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The partition dimension for a subdivision of a homogeneous firecracker

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Abstract

Finding the partition dimension of a graph is one of the interesting (and uncompletely solved) problems of graph theory. For instance, the values of the partition dimensions for most kind of trees are still unknown. Although for several classes of trees such as paths, stars, caterpillars, homogeneous firecrackers and others, we do know their partition dimensions. In this paper, we determine the partition dimension of a subdivision of a particular tree, namely homogeneous firecrackers. Let G be any graph. For any positive integer k and $e \in E(G)$, a subdivision of a graph G, denoted by $S(G(e;k))$, is the graph obtained from G by replacing an edge e with a $(k+1)$-path. We show that the partition dimension of $S(G(e;k))$ is equal to the partition dimension of G if G is a homogeneous firecracker.

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1. Introduction

Let u, v be two vertices of a connected graph $G(V, E)$. We define the *distance* between vertices u and v as the minimum length of a path connecting them. This distance is denoted by $d(u, v)$. For a set $A \subseteq V$, the *distance* from vertex u to set A, denoted by $d(u, A)$, is min$\{d(u, x) | x \in A\}$. Let $\Pi = \{A_1, A_2, \cdots, L_t\}$ is a partition with t partition classes of the vertex set of G. The

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representation of vertex u in G under Π is defined as the vector $(d(v, A_1), d(v, A_2), \cdots, d(v, A_t))$ and it is denoted by $r(v|\Pi)$. The partition Π will be called a *resolving partition* of graph G if the representations of all vertices in G are different. We define the *partition dimension* of graph G, denoted by $pd(G)$, is the least number s such that G admits a resolving s-partition. Two distinct vertices x and y are said to be *distinguished* by a subset $S \subseteq V(G)$ if $d(x, S) \neq d(y, S)$. In this case, we also call that vertices u and v are *distinguishable* by the set L.

Chartrand et al. [9, 10] introduced the concept of partition dimension of a graph and gave a very solid foundation of the concept, including deriving a lower bound of such a dimension for any graph. In addition, Juan et al. (2015) derived an upper bound of the partition dimension of any tree. From these lower and upper bounds, we still have a large interval for the value of the partition dimension of a general tree. Several authors have published the partition dimensions of certain classes of trees. Some of them have the values of smaller than this upper bound, namely for caterpillars and windmills by Darmaji et al. [12], double stars $S_{m,n}$ by Chartrand et al. [9], and homogenous firecrackers by Amrullah et al.[6]. Several results propose some constructions of the family of graphs having certain partition dimension, see for instance [14] and [3]. Recently, Baskoro and Haryeni [8] gave the characterization of all graphs G of order n (≥ 11) with partition dimension $n - 3$ and diameter 2. In earlier studies, the concept of graph metric dimension introduced by Slater [18] and Harary & Melter [13] has been extensively developed. Some new results on the metric dimension of graphs, see [1], [19], and [20].

Herein, we are going to determine the partition dimension of a graph obtained by a subdivision operation on a given graph. Let $G(V, E)$ be a connected graph, $e \in E$ and $e = uv$. Let k be a positive integer. The *subdivision* of a graph G on edge e in k times, denoted by $S(G(e; k))$, is the graph obtained from the graph G by replacing edge e with a path $u, a_1, a_2, \cdots, a_k, v$ of length $k+2$. The new vertices in the graph $S(G(e; k))$ are called *subdivision vertices* of $S(G(e; k))$. Some known results regarding the partition dimension of graphs obtained from a subdivision operation can be found in [3, 5, 4, 2].

For integers $m, r \geq 2$, define a *homogeneous firecracker* $F(m; r)$ as the graph obtained by the concatenation of m independent stars $K_{1,r}$ by linking one leaf from each star. Denote by v_i and x_i for $(i = 1, 2, \cdots, m)$ the centers and the linked leaves of above stars, respectively. Denote by $w_{i,1}, w_{i,2}, ...w_{i,r-1}$ all other leaves incident to v_i. Later, all the edges v_iw_{ij} are called *pendant* and the other edges are called *non-pendant edges*. In this paper, we determine the partition dimension of the subdivision graph $S(G(e; k))$ if G is a homogeneous firecracker $F(m; r)$.

2. Preliminaries

The following lemma is a useful in helping us to determine or estimate the value of $pd(G)$ for a connected graph G.

Lemma 2.1. *[9] Let G be a connected graph. Let Π be a resolving partition of G, and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$ then vertices u and v must be in distinct partition classes of Π.*

The following result is a direct consequence of Lemma 2.1.

Corollary 2.1. *[9] Let G be a connected graph. If G has a vertex adjacent to k leaves then $pd(G) \geq k$.*

Lemma 2.2. *[10] Let G be a connected graph of order $n \geq 2$. Then, $pd(G) = 2$ if and only if $G = P_n$.*

The partition dimension of a homogeneous firecracker has been derived by Amrullah et al. [6] as following theorem.

Theorem 2.1. *[6] Let $G \simeq F(m;r)$ for $r \geq 2$ and $m \geq 2$. Then*

$$pd(G) = \begin{cases} 2, & \text{if } m = r \text{ and } r = 2, \\ 3, & \text{if } m \geq 3 \text{ and } r = 2 \text{ or} \\ & 2 \leq m \leq 9 \text{ and } r = 3 \text{ or } m = 2 \text{ and } r = 4, \\ 4, & \text{if } m > 9 \text{ and } r = 3 \text{ or } m \geq 3 \text{ and } r = 4, \\ r-1, & \text{if } m < r \text{ and } r \geq 5, \\ r, & \text{if } m \geq r \text{ and } r \geq 5. \end{cases}$$

In the next lemma, we give a lower bound of the partition dimension of graph G whose three distinct vertices in which each vertex is adjacent to three leaves.

