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On the Balanced Case of the Brualdi-Shen Conjecture on 4-Cycle Decompositions of Eulerian Bipartite Tournaments

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Abstract

The *Brualdi-Shen Conjecture on Eulerian Bipartite Tournaments* states that any such graph can be decomposed into oriented 4-cycles. In this article we prove the balanced case of the mentioned conjecture. We show that for any $2n \times 2n$ bipartite graph $G = (U \cup V, E)$ in which each vertex has n-neighbors with biadjacency matrix M (or its transpose), there is a particular proper edge coloring of a column permutation of M denoted M^σ. This coloring has the property that the nonzero entries at each of the $first\ n$ columns are colored with elements from the set $\{n+1, n+2, \ldots, 2n\}$, and the nonzero entries at each of the $last\ n$ columns are colored with elements from the set $\{1, 2, \ldots, n\}$. Moreover, if the nonzero entry $M^\sigma_{r,j}$ is colored with color i then $M^\sigma_{r,i}$ must be a zero entry. Such a coloring will induce an oriented 4-cycle decomposition of the bipartite tournament corresponding to M. We achieve this by constructing an euler tour on the bipartite tournament that avoids traversing both pair of edges of any two internally disjoint s-t 2-paths consecutively, where s and t belong to V.

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1. Introduction

A *bipartite tournament* is an orientation of the edges of a complete bipartite graph. Given an eulerian bipartite tournament $\mathcal{G} = \overrightarrow{K}_{2n,2n}$, we study how a tour would traverse an orientation of

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a sub $\overrightarrow{K}_{2,2}$ of $\mathcal{G}$. Let U and V be the canonical partition of $\mathcal{G}$ and $\{X, Y\}$ a balanced partition of V, we call a tour of $\mathcal{G}$ that alternates between X and Y a *lateral tour*, i.e., every 2-path of the tour that originates at X ends at Y, and vice versa. If a tour of $\mathcal{G}$ alternates between some balanced partition of V with $v_1 \in X$ and $v_2 \in Y$, we refer to the orientation of $K_{2,2}$ given by either a pair of inner disjoint v_1v_2 2-paths or v_2v_1 2-paths, a *lateral torus*. A *decomposition* of a graph is the set of subgraphs induced by a partition of the edge set.

Conjecture 1. *(Brualdi, Shen) [2]*
Every eulerian bipartite tournament can be decomposed into oriented 4-cycles.

Work on the general conjecture has been carried out by Yuster [3], that gives (up to our knowledge) the best lower bounds on the number of edge-disjoint oriented 4-cycles. Our intentions are to work on the balanced case of the conjecture. We need to show that there exists a lateral tour of $\mathcal{G}$ in which for no lateral torus does the tour traverse both edges of each 2-path of the lateral torus consecutively.

2. The Hamming Distance

Given a simple undirected $n \times n$ bipartite graph $G = (U \cup V, E)$ with $U = \{u_1, u_2, \dots u_n\}$ and $V = \{v_1, v_2, \dots v_n\}$, there is an associated $n \times n$ matrix M with entries in $\{0, 1\}$ with $M_{i,j} = 1$ *if and only if* $\{u_i, v_j\} \in E$ and it is called the *biadjacency matrix* of G. The biadjacency matrix of a $n \times n$ bipartite tournament $\mathcal{G}$ has entries in the set $\{-, +\}$, $-$ for backward edges and $+$ for forward. Let M be a $m \times n$ matrix, define the *hamming distance* between columns i and j to be the number of row indexes r in which $M_{r,i} \neq M_{r,j}$.

Lemma 2.1. *Let M be the biadjacency matrix of a $2n \times 2n$ bipartite graph in which each vertex has n-neighbors. Then for M (or its transpose), there is a column permutation such that the hamming distance between any one of the $first$ n columns and any one of the $last$ n columns is at least twice the largest number of copies between any of the columns.*

Proof. Let k (m) be largest number of copies of any one column (row). Then permute the rows and columns such that the ones in the maximum repeated columns are in the last positions of the rows and columns in M. Let h (l) be the largest number of similar column (row) positions between any two rows (columns), one from the first n rows (columns) and another from the last n rows (columns), in which both entries are one. We must show that $k + l \leq n$ or $m + h \leq n$.

