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Making graphs solvable in peg solitaire

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Abstract

In 2011, Beeler and Hoilman generalized the game of peg solitaire to arbitrary connected graphs. Since then peg solitaire has been considered on quite a few classes of graphs. Beeler and Gray introduced the natural idea of adding edges to make an unsolvable graph solvable. Recently, the graph invariant $\mathrm{ms}(G)$, which is the minimal number of additional edges needed to make G solvable, has been introduced and investigated on banana trees by the authors. In this article, we determine $\mathrm{ms}(G)$ for several families of unsolvable graphs. Furthermore, we provide some general results for this number of Hamiltonian graphs and graphs obtained via binary graph operations.

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1. Introduction

In [3], Beeler and Hoilman introduced the game of peg solitaire on graphs as a generalization of the classical peg solitaire game:

Given a connected, undirected graph $G = (V, E)$, we can put pegs in the vertices of G. Given three vertices u, v, w with pegs in u and v and a hole in w such that $uv, vw \in E$, we can jump with the peg from u over v into w, removing the peg in v (see Figure 1). This jump will be denoted as $u \cdot \vec{v} \cdot w$.

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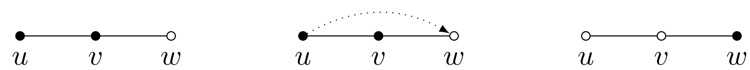

Figure 1. A jump in peg solitaire.

In general, we begin with a *starting state* $S \subset V$ of vertices that are empty (i.e., without pegs). A *terminal state* $T \subset V$ is a set of vertices that have pegs at the end of the game such that no more jumps are possible. A terminal state T is *associated* to a starting state S, if T can be obtained from S by a series of jumps. We will always assume that the starting state S consists of a single vertex.

The goal of the original game is to remove all pegs but one. This is not possible for all graphs. Therefore, we use the following notation. A graph G is called

- *solvable*, if there is some $v \in V$ such that the starting state $S = \{v\}$ has an associated terminal state consisting of a single vertex.
- *freely solvable*, if for all $v \in V$ the starting state $S = \{v\}$ has an associated terminal state consisting of a single vertex.
- k*-solvable*, if there is some $v \in V$ such that the starting state $S = \{v\}$ has an associated terminal state consisting of k vertices.
- *strictly* k*-solvable*, if G is k-solvable but not ℓ-solvable for any $\ell < k$. In that case G has *solitaire number* $\mathrm{Ps}(G) = k$.

Peg solitaire has been considered for quite a few classes of graphs, including paths, complete graphs, stars, double stars and caterpillars (for more results and variants see [7, 2, 3, 4, 5, 6, 8, 9, 11]).

In 2016 [1], Beeler and Gray considered the natural question of determining the minimum number of edges necessary to guarantee the solvability of a connected graph. Furthermore, they posed the question of how much the addition of edges can influence the solvability of a graph. In [8], the authors defined the smallest number $\mathrm{ms}(G)$ of edges that have to be added to a graph G to make it solvable and provided an example showing that the solvability might be improved arbitrarily good with the addition of just one edge. Since many unsolvable graphs exist, it seems natural trying to compute $\mathrm{ms}(G)$ for these graphs. We do this for several graph classes in Section 2 and provide general results in Section 3. First, we start with two rather obvious, but important facts.

Every complete graph is solvable except for K_1, which cannot have a starting state and an associated terminal state both of size one. Therefore, $\mathrm{ms}(G)$ exists for every graph $G \neq K_1$ (and thus we exclude the case $G = K_1$ whenever considering $\mathrm{ms}(G)$). Furthermore, we have the following relationship between $\mathrm{ms}(G)$ and $\mathrm{Ps}(G)$.

Proposition 1.1. *For every connected graph $G = (V, E)$, we have* $\mathrm{ms}(G) \leq \mathrm{Ps}(G) - 1$.