Lemma 2.3. *[5] Let G be a connected graph of order $n \geq 13$. If G has three distinct vertices x_1, x_2, x_3 where each x_i is adjacent to three leaves, then $pd(G) \geq 4$.*

Lemma 2.4. *[5] Let G be a connected graph, $e \in E$ and $e = v_1v_2$. Let v_i be a vertex adjacent to three leaves, for each $i \in [1,2]$. If $pd(G) = 3$, then $pd(S(G(e;4))) \geq 4$.*

Lemma 2.4 will be used to derive a lower bound of the partition dimension of the subdivision graph $S(G(e,2))$ where $G = F(2;4)$ and e is non-pendant edge of G.

3. Main Results

In this section, we determine the partition dimension of the subdivision graph of a homogeneous firecracker. The following lemma gives some condition of a graph $G \cong F(m;r)$ satisfying $pd(S(G(e;k))) \leq pd(G)$.

Lemma 3.1. *Let $G \cong F(m;r)$ with $m \geq 2$, and $r \geq 4$. Let $e = vw$ be a pendant edge with w be a leaf. If there is a minimum resolving partition Π of G so that v and w are in the same partition class of Π, then $pd(S(G(e;k))) \leq pd(G)$ for any positive integer k.*

Proof. Let $\Pi = \{L_1, L_2, \cdots, L_p\}$ be a minimum resolving partition of G. Let $e = vw$ be a pendant edge with w be a leaf. Let v and w be in the same partition class, say L_i, of Π. Since $r \geq 4$, there are at least two leaves other than w adjacent to v, say w_1 and w_2. Since all these leaves w, w_1, and w_2 must be in different partition classes, we may assume $w_1 \in L_2$ and $w_2 \in L_3$. Let $a_1, a_2, \cdots, a_k$ be the subdivision vertices in $S(G(e;k))$. Now, we define a partition $\Pi' = \{L'_1, L_2, \cdots, L'_p\}$ of $S(G(e;k))$, where $L'_1 = L_1 \cup \{a_1, a_2, \cdots, a_k\}$ and $L'_i = L_i$ for all $i \in [2,p]$. Let $B = \{w, a_2, a_3, \cdots, a_k\}$. Now, consider any vertices a and b in L_i for some i. If $i \geq 2$ then

$d(v, L'_i) = d(v, L_i)$ and since Π is a resolving partition of G, then we have $r(a|\Pi') = r(a|\Pi) \neq r(b|\Pi) = r(b|\Pi')$. If a and b in L_1, then a and b will be distinguished by L'_2 provided a and b are in B. Now, If $a \in B$ and $b \notin B$ then $d(a, L'_2) \geq 3$ and $d(a, L'_3) \geq 3$ but $d(b, L'_i) \leq 2$ for all $i \in [1, p-1]$ (because each v_i is adjacent to $p-1$ leaves). So we have $r(a|\Pi') \neq r(b|\Pi')$. This implies that $r(a|\Pi') \neq r(b|\Pi')$ for all pairs of different vertices a and b in $S(G(e; k))$. Thus, $pd(S(G(e; k))) \leq pd(G)$. □

Lemma 3.2. *Let $G \cong F(m; r)$, $m \geq 2$, $r \geq 5$, $m \leq r$. Let $e = uv$ be a non pendant edge and u be adjacent to other vertex w, $v \in L_b$, $u \in L_c$ and $w \in L_d$ with $c \neq d \neq b$. If v is adjacent to $w_1 \in L_c$ and $w_2 \in L_d$, then $pd(S(G(e; k))) \leq pd(G)$.*

Proof. Since $r \geq 5$, we have $pd(G) = p \geq 4$. Let $\Pi = \{L_1, L_2, \cdots, L_p\}$ be a resolving partition of G. We define a new partition Π' of $V(S(G(e; k)))$, $\Pi' = \{L'_1, L'_2, \cdots, L'_p\}$ where $L'_i = L_i$ for $i \notin \{c, d\}$, $L'_c = L_c \cup \{a_2, a_3, \cdots, a_k\}$ and $L'_d = L_d \cup \{a_1\}$.

Let $B = \{u, a_1, a_2, \cdots, a_k\}$. Since each x_i has $d(x, w_{i,j}) = 2$, $j \in [1, r-1]$, then $r(x_i|\Pi)$ is not affected by subdivision edge. So, we have $r(x|\Pi') \neq r(y|\Pi')$ for any pair of $x, y \in V(S(G(e; k)) \setminus B$. So, we consider vertices x or y in B. If $x, y \in B$ then $x, y \in L'_c$. So, the class partition L'_d or L'_b can be distinguishing of vertices x, y. If $x \in B$ and $y \notin B$ then consider $x, y \in L'_d$ or $x, y \in L'c$. For $x, y \in L'_c$, there are at least one component of $r(x|\Pi')$ have value at least '3' but all components of $r(y|\Pi')$ have value at most '2'. For $x, y \in L'_d$, we consider $k = 1$ or $k > 1$. If $k > 1$, then there are at least one component of $r(x|\Pi')$ have value at least '3' but all components of $r(y|\Pi')$ have value at most '2'. If $k = 1$, then $d(x, L'_b) = 1$ but $d(y, L'_b) = 2$. So we have $r(x|\Pi') \neq r(y|\Pi')$. This implies $r(x|\Pi') \neq r(y|\Pi')$ for all pair distinct $x, y \in S(G(e; k))$. Thus, $pd(S(G(e; k))) \leq pd(G)$. □

Lemma 3.3. *Let $G \cong F(m; r)$ with $r \geq 5$ and $m > r$. Then, $pd(S(G(e; k))) = pd(G)$, for any non pendant edge e and positive integer k.*

Proof. We consider the following two cases.