Assume, for the sake of contradiction, that $k + l > n$ and $m + h > n$. Then we have $k + l + m + h > 2n$. Since each of the last n rows have k ones belonging to the largest column copies, then $h \leq n - k$ and similarly we will have $l \leq n - m$. Combining these two inequalities, we have $k + l + m + h \leq 2n$. A contradiction. Hence $2k \leq 2(n - l)$, observe $2(n - l)$ is the smallest hamming distance possible (similarly for $2m \leq 2(n - h)$).

□

There is a one-to-one correspondence between a $2n \times 2n$ bipartite graph G in which each vertex has n-neighbors and a $2n \times 2n$ eulerian bipartite tournament $\mathcal{G}$. The former can be turned into the latter by orienting all edges of G from U into V and adding the nonedges of G between U and V and orient them from V into U. We will use: G corresponds to $\mathcal{G}$.

3. Edge Colorings Induced by Euler Tours

An *edge coloring* of a graph G is a function from the edge set E to a set of colors C, it is called *proper* if no two adjacent edges receive the same color. Edge colorings of a bipartite graph and colorings of the entries of its biadjacency matrix are to be used interchangeably.

Definition 3.1. Let M be the biadjacency matrix of a $2n \times 2n$ bipartite graph in which each vertex has n-neighbors. A *lateral edge coloring* is a proper edge coloring with colors from the set $C = \{1, 2, 3, \dots, 2n\}$ by replacing the nonzero entries of M with elements of C such that the following must be met:

(A1) The colors at each row are the column positions of the zero entries of that same row.

(A2) The colors at each column $i \in \{1, 2, \dots, n\}$ are $\{n+1, n+2, \dots, 2n\}$ and the colors at each column $j \in \{n+1, n+2, \dots, 2n\}$ are $\{1, 2, \dots, n\}$.

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad M^l = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 4 & 0 & 2 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

Figure 1. A biadjacency matrix and a lateral edge coloring

Definition 3.2. Let $\mathcal{G} = (U, V, E)$ be a $2n \times 2n$ eulerian bipartite tournament and a balanced partition $V = X \cup Y$. An euler tour on $\mathcal{G}$ is called *lateral* on $\{X, Y\}$ if every oriented sub 2-path of the tour originating at X must terminate at Y, and vice versa.

Example 3.1.

$$M^\tau = \begin{bmatrix} +_8 & -_{13} & -_7 & +_{14} \\ +_{16} & -_5 & +_6 & -_{15} \\ -_9 & +_4 & +_{10} & -_3 \\ -_1 & +_{12} & -_{11} & +_2 \end{bmatrix} \qquad M^l = \begin{bmatrix} 3 & - & - & 2 \\ 4 & - & 2 & - \\ - & 4 & 1 & - \\ - & 3 & - & 1 \end{bmatrix}$$

Figure 2. A lateral euler tour on $\{v_1, v_2\} \cup \{v_3, v_4\}$ and its induced lateral coloring

The eulerian bipartite tournament biadjacency matrix M^τ, in Figure 2, contains the following euler tour τ: $v_1u_4v_4u_3v_2u_2v_3u_1v_1u_3v_3u_4v_2u_1v_4u_2v_1$. Observe that τ is lateral on $\{X, Y\}$ where $X = \{v_1, v_2\}$ and $Y = \{v_3, v_4\}$. The matrix M^l is a lateral edge coloring of M^τ induced by the lateral tour τ. It is constructed by coloring the $+$ entry with the column position of the previous $-$ entry on the same row taken consecutively by τ. The colors at each row will be distinct. The problem we have is that this procedure can produce the same color on the same column for distinct rows, which by definition is not a proper edge coloring. In the next sections we devise a strategy to prevent this eventuality.

4. Euler Tours Traversing Special Subgraphs

In an oriented graph a vertex of in-degree zero is called a *source* and a vertex of out-degree zero is called a *sink*.

Definition 4.1. A *torus* is the orientation of the edges of the complete bipartite graph $K_{2,2} = (U \cup V, E)$ with precisely one source and one sink both on V. A torus is called *lateral* on partition $\{X, Y\}$ of V if the sink and source belong to different parts.

$$\begin{bmatrix} - & + \\ - & + \end{bmatrix} \qquad\qquad \begin{bmatrix} + & - \\ + & - \end{bmatrix}$$

Figure 3. The submatrices forming tori

Definition 4.2. Let H be a subgraph of G and let τ be an euler tour of G. We say H is *k-traverse* by τ if there is a path-cycle decomposition of H (call its elements $parts$) with k parts such that each part is a path or cycle traversed by τ and any two distinct edges in distinct parts are not taken consecutively by τ.