Proof. The cases $|V(G)| < 3$ and $\mathrm{Ps}(G) = 1$ are trivial, hence we assume $|V| \geq 3$ and $\mathrm{Ps}(G) \geq 2$. Let $T = \{t_1, t_2, \ldots, t_{\mathrm{Ps}(G)}\}$ be a terminal state of G with minimal number of pegs. We find some

$u \in V \setminus T$ with $ut_1 \in E$ or $ut_2 \in E$, w.l.o.g. assume $ut_1 \in E$. After adding the edge t_1t_2, we can jump $t_2 \cdot \vec{t_1} \cdot u$. Next we add ut_3 (unless it does already exist) and jump $t_3 \cdot \vec{u} \cdot t_1$. Continuing in this alternating manner (adding t_4t_1, t_5u and so on) yields a terminal state with one peg after adding at most $\mathrm{Ps}(G) - 1$ edges. □

2. Graph classes

In this section we determine $\mathrm{ms}(G)$ for some graph classes. We start with trivial results where $\mathrm{ms}(G)$ is either 0 or 1. These follow from known results on $\mathrm{Ps}(G)$ for the respective graphs together with Proposition 1.1.

Proposition 2.1. *Let K_n be the complete graph on n vertices, $K_{m,n}$ be the complete bipartite graph on $m + n$ vertices, P_n the path on n vertices, C_n the cycle on n vertices and $W(B)$ the windwill with B blades. Table 1 gives* $\mathrm{ms}(G)$ *for these graphs.*

G	K_n	$K_{m,n}$	P_{2n}	P_{2n+1}	C_{2n}	C_{2n+1}	$W(B)$
$\mathrm{Ps}(G)$	1	1	1	2	1	2	1
$\mathrm{ms}(G)$	0	0	0	1	0	1	0

Table 1. $\mathrm{ms}(G)$ for some graph classes.

To determine $\mathrm{ms}(G)$ for windmills with pendants and for stars, we define a more general version of generalized windmills. Let $P, B \in \mathbb{Z}_{\geq 0}$. A *general windmill* $W^*(P, B)$ with B blade vertices and P pendant vertices is a graph G with a vertex u that is adjacent to exactly P pendant vertices and B vertices that lie in blades, i.e., the induced subgraph defined by these vertices is a disjoint union of paths. Figure 2 shows a windmill $W^*(3, 7)$.

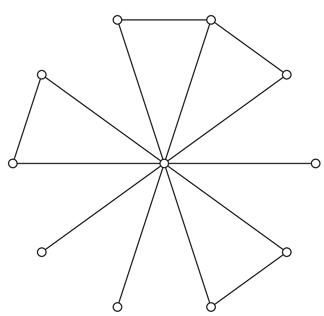

Figure 2. A general windmill $W^*(3, 7)$.

Note that the parameters B and P do not fully characterize a general windmill, since, for example, the windmill $W^*(1, 4)$ could have 4 vertices lying in one blade of length 4 or two times two vertices lying in blades of length 2. This will not be a problem for us since we are only interested in the total number of blade vertices. Note also that in [4] the windmill $W(P, B)$ has $2B$ blade vertices (which by definition induce a union of B paths of length 2), whereas $W^*(P, B)$ has only B blade vertices (this is due to the fact that we do not make assumptions on the size of the blades).

We begin with a result about solvability of general windmills. We assume $(P, B) \notin \{(1,0),(2,0)\}$ since otherwise we are dealing with the paths P_2 resp. P_3. To deal with general windmills and double stars we need the following lemma.

Lemma 2.1 ([1, Corollary 2.2]). *A graph G is not solvable if it contains a vertex which is adjacent to at least $\frac{1}{2}|V(G)|$ leaves.*

Proposition 2.2. *Let $(P, B) \notin \{(1,0),(2,0)\}$. The general windmill $W^*(P, B)$ is solvable if and only if $B \geq P$.*

Proof. If $P > B$, Lemma 2.1 immediately yields the unsolvability of $W^*(P, B)$ since every pendant vertex is a leaf. If $B \geq P$, we can use the same strategy as in [4] to solve the graph:

Start with a hole in a pendant vertex. Jump from another pendant vertex over the centre into the hole. From now on, whenever the centre is empty, jump from two adjacent blade vertices into the centre. If the centre is not empty, jump from a pendant vertex over the centre into a blade vertex (in such a way that adjacent blade vertices are filled with pegs). Iterating this process yields the solvability. □

Remark 2.1. Using the above strategy, we can also show that $\mathrm{Ps}(W^*(P, B)) = P - B + 1$ if $P > B$.

