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Totally irregular total labeling of some caterpillar graphs

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Abstract

Assume that $G(V,E)$ is a graph with V and E as its vertex and edge sets, respectively. We have G is simple, connected, and undirected. Given a function λ from a union of V and E into a set of k-integers from 1 until k. We call the function λ as a totally irregular total k-labeling if the set of weights of vertices and edges consists of different numbers. For any $u \in V$, we have a weight $wt(u) = \lambda(u) + \sum_{uy \in E} \lambda(uy)$. Also, it is defined a weight $wt(e) = \lambda(u) + \lambda(uv) + \lambda(v)$ for each $e = uv \in E$. A minimum k used in k-total labeling λ is named as a total irregularity strength of G, symbolized by $ts(G)$. We discuss results on ts of some caterpillar graphs in this paper. The results are $ts(S_{p,2,2,q}) = \left\lceil \frac{p+q-1}{2} \right\rceil$ for p, q greater than or equal to 3, while $ts(S_{p,2,2,2,p}) = \left\lceil \frac{2p-1}{2} \right\rceil, p \geq 4$.

Keywords: totally irregular total k-labeling, total irregularity strength, weight, caterpillar graph.
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1. Introduction

Graph theory is one of branch of mathematics. In this field, many real life problems can be solved, especially on optimization problem [8]. Given a graph $G(V, E)$ which is assumed as connected, simple, and undirected graph. A function that assigns a set of elements (vertex/edge) of G into a set of integers is mentioned as labeling (Wallis [12]). The labeling is said to be a total labeling if the domain is a union of vertex and edge sets.

A function $f : V \cup E \to \{1, 2, \ldots, k\}$ is named a vertex irregular total k-labeling if $wt_f(u) \neq wt_f(v)$ for each $u \neq v \in V(G)$, where $wt(u) = f(u) + \sum_{uz \in E} f(uz)$ [1]. A minimum k in which there exists a vertex irregular total k-labeling of G is named as a total vertex irregularity strength (tvs) of G. Indriati *et al.* [4] obtained tvs of generalized helm. Recently, the tvs of comb product of two cycles and two stars has been found in [10]. Meanwhile, Nurdin *et al.* [11] proved tvs of tree T which does not have vertex of degree two and has n pendant nodes, i.e.

$$tvs(T) = \left\lceil \frac{n+1}{2} \right\rceil. \tag{1}$$

Further, a total k-labeling g that assigns a union of V and E into $\{1, 2, \ldots, k\}$ is called an edge irregular when the requirement $wt(xy) \neq wt(x'y')$ is satisfied for each pair $xy \neq x'y'$ in $E(G)$ with $wt(xy) = g(x) + g(xy) + g(y)$. Bača *et al.* [1] mentioned the minimum k required in labeling g as a total edge irregularity strength (tes) of G. The exact value of tes of generalized web graphs was given in [2]. Recent research has found tes of some n-uniform cactus chain graphs and related chain graphs [6]. In addition, tes of any tree has been given in [7], i.e. $tes(T)$ is equal to

$$\max\left\{ \left\lceil \frac{(|E(T)|+2)}{3} \right\rceil, \left\lceil \frac{(\Delta(T)+1)}{2} \right\rceil \right\}. \tag{2}$$

Furthermore, the total k-labeling g becomes a totally irregular total k-labeling if the set of all weights of vertices and edges contains distinct numbers [9]. A minimum k needed in the labeling g is named as total irregularity strength (ts) of G. Marzuki, *et al.* observed

$$ts(G) \geq \max \text{ of } \{tes(G), tvs(G)\}. \tag{3}$$

Different with *tes* and *tvs*, the value of ts of tree has not been obtained. In order to find ts of tree, we have started the investigation for double stars $S_{p,q}$ and related graphs $S_{p,2,q}$ ([3], [5]). In this research, we verify ts of caterpillar graphs $S_{p,2,2,q}$ and $S_{p,2,2,2,p}$.

We use the notion of caterpillar $S_{p,2,2,q}$. It is a graph which is formed from double-star $S_{p,q}$ by putting two vertices on the path which are connected to the two centers of stars in $S_{p,q}$. The value of tes of graph $S_{p,2,2,q}$ can be found by (2), that is

$$tes(S_{p,2,2,q}) = \max\left\{ \left\lceil \frac{\max\{p,q\}+1}{2} \right\rceil, \left\lceil \frac{p+q+3}{3} \right\rceil \right\} = \left\lceil \frac{p+q+3}{3} \right\rceil. \tag{4}$$

This graph has two vertices of degree two. Therefore, (1) cannot be used for determining tvs of this graph. The next theorem gives this parameter.