Case 1. $e = x_t v_t$, for some $t \in [1, m]$.
Since e is non pendant edge then we have $pd(S(G(e; k))) \geq r$. Let $\Pi = \{L_1, L_2, \cdots, L_r\}$ be a partition of $V(S(G(e; k)))$ where $L_r = \{x_t, v_t, a_1, a_2, \cdots, a_k\}$, $L_1 = \{w_{i,1}|1 \leq i \leq m\} \cup \{x_i|i = t + 2j - 1 \leq m, j \in \mathbb{N}^+\} \cup \{x_i|1 \leq i = t - 2j + 1, j \in \mathbb{N}^+\}$, $L_2 = \{w_{i,2}|1 \leq i \leq m\} \cup \{x_i, v_i|i = t + 2j \leq m, j \in \mathbb{N}^+\} \cup \{v_i, x_i|i \leq i = t - 2j + 1, j \in \mathbb{N}^+\}$, $L_3 = \{w_{i,3}|1 \leq i \leq m\} \cup \{v_i|i = t + 2j - 1 \leq m, j \in \mathbb{N}^+\} \cup \{v_i|1 \leq i = t - 2j - 1, j \in \mathbb{N}^+\}$, and $L_i = \{w_{i,1}|1 \leq i \leq m\}$ for $i \notin \{1, 2, 3, \cdots, r\}$. We use the $\mathbb{N}^+$ as the positive integer set.

Let u and z be any two distinct vertices in the same partition class of Π. If $u, z \in L_r$, then the class partition L_1 or L_3 will distinguish u and z. If $u, z \notin L_r$, then consider $d(u, L_r)$ and $d(z, L_r)$. If $d(u, L_r) \neq d(z, L_r)$, then u, z will be distinguished by L_r. If $d(u, L_r) = d(z, L_r)$, then consider these four cases. i). Let $u = x_k$ and $z = v_i$, By definition the Π, since u, z are in same partition class, we have $u, z \in L_2$. So, the vertices u, z can be distinguished by L_3. ii.) Let $u = x_k$ and $z = w_{i,j}$, by definition Π, this means $u, z \in L_2$ or $u, z \in L_1$. If $u, z \in L_2$, then u, z can be distinguished by L_1 or L_3. If $u, z \in L_1$, then u, z can be distinguished by L_2 or L_3. iii). Let $u = v_k$

and $z = w_{i,j}$, Since u, z are in the same partition class, we obtain that the vertices $u, z \in L_2$ or $u, z \in L_3$. If $u, z \in L_2$, then u, z can be distinguished by L_1 or L_3. If $u, z \in L_3$, then u, z can be distinguished by L_1 or L_2. iv.) Let $u = w_{k,j}$ and $z = w_{i,j}$, we have that u, z can be distinguished by L_2 or L_3.

Case 2. Let $e = x_t x_{t-1}$ with $t \in [2, m]$. Let $\Pi' = \{L'_1, L'_2, \cdots, L'_r\}$be a partition of $V(S(G(e;k)))$ where $L'_r = \{x_1, v_1\}$, $L'_2 = \{w_{i,2} | 1 \leq i \leq m\} \cup \{x_i, v_i | i = 2j \leq m, j \in \mathbb{N}^+\}$, $L'_3 = \{w_{i,3} | 1 \leq i \leq m\} \cup \{x_i, v_i | i = 2j+1 \leq m, j \in \mathbb{N}^+\} \cup a_1, a_2, \cdots, x_k$, dan $L'_i = \{w_{k,i} | 1 \leq k \leq m\}$ for $i \notin \{2, 3, r\}$. Let u, z be two different vertices in the same partition class of Π'. If $u, z \in L'_r$, then u, z can be distinguished by L'_3. If $u, z \notin L'_r$, then consider $d(u, L'_r)$ and $d(z, L'_r)$. If $d(u, L'_r) \neq d(z, L'_r)$, then u, z can be distinguished by L'_r. If $d(u, L'_r) = d(z, L'_r)$, then consider u, z in two conditions. i). For $u = w_{i,j}$ and $z = v_{i+2}$, vertices u, z can be distinguished by L'_2 or L'_3. ii.) For $u = w_{i,j}$ and $z = v_{i+1}$, vertices u, z can be distinguished by L'_1. This implies $pd(S(G(e;k)) = pd(G)$. □

Lemma 3.4. *Let $G \cong F(m;r)$ with $m \geq 2$ and $r \geq 5$. If $m < r$, then $pd(S(G(e;k))) = pd(G)$.*

