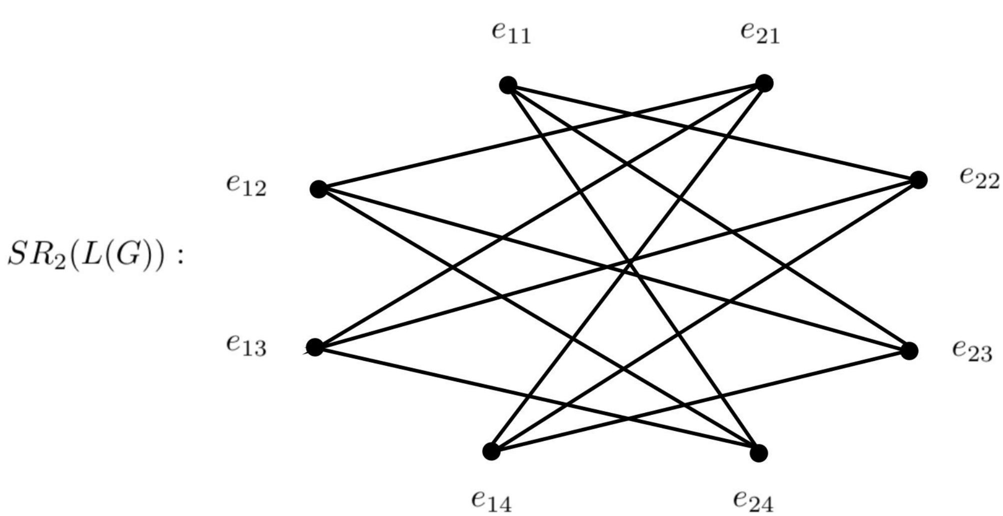

Figure 2. Steiner 2-radial graph of $L(G)$

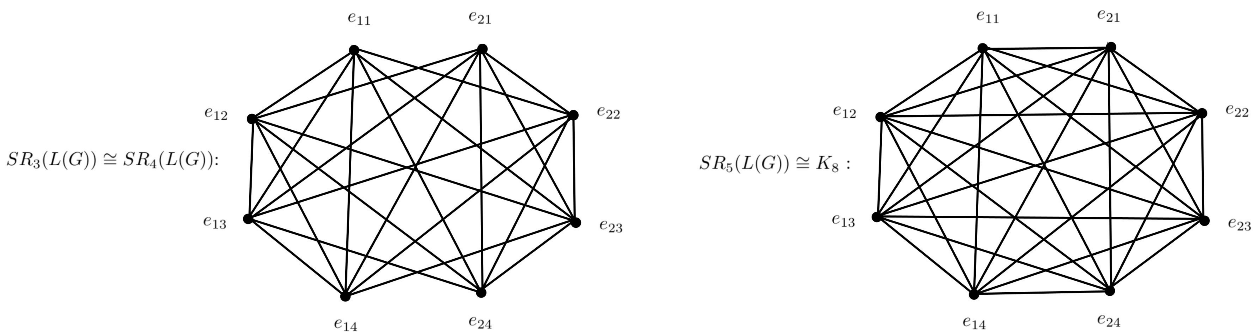

Figure 3. Steiner 3-radial graph of $L(G)$ and Steiner 5-radial graph of $L(G)$

2.1. ***Steiner radial number for Line graph of some graphs***

In the next theorem, we compute the Steiner radial number for tree.

Theorem 2.1. *For any tree T of order p other than star and bistar, $r_S(L(T)) = r_S(T)$. When T is either a star or a bistar, $r_S(L(T))$ is 2 and 3 respectively.*

Proof. Let T be any tree of order p other than star and bistar with m pendant vertices. Let $x_1, x_2, \ldots, x_m$ be the m pendant edges of T. Since $L(T)$ has $p-1$ vertices, $d(S) \leq p-2$ for any vertex subset S of $L(T)$. For any vertex e in $L(T)$, $e_n(e) = p-2$ for $n = m+1$. Let e_i and e_j be two non pendant edges of T. For any set $X \subseteq \{x_1, x_2, \ldots, x_m\}$ with $|X| = m-1$, SD of the set $\{e_i, e_j\} \bigcup X$ is less than $p-2$ and hence e_i and e_j are non-adjacent in Steiner $(m+1)$-radial of $L(T)$. Since $(m+2)$-radius of $L(T)$ is $p-2$ and $\{e_i, e_j, x_1, x_2, x_3, \ldots x_m\}$ has SD $p-2$ in $L(T)$, the Steiner $(m+2)$-radial of $L(T)$ is K_p. Therefore, $r_S(L(T)) = m+2$. By Theorem A, the result follows.