Theorem 1.1. *Let $S_{p,2,2,q}$ be a caterpillar with p, q greater than or equal to 3. The graph $S_{p,2,2,q}$ has*

$$tvs(S_{p,2,2,q}) = \left\lceil \frac{p+q-1}{2} \right\rceil.$$

Proof. Without loss of generality, we can assume that $p \leq q$. We know that $S_{p,2,2,q}$ contains $p+q-2$ pendants, two vertices with degree two, one vertex with degree p, and one vertex of degree q. The smallest weight of each vertex is at least two. Each pendant vertex has the smallest weight which is not less than $p+q-1$, i.e. the weight is a sum of two labels. Then, the largest number to label pendant vertices is not less than $\lceil \frac{p+q-1}{2} \rceil$. The graph $S_{p,2,2,q}$ consists of $V(S_{p,2,2,q}) = \{v_r^1 : 1 \leq r \leq q-1\} \cup \{v_r^4 : 1 \leq r \leq p-1\} \cup \{v^s : \ s = 1, 2, 3, 4\}$ and $E(S_{p,2,2,q}) = \{v^1 v_r^1 : 1 \leq r \leq q-1\} \cup \{v^4 v_r^4 : 1 \leq r \leq p-1\} \cup \{v^s v^{s+1} : s = 1, 2, 3\}$.
Next we will distinguish the following three cases, i.e. $p = q = 3$, $p = q \geq 4$ and $3 \leq p < q$. Assume $k = \lceil \frac{p+q-1}{2} \rceil$ for all cases, and define a total k-labeling λ on each element $x \in V(S_{p,2,2,q}) \cup E(S_{p,2,2,q})$ as follows.

x	$\lambda(x)$		Case for p, q
v_r^s	$1,$	$1 \leq r \leq p-1; s = 1$	$p = q \geq 3$
	$r,$	$1 \leq r \leq p-1; s = 4$	
	$1,$	$1 \leq r \leq k; s = 1$	$3 \leq p < q$
	$r-k+1,$	$k+1 \leq r \leq q-1; s = 1$	
	$q-k+r,$	$1 \leq r \leq p-1; s = 4$	
v^s	$1,$	s= 1, 2	$p = q = 3$
	$k,$	s= 3, 4	
	$1,$	s= 1, 3	$4 \leq p = q$
	$2,$	s= 2	
	$4,$	s= 4	
	$1,$	s= 1	$3 \leq p < q$
	$\left\lceil \frac{\lvert p-q \rvert+5}{2} \right\rceil,$	$s = 2$	
	$\left\lceil \frac{\lvert p-q \rvert}{2} \right\rceil,$	$s = 3$	
	$4,$	s= 4	

x	$\lambda(x)$		Case for p, q
$v^s v_r^s$	r,	$1 \le r \le p-1; s=1$	$p = q \ge 3$
	k,	$1 \le r \le p-1; s=4$	
	i,	$1 \le r \le k; s=1$	$3 \le p < q$
	k,	$k+1 \le r \le q-1; s=1$	
	k,	$1 \le r \le p-1; s=4$	
$v^s v^{s+1}$	k,	$s=1, 3$	$p = q = 3$
	$k-1$,	$s=2$	
	$p-1$,	$s=1,3$	$p, q \ge 4$
	p,	$s=2$	
	$\lceil \frac{p+q-4}{2} \rceil$,	$s=1$	$3 \le p < q$
	p,	$s=2$,	
	k,	$s=3$	

We observe that each vertex and each edge has been labeled with a number which is at most $k = \lceil \frac{p+q-1}{2} \rceil$. Further, each vertex $x \in V(S_{p,2,2,q})$ has a weight as follows.