Proof. If e is a pendant edge, then $pd(G) = r - 1$ by Theorem 2.1. Since there is a vertex v_i which is adjacent to $r-1$ leaves, $pd(S(G(e;k))) \geq pd(G)$. By Lemma 3.1, we have $pd(S(G(e;k))) \leq pd(G)$. Thus, $pd(S(G(e;k))) = pd(G)$. If e is a non pendant edge, then we have $pd(G) = r-1$, by Theorem 2.1. Since a vertex v_i is adjacent to $r-1$ leaves, $pd(S(G(e;k))) \geq pd(G)$. By Lemma 3.2, $pd(S(G\ (e;k))) \leq pd(G)$. Thus, $pd(S(G(e;k))) = pd(G)$. □

Lemma 3.5. *Let $G \cong F(m;r)$ with $m \geq 2$, $r \geq 5$, and e be a pendant edge of G. If $pd(G) = r$ and $m = r$ then $pd(S(G(e;k))) = pd(G) - 1$.*

Proof. Let $e = v_j w_{j,1}$ for some $j \in [1, m]$. Since $S(G(e;k))$ has a vertex v_t which is adjacent to $r-1$ leaves, then $pd(S(G(e;k))) \geq r-1$. Let $\Pi' = \{L'_1, L'_2, ..., L'_{r-1}\}$ be a partition of $V(S(G(e;k)))$. We define L'_i in two conditions of $e = v_j w_{j,1}$.
For $j = 1$ (we can use the same reason for $i = m$), we define $L'_1 = \{v_1, v_2, a_1, w_{1,2},\ x_{m-2}\} \cup \{w_{t,1} | t \neq j\}$, $L'_2 = \{a_2, ..., a_k, w_{1,1}, w_{1,3}, x_{m-1}, v_3\} \cup \{w_{t,2} | t \neq 1\}$, $L'_3 = \{x_1, x_m, w_{1,4}, v_4\} \cup \{w_{t,3} | t \neq j\}$ and $L'_i = \{x_{i-2}, v_{i+1}\} \cup \{w_{t,i} | t \neq j\} \cup \{w_{j,i+1} | i < r-1\}$, for $4 \leq i \leq r-1$.
For $2 \leq j \leq m-1$, we define $L'_1 = \{v_1, v_j, a_1, w_{j,2}, x_j\} \cup \{w_{t,1} | t \neq j\}$, $L'_2 = \{a_2, ..., a_k, w_{j,1}, w_{j,3}, x_{j+1}\} \cup \{w_{t,2} | t \neq 1\} \cup \{v_2 | j \neq 2\} \cup \{v_2 | j = 2\} \cup \{x_1 | j = m-1\}$, and $L'_i = \{w_{j,i+1}\} \cup \{w_{t,i} | t \neq j\} \cup \{v_i | j < i\} \cup \{v_{i+1} | j > i\} \cup \{x_{j-1+i} | j-1+i < m\} \cup \{x_{j-1+i}, x_1 | m = j-1+i\} \cup \{x_{j+i-m} | j-1+i > m\}$, for $3 \leq i \leq r-1$.

Let u, z be two distinct vertices in the same partition class L'_t of Π'. If u is a leaf and z is a non-leaf, then the components of $r(u|\Pi')$ appear at most one value '1', but the $r(z|\Pi')$ appear at least two value '1'. This implies that $r(u|\Pi') \neq r(z|\Pi')$. If u, z are two leaves, $u = w_{a,i_1}$ and $z = w_{b,i_2}$ with $a \neq b$, then they can be distinguished by v_a or v_b for $a \neq 1\ and\ b \neq 1$.If the same condition of u, z and $a = b = 1$ then they can be distinguished by L'_{r-1}. This implies that $r(u|\Pi') \neq r(z|\Pi')$.

If u, z are two non-leaves then we have three cases: (i) $u = v_{i_1}$ and $z = v_{i_2}$. So, we have $u, z \in L'_1$. They can be distinguished by L'_{r-1}. This implies that $r(u|\Pi') \neq r(z|\Pi')$. (ii) $u = x_{i_1}$ and $z = x_{i_2}$. So, we have $u, z \in L'_3$. They can be distinguished by L'_2 or L'_4 . This implies that $r(u|\Pi') \neq r(z|\Pi')$. (iii) $u = x_{i_1}$ and $z = v_{i_2}$. The components of $r(u|\Pi')$ appear twice component having value '1' but in the $r(z|\Pi')$ appear at least three component having value '1'. This implies that $r(u|\Pi') \neq r(z|\Pi')$. Therefore, $pd(S(G(e; k))) = r - 1 = pd(G) - 1$. □

Lemma 3.6. *Let $G \cong F(m; r)$ with $r \geq 4$, and e be a non pendant edge of G. If $pd(G) = r$, then $pd(S(G(e; k))) = pd(G)$.*

Proof. According Theorem 2.1, we know that $pd(G) = r$ if ($r = 4$ and $m \geq 3$) or ($r > 4$ and $m \geq r$). First, we show that $pd(S(G(e; k))) \geq r$. For $r = 4$, since e is non pendant edge of G, then by Lemma 2.3 $pd(S(G(e; k))) \geq 4 = r$. For $r > 4$ and $m \geq r$, since e is a non pendant edge, then there are at least r vertices v_i where each v_i is adjacent to $r - 1$ leaves. So, $pd(S(G(e; k))) \geq r$. By Lemma 3.2, we have $pd(S(G(e; k))) \leq pd(G)$. Thus, $pd(S(G(e; k))) = pd(G)$. □

By Lemmas 3.7-3.9, we determine the partition dimension of the subdivision of $G = F(m; 4)$ with $m \geq 2$. Lemma 3.7 gives the partition dimension of the subdivision of $F(2; 4)$.