When T is a star, $L(T)$ is a complete graph on $p-1$ vertices and hence $r_S(L(T)) = 2$. When T is a bistar $B(m, n)$ on $m+n+2$ vertices, $L(T)$ is a graph obtained by identifying a vertex of K_{m+1} with a vertex of K_{n+1}. This shows that $L(T) \in F_{12}$ and by proposition 2.1 , $r_S(L(T)) = 3$. □

In the next theorem, we compute the Steiner radial number for complete bipartite graph.

Proposition 2.2. *For any complete bipartite graph K_{p_1,p_2} with $p_1 \leq p_2$ and $p_1 \neq 1$, $r_S(L(K_{p_1,p_2})) = p_2 + 1$.*

Proof. Let $\{u_1, u_2, \ldots, u_{p_1}\}$ and $\{v_1, v_2, \ldots, v_{p_2}\}$ be the two partitions of K_{p_1,p_2}. Let $\{e_{i,j} : 1 \leq i \leq p_2, 0 \leq j \leq p_1 - 1\}$ be the vertices of $L(K_{p_1,p_2})$. For each vertex $e_{i,j}$,

$$e_n(e_{i,j}) = \begin{cases} 2n-2 \quad , & \text{if} \quad n \leq p_1 \\ n + p_1 - 2, & \text{if} \quad p_1 + 1 \leq n \leq p_2 \\ p_1 + p_2 - 1, & \text{if} \quad p_2 + 1 \leq n \leq p_1 p_2 \end{cases}$$

Since any set of p_2 vertices having the vertices $e_{1,0}$ and $e_{1,1}$ has SD less than n-radius, $r_S(L(K_{p_1,p_2})) > p_2$. By graph symmetry, if $e_{1,0}$ is adjacent to all the remaining vertices in $SR_{p_2+1}(L(K_{p_1,p_2}))$, then the result follows. By division algorithm, $p_2 = kp_1 + r$. The set $S = \{e_{i,j} : 1 \leq i \leq kp_1 + r$

and $j = (i-1)\ mod\ p_1\}$ is a set of p_2 vertices whose SD is $p_1 + p_2 - 2$. By adding any vertex $e_{i,j} \in V(L(K_{p_1,p_2})) - S$, SD of $S \cup e_{i,j}$ is $p_1 + p_2 - 1$. Hence $e_{1,0}$ is adjacent all the vertices of $SR_{p_2+1}(L(K_{p_1,p_2}))$. □

Steiner radial numbers highlight distinct variations between a graph and its line graph. The following Theorem demonstrates the existence of graphs with specified Steiner radial numbers for both the original graph and its line graph

Theorem 2.2. *For any pair of positive integers $a, b \geq 3$ with $a \leq b$ there exists a graph whose Steiner radial number is a and Steiner radial number of its line graph is b.*

Proof. By taking $p_1 = a - 1$ and $p_2 = b - 1$, using Theorem B and proposition 2.2, the result follows. □

2.2. *Steiner radial number for Total graph of a tree*

In the upcoming theorem, we determine the Steiner radial number for the total graph of a tree.

Theorem 2.3. *For any tree T of order p with $m(\neq p-1)$ pendant vertices, $r_S(T(T)) = 2m + 2$.*

Proof. Let T be a tree of order p with m number of pendant vertices and n number of pendent edges. If $n \geq 2m$, then the n-eccentricities of all the vertices are equal. Since a tree other than star has at least two non-pendant vertices, SD of set of $2m$ or $2m + 1$ vertices with two non-pendant vertices is less than n-radius. But the set of $2m + 2$ vertices having all pendant vertices and edges is equal to n-radius. Hence the result follows. □

In the next proposition, we compute the Steiner radial number for power graph.

Proposition 2.3. *Let G be a graph on $p \geq 3$ vertices with radius r. Then*

$$r_S(G^k) = \begin{cases} 3, & \text{for} \quad r \leq k < d \\ 2, & \text{for} \quad r = d. \end{cases}$$

Proof. When $r \leq k < d$, $G^k \in F_{12}$ and by proposition 2.1, $r_S(G^k) = 3$. When $r = d$, $G^k \cong K_p$ and hence $r_S(G^k) = 2$. □

In the next proposition, we compute the Steiner radial number for join of two graphs.