x	$wt(x)$		Case for p, q
v_r^s	$r+1$,	$1 \le r \le p-1; s=1$	$p = q \ge 3$
	$p+r$,	$1 \le r \le p-1; s=4$	
	$r+1$,	$1 \le r \le q-1; s=1$	$p < q;\ p, q \ge 3$
	$q+r$,	$1 \le r \le p-1; s=4$	
v^s	7,	$s=1$	$p = q = 3$
	6,	$s=2$	
	8,	$s=3$	
	12,	$s=4$	
	$\frac{p(p+1)}{2}$,	$s=1$	$p = q \ge 4$
	$2p+1$,	$s=2$	
	$2p$,	$s=3$	
	p^2+3,	$s=4$	
	$-\frac{k^2}{2} + k(q-1/2) + 1 + \lceil \frac{p+q-4}{2} \rceil$,	$s=1$	$p < q; p, q \ge 3$
	$p+q+1$,	$s=2$	
	$p+q$,	$s=3$	
	$4+kp$,	$s=4$	

It is shown above, each vertex has a distinct weight under total labeling f. Therefore, $tvs(S_{p,2,2,q}) = k = \lceil \frac{p+q-1}{2} \rceil$. □

Furthermore, an exact value of ts of $S_{p,2,2,q}$ is proved in the next theorem.

Theorem 1.2. *Given a caterpillar $S_{p,2,2,q}$ with p, q greater than or equal to 3. We get*

$$ts(S_{p,2,2,q}) = \left\lceil \frac{p+q-1}{2} \right\rceil.$$

Proof. According to (3), by using Equality (4) and Theorem 1.1, the lower bound is as follows:

$$ts(S_{p,2,2,q}) \geq \max\left\{ \left\lceil \frac{p+q+3}{3} \right\rceil, \ \left\lceil \frac{p+q-1}{2} \right\rceil \right\} = \left\lceil \frac{p+q-1}{2} \right\rceil. \tag{5}$$

Furthermore, we use total k-labeling λ constructed in Theorem 1.1 to get a totally irregular total k-labeling. Under labeling λ, we obtain the edge-weights below.

Case 1: For $p = q = 3$.

$$wt(v^s v_r^s) = \begin{cases} r+2, & 1 \leq r \leq p-1, \ s=1, \\ 2k+r, & 1 \leq r \leq p-1, \ s=4. \end{cases}$$

$$wt(v^s v^{s+1}) = \begin{cases} k+2, & s=1, \\ 2k, & s=2, \\ 3k, & s=3. \end{cases}$$

Case 2: For $p = q \geq 4$ and $3 \leq p < q$.

$$wt(v^s v_r^s) = \begin{cases} r+2, & 1 \leq r \leq q-1, \ s=1, \\ q+4+r, & 1 \leq r \leq p-1, \ s=4. \end{cases}$$

$$wt(v^s v^{s+1}) = \begin{cases} q+2, & s=1, \\ q+3, & s=2, \\ q+4, & s=3. \end{cases}$$

It can be seen that each edge has a different weight. This concludes that λ is totally irregular total k-labeling. Thus, $ts(S_{p,2,2,q}) = k = \left\lceil \frac{p+q-1}{2} \right\rceil$. □

2. A graph $S_{p,2,2,2,p}$

A graph that is formed from the double-star $S_{p,p}$ by inserting three vertices on the path connecting two centers of the two stars in $S_{p,p}$ is called as a caterpillar $S_{p,2,2,2,p}$. Hence, $S_{p,2,2,2,p}$ is a kind of tree with $|E(S_{p,2,2,2,p})| = 2p+2$ and it has maximal degree $\Delta = p$. Based on (2), tes of $S_{p,2,2,2,p}$ is

$$tes(S_{p,2,2,2,p}) = \max\left\{ \left\lceil \frac{p+1}{2} \right\rceil, \left\lceil \frac{2p+4}{3} \right\rceil \right\} = \left\lceil \frac{2p+4}{3} \right\rceil. \tag{6}$$

Meanwhile, tvs of $S_{p,2,2,2,p}$ is given in Theorem 2.1.