This gives the following result about $\mathrm{ms}(G)$ for general windmills.

Proposition 2.3. *Let $(P, B) \notin \{(1,0),(2,0)\}$. We have*

$$\mathrm{ms}(W^*(P, B)) = \max\left\{\left\lceil \frac{P-B}{4} \right\rceil, 0\right\}. \tag{1}$$

Proof. Note that if we add any non-existent edge to a graph $W^*(P, B)$, there are four possibilities:

1. We add an edge between two pendant vertices. This results in a graph $W^*(P-2, B+2)$.
2. We add an edge between a pendant vertex and a blade vertex. This yields a graph $W^*(P-1, B+1)$.
3. We add an edge between two blade vertices lying in disjoint blades. This gives us a graph $W^*(P, B)$.
4. We add an edge between two blade vertices lying in the same blade. The resulting graph will not be a general windmill any more but has the same number of leaves.

Due to Lemma 2.1, we need to add edges such that the number of leaves gets reduced. This means that the fourth option will never increase the solvability. Hence, we only need to consider the first three types of edges. Therefore, adding an edge will always yield another general windmill. Then, $\mathrm{ms}(W^*(P, B))$ is the least number of edges that result in a graph $W^*(P', B')$ with $B' \geq P'$. This implies that the best option is to join two pendant vertices with each edge added. Every such edge lowers $\mathrm{Ps}(W^*(P, B))$ by 4 (except for potentially the last edge). Thus, $\mathrm{ms}(W^*(P, B))$ is the smallest k such that $P - B + 1 - 4k \leq 1$, i.e., $\mathrm{ms}(W^*(P, B)) = \left\lceil \frac{P-B}{4} \right\rceil$. □

Since $K_{1,n}$ is a general windmill $W^*(n, 0)$ and $W(P, B)$ is a general windmill $W^*(P, 2B)$, we get the following result.

Corollary 2.1. • *Let $K_{1,n}$ be the star graph with n leaves. If $n \geq 3$, we have* $\operatorname{ms}(K_{1,n}) = \lceil \frac{n}{4} \rceil$.

• *We have* $\operatorname{ms}(W(P,B)) = \max\left\{\lceil \frac{P-2B}{4} \rceil, 0\right\}$.

We now turn our attention to double stars. The double star $DS(L,R)$ is the union of the stars $K_{1,L}$ and $K_{1,R}$ together with an edge connecting the centres of the two stars. The following result on their solvability is known.

Proposition 2.4 ([4, Theorem 3.1])**.** *Given $L, R \in \mathbb{N}$ with $R \geq L \geq 1$, the double star $DS(L,R)$ is solvable if and only if $R \leq L+1$.*

Proposition 2.5. *For every $L, R \in \mathbb{N}$ with $R \geq L \geq 1$ we have*

$$\operatorname{ms}(DS(L,R)) = \left\lceil \frac{R-L-1}{4} \right\rceil.$$

Proof. Assume $R \geq L+2$ (since otherwise $DS(L,R)$ is solvable and we get $\operatorname{ms}(DS(L,R)) = 0$). First we show that at least $\lceil \frac{R-L-1}{4} \rceil$ edges need to be added. The idea is similar to the proof of Proposition 2.3. The right centre of the graph $DS(L,R)$ has R adjacent leaves. Since $R \geq L+2$ holds, we have $R \geq \frac{1}{2}(R+L+2) = \frac{1}{2}|V|$, hence Lemma 2.1 yields that $DS(L,R)$ is unsolvable. Moreover, we have to reduce the number of leaves adjacent to the right centre. Adding at most $\lceil \frac{R-L-1}{4} \rceil - 1$ edges results in a graph such that the right centre has at least $R + 2 - 2\lceil \frac{R-L-1}{4} \rceil$ adjacent leaves. By considering the four possible cases $R - L \equiv 0,1,2,3 \mod 4$, we see that this quantity is at least $\frac{1}{2}(R+L+2) = \frac{1}{2}|V|$. Hence the graph is still unsolvable, so $\operatorname{ms}(DS(L,R)) \geq \lceil \frac{R-L-1}{4} \rceil$.