Lemma 3.7. *Let $G \cong F(2; 4)$. Then*

$$pd(S(G(e;k))) = \begin{cases} pd(G)+1, & \text{if } e \text{ is a non pendant edge and } k = 2, \\ pd(G), & \text{otherwise.} \end{cases}$$

Proof. First, by [6] we have that $pd(G) = 3$. If e is a non-pendant edge and $k = 2$, then we have that $pd(S(G(e; 2))) \geq 4$ by Lemma 2.4. In the Figure 1(b) we give a resolving 4-partition for $S(G(e; 2))$. So, we have that $pd(S(G(e; 2))) = pd(G) + 1$.

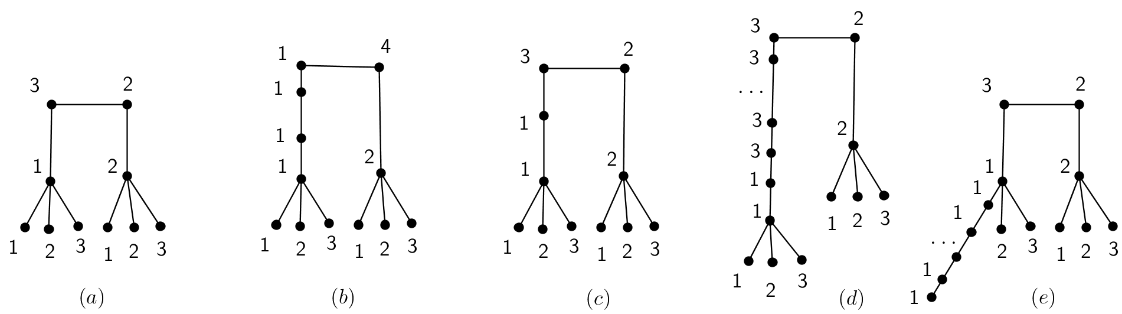

Figure 1. (a.) a resolving partition of $G = F(2; 4)$, (b-e.) a resolving partition of $S(G(e; k))$ where $G = F(2; k)$, e is a non pendant edge and (b.) $k = 2$, (c.) $k = 1$, (d.) $k \geq 3$, (e.) e is a pendant edge and $k \geq 1$.

Second, we know that $S(G(e; k))$ is not a path, then $pd(S(G(e; k))) \geq 3$. If e is a non-pendant edge and $k = 1$ or $k \geq 3$, then we give a resolving 3-partition for $S(G(e; k))$ in Figures 1(c-d). If e is a pendant edge, then we have a resolving 3-partition for $S(G(e; k))$ in Figure 1(e). Therefore, $pd(S(G(e; k))) = pd(G)$. □

Lemma 3.8. *If* $G \cong F(3;4)$*, then*
$$pd(S(G(e;k))) = \begin{cases} pd(G)-1, & if\ e\ is\ a\ pendant\ edge, \\ pd(G), & otherwise. \end{cases}$$

Proof. By [6], we obtain $pd(G) = 4$. Let e be a pendant edge and let $e = v_2w_{2,1}$. Since $S(G(e;k))$ is not a path, $pd(S(G(e;k)) \geq 3$. We have a resolving partition $\Pi = \{L_1, L_2, L_3\}$ of $V(S(G(e;k)))$ where $L_1 = \{v_3, w_{1,1}, w_{3,1}\}$, $L_2 = \{v_1, v_2, x_1, x_2, w_{1,2}, w_{2,2}, w_{3,2}, a_1\}$, $L_3 = \{x_3, w_{1,3}, w_{2,1}, w_{2,3}, w_{3,3}\} \cup \{a_i | 2 \leq i \leq k\}$. Thus, we obtain $pd(S(G(e;k))) = 3$. This implies that $pd(S(G(e;k))) = pd(G) - 1$.

Now consider if e is a non-pendant edge of G. Then, by Lemma 2.3 we obtain $pd(S(G(e;k))) \geq 4$. Next, let $e = uv$, $u \in L_t$, $u \in L_q$ and $t \geq q$. We have a resolving partition $\Pi' = \{L'_1, L'_2, L'_3, L'_4\}$ of $S(G(e;k))$ where $L'_i = L_i$ for $i \neq t$ and $L'_t = L_t \cup \{a_1, a_2 \cdots, a_k\}$. Thus, $pd(S(G(e;k))) \leq 4$. Hence, we have $pd(S(G(e;k))) = pd(G)$. □

Lemma 3.9. *Let* $G \cong F(m;4)$ *where* $m \geq 4$*. Then,* $pd(S(G(e;k))) = pd(G)$*.*

Proof. By [6], we have $pd(G) = 4$. Let $\Pi = \{L_1, L_2, L_3, L_4\}$ be a resolving partition of G where $L_1 = \{x_1, v_1, v_2, \cdots, v_m, w_{2,1}, w_{3,1}, \cdots, w_{m,1}\}$, $L_2 = \{x_{2i} | 1 \leq i \leq \lfloor \frac{m}{2} \rfloor\} \cup \{w_{1,2}, w_{2,2}, \cdots, w_{m,2}\}$, $L_3 = \{x_{2i+1} | 1 \leq i \leq \lfloor \frac{m}{2} \rfloor\} \cup \{w_{1,3}, w_{2,3}, \cdots, w_{m,3}\}$, and $L_4 = \{w_{1,1}\}$.