Theorem 2.1. *If $S_{p,2,2,2,p}$, $p \geq 4$ is a caterpillar with $p \geq 4$, then*

$$tvs(S_{p,2,2,2,p}) = p.$$

Proof. The graph $S_{p,2,2,2,p}$ is a tree that consists of $2p-2$ pendant vertices, three vertices of degree two, and it has two vertices with degree $p \geq 4$. By the similar reason as in Theorem 1.1 we get $tvs(S_{p,2,2,2,p}) \geq p$. Let $V(S_{p,2,2,2,p}) = \{v_r^s : 1 \leq r \leq p-1, s = 1, 5\} \cup \{v^s : \ s = 1, 2, 3, 4, 5\}$ and $E(S_{p,2,2,2,p}) = \{v^s v_r^s : 1 \leq r \leq p-1, s = 1, 5\} \cup \{v^s v^{s+1} : s = 1, 2, 3, 4\}$. To find tvs of $S_{p,2,2,2,p}$, we create a total labeling f of an element x, $x \in V(S_{p,2,2,2,p}) \cup E(S_{p,2,2,2,p})$ as follows.

x	$f(x)$		Case for p
v_r^s	1,	$1 \leq r \leq p-1; s = 1$	$p \geq 4$
	r,	$1 \leq r \leq p-1; s = 5$	
$v^s v_r^s$	r,	$1 \leq r \leq p-1; s = 1$	$p \geq 4$
	$\left\lceil \frac{2p-1}{2} \right\rceil$,	$1 \leq r \leq p-1; s = 5$	
v^s	1,	$s = 1, 2$	$p = 4$
	2,	$s = 3$	
	4,	$s = 4, 5$	

x	$f(x)$		Case for p
	1,	$s = 1, 2$	$p \geq 5$
	2,	$s = 3, 4$	
	5,	$s = 5$	
$v^s v^{s+1}$	p,	$s = 1, 2, 4$	$p = 4$
	$p-2$,	$s = 3$	
	p,	$s = 1, 2, 3$	$p \geq 5$
	$p-2$,	$s = 4$	

Under labeling f, we can see that each vertex has label at most $\lceil \frac{2p-1}{2} \rceil$.

x	$wt(x)$		Case for p
v_r^s	$r+1,$	$1 \leq r \leq p-1; s=1$	$p \geq 4$
	$p+r,$	$1 \leq r \leq p-1; s=5$	
v^s	$1/2(p^2+p)+1,$	$s=1$	$p \geq 4$
	$2p+1,$	$s=2$	$p \geq 4$
	$2p,$	$s=3$	$p=4$
	$2p+2,$	$s=4$	
	$5p,$	$s=5$	
	$2p+2,$	$s=3$	$p \geq 5$
	$2p,$	$s=4$	
	$p^2+3,$	$s=5$	

Moreover, the weight for each $x \in V(S_{p,2,2,2,p})$ is shown above. We can see that the each vertex has a distinct weight. Therefore, $tvs(S_{p,2,2,2,p}) = k = \lceil \frac{2p-1}{2} \rceil$. □

The exact value of ts of $S_{p,2,2,2,p}$ is discussed in the next theorem.

Theorem 2.2. *If $S_{p,2,2,2,p}$ is a caterpillar with $p \geq 4$, then*

$$ts(S_{p,2,2,2,p}) = p.$$

Proof. Based on (3), by using Theorem 2.1 and Equality (6) we get the lower bound of ts of $S_{p,2,2,2,p}$ as follows:

$$ts(S_{p,2,2,2,p}) \geq \max\left\{ \left\lceil \frac{2p+4}{3} \right\rceil,\ p \right\} = p.$$

To construct totally irregular total k-labeling, we use the vertex irregular total k-labeling f defined in Theorem 2.1. Under labeling f, we get the edge-weights as follows.

xy	$wt(xy)$		Case for p
$v^s v_r^s$	$r+2,$	$1 \leq r \leq p-1; s=1$	$p \geq 4$
	$2p+r,$	$1 \leq r \leq p-1; s=5$	$p=4$
	$p+5+r,$	$1 \leq r \leq p-1; s=5$	$p \geq 5$
$v^s v^{s+1}$	$p+2,$	$s=1$	$p \geq 4$
	$p+3,$	$s=2$	
	$p+4,$	$s=3$	
	$3p,$	$s=4$	$p=4$
	$p+5,$	$s=4$	$p \geq 5$

It is obvious that each edge has a different weight. Hence, the labeling f is desired a totally irregular total k-labeling with $ts(S_{p,2,2,2,p}) = k = p, \quad (p \geq 4)$. □

Conjecture 1. *For $p, q \geq 4$: ts of $S_{p,2,2,2,q}$ is $\lceil \frac{p+q-1}{2} \rceil$.*

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