To show that adding $\lceil \frac{R-L-1}{4} \rceil$ edges suffices, we start with a hole in the left centre, denoted by c_L. Again, we distinguish four possible cases. If $R - L \equiv 2,3 \mod 4$, we start with solving a subgraph $DS(L,L)$ of $DS(L,R)$. This leaves a subgraph $K_{1,R-L+1}$ with a hole in a pendant vertex (in c_L, to be more specific). Then Corollary 2.1 implies

$$\operatorname{ms}(DS(L,R)) \leq \operatorname{ms}(K_{1,R-L+1}) = \left\lceil \frac{R-L+1}{4} \right\rceil = \left\lceil \frac{R-L-1}{4} \right\rceil.$$

This idea does not work for $R - L \equiv 0,1 \mod 4$ (let d denote this remainder) since the last equation does not hold in those cases. We modify the proof slightly and use the fact $\operatorname{ms}(DS(1,5)) = \operatorname{ms}(DS(1,6)) = 1$. In both cases we start with solving a copy of $DS(L-1,L-1)$, leaving pegs in one left pendant vertex, say ℓ_1, in $R-L+1$ right pendant vertices, denoted by $r_1, r_2, \ldots, r_{R-L+1}$, and in the right centre c_R. We jump $r_1 \cdot \vec{c}_R \cdot c_L$. Using the idea from Theorem 2.2 in [4] (see the proof of Proposition 2.2), we can, after adding $\frac{R-L-d}{4}$ edges $r_{R-L+1}r_{R-L}, r_{R-L-1}r_{R-L-2}, \ldots$, reduce the (solvable) windmill subgraph, which is induced by the vertices c_R and r_i for $i \in [2+d, R-L+1]$, such that only pegs in $r_{2+d}, r_{3+d}, r_{R-L+1}, r_{R-L}$ remain. The subgraph induced by ℓ_1, c_L, c_R and r_i for $i \in [2, 3+d] \cup \{R-L+1, R-L\}$, which contains the last remaining pegs and has a hole in c_R, is solvable. $\square$

3. General results

Note that $\mathrm{ms}(G) \leq \mathrm{ms}(H)$ holds if H is a spanning subgraph of G. On the other hand, if $\mathrm{Ps}(G) \leq \mathrm{Ps}(G')$, but no relationship between G and G' is known, we cannot conclude anything about the relationship of $\mathrm{ms}(G)$ and $\mathrm{ms}(G')$. For example, if $B_{n,k}$ denotes the banana tree on n stars, we have $\mathrm{Ps}(B_{2,k}) = 2k - 2$ and $\mathrm{ms}(B_{2,k}) \leq 2$ (see [8]), but $\mathrm{Ps}(K_{1,k}) = k - 1$ and $\mathrm{ms}(K_{1,k}) = \lceil \frac{k}{4} \rceil$.

Furthermore, it is not true that the edges which have to be added to make G solvable have to connect vertices of a best possible terminal state. Therefore, in general, we have to start with adding edges instead of solving the original graph first and adding edges later (again the banana tree $B_{2,k}$ is a nice example for this phenomenon).

Now we turn to some general bounds for $\mathrm{ms}(G)$. Since every path is at least 2-solvable, the following result is immediate (see [3, Corollary 2.5] for some connections between solvability and the existence of a Hamiltonian cycle).

Proposition 3.1. *If G has a Hamiltonian path after adding k edges, we have* $\mathrm{ms}(G) \leq k + 1$.

This raises the question of how many edges have to be added to a graph to get a Hamiltonian path. There are various criteria (found in many books and web sources) for a graph G which guarantee that G has a Hamiltonian path. Together with Proposition 3.1 these may be used to give bounds on $\mathrm{ms}(G)$ for a given graph G. We will only show a connection to the path partition number of a graph (there are certainly many more). A *path partition* of a graph G is a set of paths such that every vertex of G belongs to exactly one path; the minimum cardinality, denoted by $\pi_p(G)$, of such a partition is called the *path partition number* of G [12].