Proposition 2.4. *If G_1 and G_2 are non-trivial trees with p_1 and p_2 vertices respectively and none of them is a star graph, then $r_S(G_1 + G_2) = min\{p_1, p_2\}$.*

Proof. Assume $p_1 \leq p_2$. Let $u_1, u_2, \ldots, u_{p_1}$ be the vertices of G_1 and $v_1, v_2, \ldots, v_{p_2}$ be the vertices of G_2. In $G_1 + G_2$, $e_n(u_i) = \left\{\begin{smallmatrix} n, & \text{if } n<p_1 \\ n-1, & \text{if } n\geq p_1 \end{smallmatrix}\right.$ and $e_n(v_j) = \left\{\begin{smallmatrix} n, & \text{if } n<p_2 \\ n-1, & \text{if } n\geq p_2 \end{smallmatrix}\right.$ for any i and j, $1 \leq i \leq p_1$ and $1 \leq j \leq p_2$ respectively. If $n < p_1$, $rad_n(G_1 + G_2) = n$ and in this case $SR_n(G_1 + G_2)$ is not complete on $p_1 + p_2$ vertices. No one of the vertex u_i is adjacent to any one of v_j, $1 \leq i \leq p_1$ and $1 \leq j \leq p_2$. Hence $r_S(G_1 + G_2) \geq n + 1$ whenever $n < p_1$. Suppose that $n \geq p_1$. In this case $rad_n(G_1 + G_2) = n - 1$. Let X be a subset of $V(G_1 + G_2)$ with n-elements. If X contains at least one element from each G_1 and G_2, then SD of X is $n - 1$. This implies that any two vertices in $SR_n(G_1 + G_2)$ are adjacent. Hence the result follows. □

2.3. Steiner radial number for complement of a tree

In the next result, we obtain a Steiner radial number for complement of a tree.

Theorem 2.4. *Let T be a tree other than star. If T has either a pendant vertex which is adjacent to a vertex of degree 2 or a vertex and its neighbours all of degree 2, then $r_S(\overline{T}) = n$. Otherwise $r_S(\overline{T}) = d + 2$ where d is the minimum degree among the vertices of degree ≥ 3.*

Proof. Let T be a tree and v be a vertex of T. If v is a pendant vertex which is adjacent to a vertex of degree 2, then $e_n(v, \overline{T}) = n$ while $n \leq 3$ and $n - 1$ while $n \geq 4$. If v is a vertex of degree 2 whose neighbours are also of degree 2, then $e_n(v, \overline{T}) = n$ while $n \leq 3$ and $n - 1$ while $n \geq 4$. If T has a vertex and its neighbours all of degree 2, then $rad_n(\overline{T}) = n$ while $n \leq 3$ and $n - 1$ while $n \geq 4$. In case of $n \leq 3$, $SR_n(\overline{T})$ is not complete. Since there is no set of n-elements containing at least two peripheral vertices of T with SD n. When $n = 4$, $rad_n(\overline{T}) = 3$ and $SR_n(\overline{T})$ is complete.

Assume that T has neither a pendant vertex adjacent to a vertex of degree 2 nor a vertex and its neighbours all of degree 2. In this case every vertex v of T is adjacent to at least one vertex say u of degree ≥ 3. Among all the vertices of T with degree at least 3, let v has the smallest degree d. Any set S having the vertex v and a non-neighbour of v is of SD $|S| - 1$. Any set S having the vertices from $N[v]$ is of SD $|S|$. Hence for each vertex u_i of T with degree d_i, $e_n(u_i, \overline{T}) = \begin{cases} n, & \text{if } n \leq d_i + 1 \\ n-1, & \text{if } n \geq d_i + 2 \end{cases}$ and for each vertex $w \in N(u_i)$, $e_n(w, \overline{T}) = \begin{cases} n, & \text{if } n \leq d_i + 1 \\ n-1, & \text{if } n \geq d_i + 2 \end{cases}$. This implies that $rad_n(\overline{T}) = n$ if $n \leq d + 1$ and $n - 1$ if $n \geq d + 2$. In $\overline{T}$, any set S on $(d + 1)$-elements having at least two peripheral vertices is of SD d. Hence $SR_{d+1}(\overline{T})$ is not complete. Take $n = d + 2$. Since any set S of size $d + 2$ having any two pendant vertices of T and at least two peripheral vertices of T is of SD $d + 1$ in $\overline{T}$, $SR_n(\overline{T})$ is complete. Hence the result follows. □

If T is a star on p vertices, then $\overline{T}$ is $K_{p-1} \cup K_1$ and by Theorem C, $r_S(\overline{T}) = 3$.