Since $S(G(e;k))$ has the condition satisfying Lemma 2.3, then $pd(S(G(e;k))) \geq 4 = pd(G)$. Since G satisfies the condition of Lemma 3.2, then $pd(S(G(e;k))) \leq pd(G)$. This implies that $pd(S(G(e;k))) = pd(G)$. □

In the use of Lemma 3.10, let consider $G = F(9;3)$ and $S(G(e;k))$. Define a notation $\triangle_i = \{x_i, v_i, w_{i,1}, w_{i,2}\}$ if $e \neq x_iv_i$, and $\triangle_i = \{x_i, v_i, w_{i,1}, w_{i,2}, a_1, a_2, \cdots a_k\}$ if $e = x_iv_i$ for $i \in [1,9]$. Let $\Pi = \{L_1, L_2, \cdots, L_p\}$ be a resolving partition of $S(G(e;k))$. For $i \in \{1, 2, \cdots, 9\}$, define $\Pi\{i\}$ as $\Pi|_{\Delta_i}$. For simple notation, use $\Pi\{i,j\} = \Pi|_{\Delta_i \cup \Delta_j}$. So, we have $\Pi\{1, 2, \cdots, m\} = \Pi$. In Figure 2, for $i = 1, 2, \cdots, 5$, by symmetry, we provide a resolving 3-partition Π for $G_i = S(G(e;k))$ if $G = F(9;3)$ and $e = x_iv_i$.

Now for any $m \in [2,8]$ and $j \in [1,8]$, we will construct a resolving 3-partition of $S(G(e;k))$ with $G = F(m;3)$ and $e = x_jv_j$ by using the restriction of the partition Π_i of G_i to some $A \subseteq [2,9]$, for some i. For example, $\Pi_1\{1,2\}$ is a resolving 3-partition of $S(G(e;k))$ with $G = F(2;3)$ and $j = 1$; $\Pi_1\{1,2,3\}$ is a resolving 3-partition of $S(G(e;k))$ with $G = F(3;3)$ and $j = 1$; and $\Pi_1\{1,2,4\}$ is a resolving 3-partition of $S(G(e;k))$ with $G = F(3;3)$ and $j = 2$ as in Figure 3.

Lemma 3.10. *Let* $G \cong F(m;3)$ *for* $m \geq 2$*. Then,* $pd(S(G(e;k))) = pd(G)$*.*

Proof. Let $G \cong F(m;3)$. According to Theorem 2.1, we have $pd(G) = 3$ for $2 \leq m \leq 9$ and $pd(G) = 4$ for $m \geq 9$. If e is a non-pendant edge of G, then $pd(S(G(e;k))) \geq 3$. The resolving partitions in Figure 2 show that $pd(S(G(e;k))) \leq 3$, if $G = F(9;3)$. So, we obtain $pd(S(G(e;k))) = 3$.

For $2 \leq m \leq 8$, consider the graph $S(G(e;k))$ with $G = F(m;3)$ and $e = x_iv_i$. Now, define a resolving 3-partition of $S(G(e;k))$ as the restriction of the partition Π_j of G_j to some $A \subseteq [2,9]$, for some j as shown in Table 1. For an illustration, in Figure 3, we present a resolving 3-partition

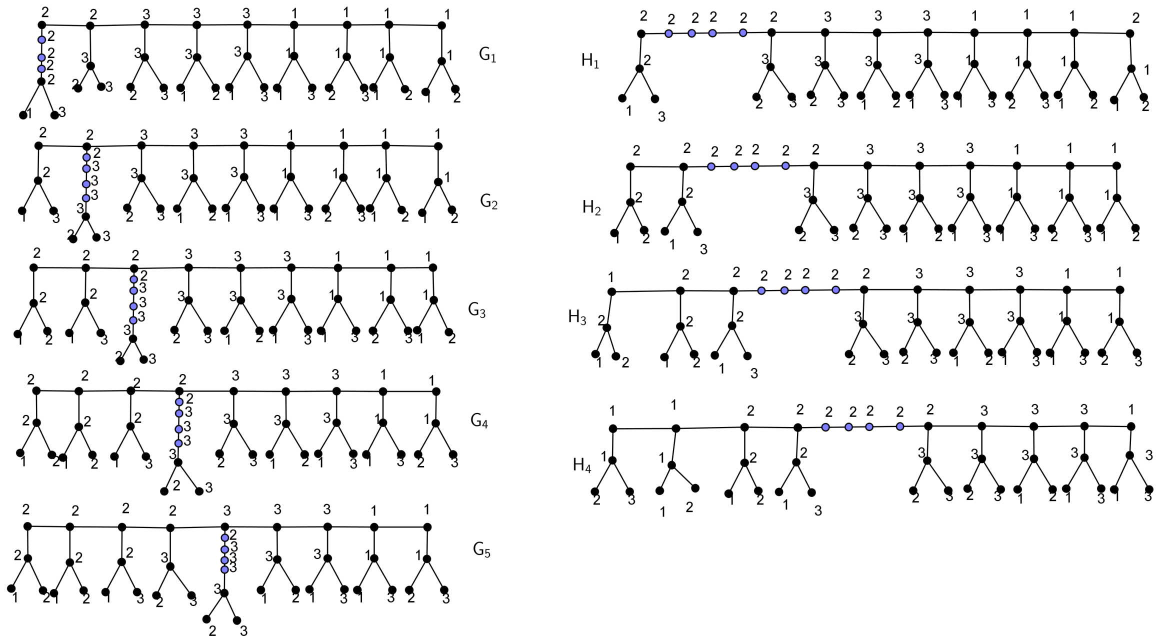

Figure 2. Graph $G_i = S(G(e;k))$ where $G = F(9;3)$ and $e = x_i v_i$, and $H_i = S(G(e;k))$ where $G = F(9;3)$ and $e = x_i x_{i+1}$