Proposition 3.2. *For every graph $G = (V, E)$, we have*

$$\mathrm{ms}(G) \leq \pi_p(G).$$

Proof. Start with a minimal path partition $\mathcal{P} = \{P^{(1)}, P^{(2)}, \ldots, P^{(k)}\}$ of G. For each $i = 1, 2, \ldots, k$, let $P^{(i)} = \{p_1^{(i)}, p_2^{(i)}, \ldots, p_{t_i}^{(i)}\}$ with $p_j^{(i)} p_{j+1}^{(i)} \in E$ for $j = 1, 2, \ldots, t_i - 1$. Adding the edges $p_{t_i}^{(i)} p_1^{(i+1)}$ for $i = 1, 2, \ldots, k-1$ yields a Hamiltonian path which is solvable after adding at most one more edge. □

To end the section, we will give some results about binary graph operations. Since the join $G + H$, which is $G \cup H$ together with additional edges connecting every pair of vertices g, h with $g \in V(G)$ and $h \in V(H)$, of any two graphs G and H with $|V(G)|, |V(H)| \geq 2$ is solvable [3, Theorem 2.7], we have $\mathrm{ms}(G + H) = 0$ in that case. The special case, where at least one of the graphs is K_1, can be dealt with in the following way. As usual $G[W]$ denotes the subgraph of G induced by some set $W \subseteq V(G)$.

Lemma 3.1. *The vertex set of a non trivial connected graph G can be partitioned into $V_1, V_2, \ldots, V_\ell$ such that $G[V_i]$ contains a spanning star on at least two vertices for every $i \in \{1, \ldots, \ell\}$.*

Proof. W.l.o.g. we may assume G to be a tree. Let u be a vertex of degree 1 and v be a neighbour of u. Then $G[\{v\} \cup N_v]$, where N_v are the neighbours of v with degree 1, is a star. Iterating this process on the components of $G[V(G) \setminus (\{v\} \cup N_v)]$ yields the statement of the lemma. □

Lemma 3.2. *Let G be a star, the union of two stars or the union of at least 3 non trivial stars. Then $G + K_1$ is solvable.*

Proof. The special case $G = 2K_1$ yields $G + K_1 = P_3$, which is solvable, hence from now on G is not of this form.

Let w denote the vertex corresponding to K_1 and $V_1, V_2, \ldots, V_\ell$ be the vertex sets of the stars from G such that $V_i = \{v_{1,i}, v_{2,i}, \ldots, v_{s_i,i}\}$ ($s_i = |V_i| \geq 2$) and $v_{1,i}$ is adjacent to all the vertices in V_i. Start with a hole in w.

As long as V_i and V_j ($i \neq j$) with each of them containing at least 3 pegs exist, we carry out a double star purge on $V_i \cup V_j \cup \{w\}$ (with centres w and $v_{1,i}$ or $v_{1,j}$, depending on the size of V_i and V_j) until there is a hole in w and at least one of V_i and V_j contains exactly two pegs (one of them being in the star centre $v_{1,i}$ resp. $v_{1,j}$). This results in a configuration, where we have a hole in w and all V_i with $i \neq j$ for some $j \in \{1, \ldots, \ell\}$ contain exactly two pegs.

If V_j contains exactly two pegs, we are done, since we find a solvable windmill subgraph. Otherwise, we can easily reduce the number of pegs in V_j using w and induced cycles of length 3. Again, we obtain a solvable windmill subgraph. □

Proposition 3.3. *Let G be a graph with k isolated vertices. Then $G + K_1$ is solvable if and only if $k \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.*

Proof. The necessity is an immediate consequence of Lemma 2.1. Let us now consider a graph G with $k \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ and let $u_1, u_2, \ldots, u_k$ denote the isolated vertices of G.

The vertex set (ignoring isolated vertices) of G may be partitioned using Lemma 3.1 into $V_1, V_2, \ldots, V_\ell$ such that $V_i = \{v_{1,i}, v_{2,i}, \ldots, v_{s_i,i}\}$ ($s_i = |V_i| \geq 2$) and $v_{1,i}$ is adjacent to all the vertices in V_i. Note that

$$k \leq \sum_{i=1}^{\ell} s_i \tag{2}$$

holds.

If $s_i = 2$ for every $i \in \{1, \ldots, \ell\}$, $G + K_1$ contains a solvable windmill (where w, the vertex corresponding to K_1 in $G + K_1$, is the centre and every V_i forms a blade) because of (2).