Corollary 2.1. *For any positive integer n,* $r_S(\overline{P_n}) = \begin{cases} 4, & \text{if } \quad n \geq 4, \\ n, & \text{if } \quad n = 1, 2, 3. \end{cases}$

Proof. By 2.4, the result is true for $n \geq 5$. When $n = 4$,$\overline{P_n} \cong P_4$ and hence by Theorem A, $r_S(\overline{P_4}) = 4$.When $n = 3$, $\overline{P_n} \cong P_2 \cup K_1$ and hence by Theorem C, $r_S(\overline{P_3}) = 3$.When $n = 2$, $\overline{P_n}$ is totally disconnected and hence by Theorem D, $r_S(\overline{P_2}) = 2$. When $n = 1$, the result is trivial. □

Corollary 2.2. *For any positive integers m_1 and m_2 with $m_1 \leq m_2$, $r_S(\overline{B_{m_1,m_2}}) = m_1 + 3$*

2.4. Steiner radial number for Mycielskians of some standard graphs

In the following results, we shall explore the computation of the Steiner radial number for the Mycielskians of a certain family of graphs.

Theorem 2.5. *For any complete bipartite graph $K_{m,n}$, $r_S(\mu(K_{m,n})) = 3$.*

Proof. Let A=$\{a_1, a_2, \ldots, a_m\}$ and B=$\{b_1, b_2, \ldots, b_n\}$ be the two partitions of vertex set V and let E be edge set of $K_{m,n}$. Then the Mycielskian of $K_{m,n}$ is $\mu(K_{m,n})$ with vertex set V'=$A \cup B \cup A' \cup B' \cup \{u\}$, where A'=$\{a'_i : a_i \in A\}$ and B'=$\{b'_i : b_i \in B\}$ and edge set E'=$E \cup \{a_i b'_j : a_i b_j \in E\} \cup \{b_i a'_j : b_i a_j \in E\} \cup \{a'_i u : a'_i \in A\} \cup \{b'_i u : b'_i \in B\}$. Since $\mu(K_{m,n})$ is not isomorphic to

$K_{2(m+n)+1}$, by theorem D, $r_S(\mu(K_{m,n})) \geq 3$.For any vertex v in $\mu(K_{m,n})$, $e_3(v) = 3$ and hence $3-$radius is 3. The Steiner 3-radial graph of $\mu(K_{m,n})$ is $SR_3(\mu(K_{m,n}))$ has the vertex set V' and edge set can be obtained from the following cases.

Case 1. Each of the sets $\{a_i, a_j, u\}$, $\{b_i, b_j, u\}$, $\{a'_i, a'_j, a_i\}$ and $\{b'_i, b'_j, b_i\}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$ with $i \neq j$ have SD 3 respectively. Hence the subgraphs $[A], [B], [A']$ and $[B']$ are complete in $SR_3(\mu(K_{m,n}))$.

Case 2. Each of the sets $\{a_i, b_j, u\}$, $\{a_i, a'_i, u\}$, $\{a_i, b'_j, a'_i\}$, $\{b_j, b'_j, u\}$, $\{b_j, b'_j, a'_i\}$ and $\{b'_j, b'_k, a'_i\}$ for all $1 \leq i \leq m$, $1 \leq j, k \leq n$ with $j \neq k$ have SD 3 respectively. Therefore, all the elements in A, B, A', B' and $\{u\}$ are mutually adjacent in $SR_3(\mu(K_{m,n}))$. Thus Steiner 3-radial of $\mu(K_{m,n})$ is $K_{2(m+n)+1}$. □

Theorem 2.6. *'Let G be a graph of order n with diameter at most 2 and $\Delta(G) = n - 1$. Then $r_S(\mu(G)) = 3$.*

Proof. Let V=$\{v_1, v_2, \ldots, v_n\}$ be the vertex set and E be the edge set of G.Then the Mycielskian of G is $\mu(G)$, has the vertex set $V \cup V' \cup \{u\}$, where $V' = \{v'_i : v_i \in V\}$ and edge set $E \cup \{v_i v'_j : v_i v_j \in E\} \cup \{v'_i u : v'_i \in V'\}$. Since $\Delta(G) = n - 1$, there exists a vertex v in G such that $deg(v) = n - 1$, for that v, $e_3(v) = 3$ in $\mu(G)$. Hence $3-$radius of $\mu(G)$ is 3.