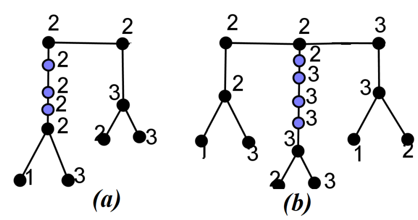

Figure 3. (a) Graph $G'_1(1,2) = S(G(x_1v_1;k))$ with $G = F(2;3)$ (b) $G'_2(1,2,4) = S(G(x_2v_2,k)$ with $G = F(3;3)$.

of $S(G(e;k))$ with $G = F(2;3)$ and $e = x_1v_1$, and a resolving 3-partition of $S(G(e;k))$ with $G = F(3;3)$ and $e = x_2v_2$.

Now, for $j = 1,2,3,4$ (by symmetry), define the graph $H_j = S(G(e;k)$ with $G = F(9;3)$ and $e = x_i x_{i+1}$. The resolving 3-partitions of H_j are provided in Figure 2. For $2 \leq m \leq 8$, consider the graph $S(G(e;k))$ with $G = F(m;3)$ and $e = x_i x_{i+1}$. Similarly, define a resolving 3-partition of $S(G(e;k))$ as the restriction of the partition Π'_j of H_j to some $A \subseteq [2,9]$, for some j as shown in Table 2.

For e is a pendant edge, by Lemma 3.4, we obtain $pd(S(G(e;k)) = pd(G)$. This implies $pd(S(G(e;k)) = pd(G)$ for $G = F(m;3)$, $2 \leq m \leq 9$.

Next, for $m \geq 10$, by [6], we have $pd(G) = 4$. We will show $pd(S(G(e;k))) \geq 4$. For a contradiction, let $\Pi' = \{L'_1, L'_2, L'_3\}$ be a resolving partition of $S(G(e;k))$. For $m \geq 10$, since there are

Table 1. The resolving partitions of $S(G(e;k))$ with $G = F(m;3)$ and $e = x_i v_i$.

m	$e = x_i v_i$
2	$\Pi_1\{1,2\}$,i=1
3	$\Pi_1\{1,2,3\}$,$\Pi_2\{1,2,4\}$,i=1,2
4	$\Pi_1\{1,2,3,4\}$, $\Pi_2\{1,2,3,4\}$,i=1,2
5	$\Pi_1\{1,\cdots,5\}$,$\Pi_2\{1,\cdots,5\}$, $\Pi_3\{1,\cdots,5\}$, i=1,2,3
6	$\Pi_1\{1,\cdots,6\}$, $\Pi_4\{3,\cdots,7,9\}$, $\Pi_5\{3,\cdots,7,9\}$, i=1,2,3
7	$\Pi_1\{1,...,6,8\}$, $\Pi_2\{1,...,6,8\}$, $\Pi_3\{1,...,6,8\}$, $\Pi_4\{1,...,7\}$,$i \in [1,4]$
8	$\Pi_1\{1,\ldots,8\}$, $\Pi_2\{1,\ldots,8\}$, $\Pi_3\{1,\ldots,8\}$, $\Pi_5\{1,\ldots,8\}$, $i \in [1,4]$

Table 2. The resolving partitions of $S(G(e;k))$ with $G = F(m;3)$ and $e = x_i x_{i+1}$.

m	$e = x_i x_{i+1}$
2	$\Pi'_1\{1,2\}$,i=1
3	$\Pi'_2\{1,2,3\}$,i=1
4	$\Pi'_1\{1,2,3,4\}$, $\Pi'_2\{1,2,3,4\}$,i=1,2
5	$\Pi'_1\{1,\cdots,5\}$,$\Pi'_2\{1,\cdots,5\}$, i=1,2
6	$\Pi'_2\{2,\cdots,6,8\}$, $\Pi'_2\{1,\cdots,6\}$, $\Pi'_3\{1,\cdots,6\}$, i=1,2,3
7	$\Pi'_1\{1,\cdots,7\}$, $\Pi'_4\{3,\cdots,9\}$, $\Pi'_3\{1,\cdots,7\}$, i=1,2,3
8	$\Pi'_2\{2,\ldots,9\}$, $\Pi'_2\{1,...,8\}$, $\Pi'_3\{1,...,7,9\}$, $\Pi'_4\{1,...,8\}$, i=1,2,3,4

at least nine $\triangle_i s$ that do not have an edge subdivision and there are at most tree $\triangle_a, \triangle_b, \triangle_c$ which are covered by one 2-subset $C = \{L_1, L_2\} \subseteq \Pi$. Let $v_a, v_b \in L_1$ and $v_c \in L_2$. So, we obtain than there are tree vertices in L_1 which are adjacent to vertex in L_2, namely $v_a, v_b, w_{c,1}$. Therefore, the representation of v_a, v_b or $w_{c,1}$ is one of $\{(0,1,1),(0,1,2),(0,1,3),(0,1,4),(0,1,5)\}$, but the coordinate of $(0,1,5)$ cannot be used for the representation of any non-leaf vertex. As a consequence, we have only 8 possible pairs of the coordinates for vertices $v_a, v_b, w_{c,1}$. If $r(v_a|\Pi) = (0,1,1)$, $r(v_b|\Pi) = (0,1,2)$ and $r(w_{c,1}|\Pi) = (0,1,3)$, then we will obtain $r(w_{c,2}|\Pi) = r(v_c|\Pi)$, a contradiction. For the seven remaining possibilities, we will also obtain two vertices with the same coordinate as summarized in Table 3, a contradiction. This implies that $pd(S(G,e,k)) \geq 4$.