Otherwise we can, starting with a hole in u_1 and jumping $u_2 \cdot \vec{w} \cdot u_1$, remove pegs from the sets V_i with $s_i \geq 3$ and from leaves of $G + K_1$ using double star purges on the subgraphs induced by $\{u_j, u_{j+1}, \ldots, u_{j+s_i-2}\} \cup \{w\} \cup (V_i \setminus \{v_{2,i}\})$ (pick the smallest j such that u_j contains a peg) until one of two configurations is reached.

- No V_i with more than two pegs exists. Again, we find some solvable windmill subgraph (which is a subgraph of the graph induced by all vertices with pegs together with w).
- There is a hole in every u_i and at least one V_i contains more than 2 pegs. This can be solved by Lemma 3.2.

□

Combining this result and Lemma 2.1, we get the following proposition.

Proposition 3.4. *Let G be a graph with k isolated vertices. Then*

$$\mathrm{ms}(G + K_1) = \max\left\{\left\lceil\frac{2k - |V(G)|}{4}\right\rceil, 0\right\}.$$

For graphs G and H we denote the Cartesian product of G and H by $G \square H$ and use the (common) notation $(g, h) \in V(G \square H)$ for the vertex induced by $g \in V(G)$ and $h \in V(H)$. Beeler and Hoilman showed that $G \square H$ is solvable if G and H are solvable or if G is solvable and H is distance 2-solvable (meaning it is 2-solvable and the terminal vertices are at distance 2) or if both are 2-solvable [3]. The authors of this paper proved that $P_2 \square G$ is solvable for any connected graph G and conjectured $\mathrm{Ps}(G \square H) = 1$ for any two non trivial connected graphs G and H [10]. Hence, we suggest $\mathrm{ms}(G \square H) = 0$ in that case. This statement seems out of reach at the moment. Since $\mathrm{ms}(G\square H)$ can be a lot smaller than $\mathrm{ms}(G)$ and $\mathrm{ms}(H)$ (for example if both are stars [10]), lower bounds seem difficult to achieve. Upper bounds are mostly trivial, hence we will not continue exploring Cartesian products in this article (although the above mentioned conjecture should definitely be investigated further).

If $G \cup H$ denotes the union of G and H, we have $\mathrm{ms}(G \cup H) \leq \mathrm{ms}(G) + \mathrm{ms}(H) + 2$. To verify this, start with solving G using $\mathrm{ms}(G)$ additional edges. Let $w \in H$ be any vertex such that H can be solved starting with a hole in w when adding $\mathrm{ms}(H)$ edges (and add these to $G \cup H$). Let $v \in H$ be any neighbour of w. Connect the terminal vertex t of G with v by an additional edge and jump $v \cdot \vec{t} \cdot u$ for some $u \in G$. Next, add an edge between u and w and jump $u \cdot \vec{w} \cdot v$. Now solve H.

Iterating this process gives the following result.

Proposition 3.5. *Let G be a graph with connected components $G_1, G_2, \ldots, G_k$. Then*

$$\mathrm{ms}(G) \leq 2(k-1) + \sum_{i=1}^{k} \mathrm{ms}(G_i).$$

4. Open problems

Since the definition of $\mathrm{ms}(G)$ is new, there are some more questions that naturally arise. For instance, $\mathrm{ms}(G)$ could be determined for other classes of graphs (for example caterpillars, banana trees, or trees of diameter 4).

One might also define the number $\mathrm{ms}^*(G)$ to be the minimal number of edges that have to be added to make a graph freely solvable (since K_n is freely solvable, this number exists and is clearly greater than or equal to $\mathrm{ms}(G)$). It would be interesting to examine this number for certain graph classes, get general results and see how this quantity relates to $\mathrm{ms}(G)$.

It would also be interesting to connect $\mathrm{ms}(G)$ to the edge-critical graphs defined in [1].

Moreover, adding edges may yield solvable graphs even if the original graphs are disconnected. This gives the possibility to study peg solitaire on graphs for which it was previously not possible. Obtaining more results on disconnected graphs than the ones in the previous section would be a desirable goal.

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