Case 1. Assume that diameter of G is 1. Then each of the sets $\{v_i, v_j, u\}$, $\{v_i, v'_i, u\}$ and $\{v_i, v'_i, v'_j\}$ in $\mu(G)$ have SD 3 and forms a K_3 in the corresponding Steiner $3-$radial graph. Thus $SR_3(\mu(G)) = K_{2n+1}$.

Case 2. Assume that diameter of G is 2. Then each of the sets $\{v_i, v_j, u\}$, $\{v'_i, v'_j, v'_k\}$ and $\{v_i, v'_i, u\}$ have SD 3. Hence in $SR_3(\mu(G))$ the subgraph $[V] \cup \{u\}$ and $[V'] \cup \{u\}$ are complete. Hence to prove $SR_3(\mu(G))$ is complete, it is enough to show that every vertex in V is adjacent to all the vertices in V'.

Subcase a. If v_i is a full degree vertex in G, the sets $\{v_i, v'_i, v'_j\}$ having SD 3. Thus the subgraph $[V \cup V']$ is complete in $SR_3(\mu(G))$. Hence the result follows.

Subcase b. If v_i is not a full degree vertex, then there exists a vertex v_j not adjacent with v_i. Since diameter of G is 2, there exists a vertex v_k common to v_i and v_j. Then the sets $\{v_i, v'_k, v'_j\}$ having SD 3 and hence the subgraph $[V \cup V']$ is complete in $SR_3(\mu(G))$. Hence the result follows. □

Theorem 2.7. *For any path P_n with $n \geq 7$ vertices, $SR_3(\mu(P_n)) = K_{2n+1} - K_{1,n}$*

Proof. Let V=$\{v_0, v_1, v_2, \ldots, v_{n-1}\}$ be the vertex set and E be the edge set of P_n.Then the Mycielskian of P_n is $\mu(P_n)$, has the vertex set $V \cup V' \cup \{u\}$, where $V' = \{v'_i : v_i \in V\}$ and edge set $E \cup \{v_i v'_j : v_i v_j \in E\} \cup \{v'_i u : v'_i \in V'\}$. Since $\mu(P_n)$ is not isomorphic to K_{2n+1}, by theorem D, $r_S(\mu(P_n)) \geq 3$. For any vertex in $\mu(P_n)$,$e_3(v_i) = 6$, $e_3(v'_i) = 5$ and $e_3(u) = 4$ for all $0 \leq i \leq n-1$ and hence $3-$radius is 4. $SR_3(\mu(P_n))$ has the vertex set as in $\mu(P_n)$ and edge set can be obtained from the following cases.

Case 1. The sets $\{v_i, v_j, u : d(v_i, v_j) \geq 3 \ in \ P_n\}$ and $\{v_i, v_j, v_k : d(v_i, v_j) = 2 \ and \ d(v_i, v_k) = 2 \ or \ d(v_j, v_k) = 2 \ in \ P_n\}$ have SD 4 and forms K_3 in $SR_3(\mu(P_n))$. Let v_k be the eccentric vertex of v_i in P_n. Then $\{v_i, v_j, v'_k : d(v_i, v_j) = 1 \ in \ P_n\}$ has SD 4. Hence in $SR_3(\mu(P_n))$, the subgraph $[V] \cup \{u\}$ is complete.

Case 2. For any two vertices v'_i and v'_j in V', there exists a vertex $v_k \in V$ such that $d(v'_j, v_k) = 2$

or $d(v_i', v_k) = 2$. Hence the sets $\{v_i', v_j', v_k\}$ forms K_3 in $SR_3(\mu(P_n))$. Thus the subgraph $[V']$ is complete in $SR_3(\mu(P_n))$.
Case 3. Let v_k be the eccentric vertex of v_i in P_n. Then the sets $\{v_i, v_j', v_k\}$ and $\{v_i, v_j', v_k'\}$ have $SD\ 4$ when v_i and v_j are adjacent and non-adjacent respectively. Hence the subgraphs $[V]$ and $[V']$ are mutually adjacent in $SR_3(\mu(P_n))$.
Case 4. For any two vertices $\{v_i', u\}$, there is no vertex w in $\mu(P_n)$ such that $d(\{v_i', u, w\}) = 4$. Hence the vertices v_i' and u are not adjacent in $SR_3(\mu(P_n))$. Hence the result follows. □