Table 3. Two vertices with the same coordinate

No	$(r(v_a|\Pi), r(v_b\Pi), r(w_{c,1}\Pi))$	the same coordinate
1	$((0,1,1),(0,1,2),(0,1,3))$	$r(w_{a,2}|\Pi) = r(v_c|\Pi)$
2	$((0,1,1),(0,1,2),(0,1,4))$	$r(w_{b,2}|\Pi) = r(v_c|\Pi)$
3	$((0,1,2),(0,1,3),(0,1,4))$	$r(w_{a,2}|\Pi) = r(v_c|\Pi)$
4	$((0,1,2),(0,1,3),(0,1,5))$	$r(w_{b,2}|\Pi) = r(v_c|\Pi)$
5	$((0,1,3),(0,1,4),(0,1,5))$	$r(w_{a,2}|\Pi) = r(v_c|\Pi)$
6	$((0,1,1),(0,1,3),(0,1,5))$	$r(w_{a,2}|\Pi) = r(v_c|\Pi)$
7	$((0,1,1),(0,1,2),(0,1,4))$	$r(x_b|\Pi) = r(x_a|\Pi)$
8	$((0,1,1),(0,1,4),(0,1,5))$	$r(v_{b-1}|\Pi) = r(v_c|\Pi)$

Next, we show $pd(S(G(e;k))) \leq 4$. Let $\Pi = \{L_1, L_2, L_3, L_4\}$ be a partition of $S(G,e,k)$ for $G = F(m,3)$ where $m \geq 10$. If $e = x_i x_{i+1}$ or $e = v_i w_{i,1}$, then define $L_1 = \{w_{1,1}, w_{2,1}, \cdots, w_{m,1}, a_1, a_2, \cdots, a_k\} \cup \{x_{2i} | 1 \leq i \leq \lfloor \frac{m}{2} \rfloor\}$, $L_2 = \{w_{1,2}, w_{2,2}, \cdots, w_{m,2}\} \cup \{x_{2i+1} | 1 \leq i \leq \lfloor \frac{m}{2} \rfloor\}$, $L_3 = \{v_1, v_2, \cdots, v_m\}$, and $L_4 = \{x_1\}$. If $e = x_i v_i$, then we define $L_1 = \{v_i\} \cup \{w_{1,1}, w_{2,1}, \cdots, w_{m,1}, a_1, a_2, \cdots, a_k\} \cup \{x_{2i} | 1 \leq i \leq \lfloor \frac{m}{2} \rfloor\}$, $L_2 = \{w_{1,2}, w_{2,2}, \cdots, w_{m,2}\} \cup \{x_{2i+1} | 1 \leq i \leq \lfloor \frac{m}{2} \rfloor\}$, $L_3 = \{v_1, v_2, \cdots, v_m\} - \{v_i\}$, and $L_4 = \{x_1\}$. It is easy to show that Π is a resolving partition of $S(G(e,k))$. Thus, $pd(S(G(e,k))) = pd(G)$. □

If $G = F(m;2)$, then the partition dimension of the subdivision graph $S(G(e;k))$ is given in the following lemma.

Lemma 3.11. *Let $G \cong F(m;2)$ for $m \geq 3$. Then, $pd(S(G(e;k))) = pd(G)$.*

Proof. By Theorem 2.1, we have $pd(G) = 3$. Since $S(G(e;k)$ is not a path, then $pd(S(G(e;k)) \geq 3$. Let $\Pi = \{L_1, L_2, L_3\}$ be a partition of $V(S(G(e;k)))$ defined as follows.
For edge $e = v_i w_{i,1}$ where $1 \leq i \leq m-1$, we define $L_1 = \{w_{1,1}, w_{2,1}, \cdots, w_{m-1,1}\} \cup \{a_1, a_2, \cdots, a_k\}$, $L_2 = \{x_i, v_1 | 1 \leq i \leq m\}$, and $L_3 = \{w_{m,1}\}$. For edge $e \neq v_i w_{i,1}$ where $1 \leq i \leq m-1$, we define $L_1 = \{w_{1,1}, w_{2,1}, \cdots, w_{m-1,1}\}$, $L_2 = \{x_i, v_i | 1 \leq i \leq m\} \cup \{a_1, a_2, \cdots, a_k\}$, and $L_3 = \{w_{m,1}\}$. It is easy to verify that Π is a resolving partition of $S(G(e,k))$. So, $pd(S(G(e;k))) = pd(G)$. □

We summarize all the above results in the following theorem.

Theorem 3.1. *Let $G \cong F(m;r)$ with $m, r \geq 2$, $e \in V(G)$ and $k \geq 1$. Then,*

$$pd(S(G(e;k))) = \begin{cases} pd(G)+1 & \text{if } e \text{ is a non pendant edge and} \\ & k=2, m=2 \text{ and } r=4, \\ pd(G)-1 & \text{if } e \text{ is a pendant edge and} \\ & (m=r \text{ and } r \geq 5) \text{ or } (m=3 \text{ and } r=4), \\ pd(G) & \text{otherwise.} \end{cases}$$

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