Observation 2.2. *For any path P_n, $r_S(\mu(P_n)) = 3$, for $1 \leq n \leq 4$.*

Theorem 2.8. *For any path P_n with $n \geq 5$ vertices, $r_s(\mu(P_n)) = \lceil \frac{n}{3} \rceil + 2$.*

Proof. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set and E be the edge set of P_n. Then the Mycielskian of P_n is $\mu(P_n)$, which has the vertex set $V \cup V' \cup \{u\}$ where $V' = \{v_i' : v_i \in V$ and edge set $E \cup \{v_i v_j' : v_i v_j \in E\} \cup \{v_i' u : v_i' \in V'\}$. For each vertex $w \in V \cup V'$, $e_{\lceil \frac{n}{3} \rceil + 1}(w) = 2\lceil \frac{n}{3} \rceil + 1$ and $e_{\lceil \frac{n}{3} \rceil + 1}(u) = 2\lceil \frac{n}{3} \rceil$. Hence $(\lceil \frac{n}{3} \rceil + 1)-$ radius is $2\lceil \frac{n}{3} \rceil$.

Consider the set S with $\lceil \frac{n}{3} \rceil$ vertices from V such that $d(v_i, v_j) \geq 3$ if $v_i, v_j \in S$. Then the set $S - \{v_i\} \cup \{u, v_i'\}$ with $(\lceil \frac{n}{3} \rceil + 1)$ vertices has $SD\ 2(\lceil \frac{n}{3} \rceil - 1) + 1 < 2\lceil \frac{n}{3} \rceil$. Therefore, u is not adjacent with v_i' in Steiner $(\lceil \frac{n}{3} \rceil + 1)$ - radial graph of $\mu(P_n)$. Hence, $r_s(\mu(P_n)) = \lceil \frac{n}{3} \rceil + 2$.

Now $e_{\lceil \frac{n}{3} \rceil + 2}(u) = 2\lceil \frac{n}{3} \rceil + 1$ and $e_{\lceil \frac{n}{3} \rceil + 2}(w) = 2\lceil \frac{n}{3} \rceil + 2$ where $w \in V \cup V'$ and hence $(\lceil \frac{n}{3} \rceil + 2)$ - radius is $2\lceil \frac{n}{3} \rceil + 1$. Consider the set $S' = \{u\} \cup \{v_j'\} \cup S_i$ where $S_i = \{v_i, v_{i+3}, v_{i+6}, \ldots\}$ with $|S_i| = \lceil \frac{n}{3} \rceil$ for $i = 1, 2, 3$ and $v_j \in S_i$. Clearly, SD of the set S' is $2\lceil \frac{n}{3} \rceil + 1$. Hence it gives u is adjacent to all vertices of V and V' and all vertices of V is adjacent to V'.

Hence, to prove $SR_{\lceil \frac{n}{3} \rceil + 2}(\mu(P_n)) = K_{2n+1}$, it is enough to prove the subgraphs $[V]$ and $[V']$ are complete in $SR_{\lceil \frac{n}{3} \rceil + 2}(\mu(P_n))$.
Case 1. For every vertices $v_j \notin S_i$ and $v_j \in V$, the set $S_i \cup \{v_j, v_j'\}$ has $SD\ 2\lceil \frac{n}{3} \rceil + 1$. Hence subgraph $[V]$ is complete in $SR_{\lceil \frac{n}{3} \rceil + 2}(\mu(P_n))$.
Case 2. Let $S_i' = \{v' \in V' : v \in S_i \subseteq V\}$.
Subcase a. For any $v_j', v_k' \in S_i'$, the sets $S_i - \{v_j\} \cup \{v_{j-1}', v_j', v_k\}$ or $S_i - \{v_j\} \cup \{v_{j+1}', v_j', v_k'\}$ have $SD\ 2\lceil \frac{n}{3} \rceil + 1$.
Subcase b. For any v_j' and v_k' in different S_i', then clearly either v_{k+1} or v_{k-1} in S_i'. Hence, the sets $S_i - \{v_{k+1}\} \cup \{v_j', v_k', v_{k+1}'\}$ or $S_i - \{v_{k-1}\} \cup \{v_j', v_k', v_{k-1}'\}$ have $SD\ 2\lceil \frac{n}{3} \rceil + 1$. Therefore, the above two subcases make $[V']$ is complete in $SR_{\lceil \frac{n}{3} \rceil + 2}(\mu(P_n))$. Hence the result follows. □

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