Observe the only 1-traverse graphs are paths and cycles. A torus can be:
2-traverse if the edges of each of the two 2-paths are consecutive in the tour,
3-traverse if the edges of precisely one of the 2-paths are consecutive in the tour,
4-traverse if no edges of each of the two 2-paths are consecutive in the tour.

With these definitions the following lemma will give a sufficient condition for the existence of a 4-cycle decomposition of an eulerian $2n \times 2n$ bipartite tournament.

Lemma 4.1. *Let $\mathcal{G} = (U \cup V, E)$ be a $2n \times 2n$ eulerian bipartite tournament and a lateral euler tour τ on a balanced partition $\{X, Y\}$ of V. If no lateral torus (on the same partition as the tour) of $\mathcal{G}$ is 2-traverse by τ, then τ induces a lateral edge coloring of G (corresponding to $\mathcal{G}$).*

Proof. Start at a $-$ edge in the biadjacency matrix M of $\mathcal{G}$ and assume it is a backward edge coming from a vertex in X say labelled by v_i such that $i \in \{1, 2, \ldots, n\}$. Then, τ takes $+$ at Y in v_j such that $j \in \{n+1, n+2, \ldots, 2n\}$ both on the same row k of M. Color entry $M_{k,j}$ with color i, which is the column position of the previous edge on τ. Continue until finishing the tour.

$$\begin{array}{ccc} & i & j \\ k & - & + \\ l & - & + \end{array} \qquad \begin{array}{cc} - & i \\ - & i \end{array} \qquad \begin{array}{ccc} & i & j \\ k & + & - \\ l & + & - \end{array} \qquad \begin{array}{cc} j & - \\ j & - \end{array}$$

Figure 4. The 2-traverse tori together with their induced bad colorings

But since no lateral torus of $\mathcal{G}$ on $\{X, Y\}$ is 2-traverse by τ, then we do not even get to this eventuality. Therefore, the coloring induced by τ will not have the same color on the same column, hence proper.

□

5. Main Result

Given an oriented *3-cycle* in an oriented graph we will refer to it as a *triangle* of the graph.

Theorem 5.1. *Let M be the biadjacency matrix of a $2n \times 2n$ bipartite graph in which each vertex has n-neighbors. Then there is a lateral edge coloring of some column permutation of M (or its transpose).*

Proof. Let M^σ (assume without loss of generality) be the column permutation of M according to Lemma 2.1. Let T be the matrix obtained from M^σ by replacing the zeros to $-$ and the ones to $+$. Let $\mathcal{G} = (Y \cup W, E)$ be the eulerian bipartite tournament with biadjacency matrix T and canonical partition $\{Y, W\}$, these are the row and column vertices respectively, and a partition $X = \{w_1, w_2, \ldots, w_n\}$ and $Z = \{w_{n+1}, w_{n+2}, \ldots, w_{2n}\}$ of W. These labels are given by the natural labelling of T.

Construct a tripartite oriented graph $\mathcal{F} = (X \cup Y \cup Z, E_1)$ as follows:
edges between vertices in X and Y are the edges in $\mathcal{G}$ oriented from X to Y,
edges between vertices in Y and Z are the edges in $\mathcal{G}$ oriented from Y to Z,
edges between vertices in Z and X are the edges of $K_{n,n}$ oriented from Z to X.

Also the tripartite oriented graph $\mathcal{B} = (X \cup Y \cup Z, E_2)$:
edges between vertices in Z and Y are the edges in $\mathcal{G}$ oriented from Z to Y,
edges between vertices in Y and X are the edges in $\mathcal{G}$ oriented from Y to X,
edges between vertices in X and Z are the edges of $K_{n,n}$ oriented from X to Z.

Observe that the graph $\mathcal{F}$ is eulerian and moreover by the conclusion of Lemma 2.1, we have at least one oriented 2-path from any vertex in X into any vertex in Z, i.e., every edge belongs to at least one triangle. This implies that we can decompose the edge set E_1 into triangles and call this decomposition D_{XZ}. Since $\mathcal{B}$ also has these properties, we can decompose the edge set E_2 into triangles and call this other decomposition D_{ZX}.

We now construct an euler tour τ of $\mathcal{G}$ starting at a vertex in X and taking an arbitrary backwards edge e_1 into Y, then take the consecutive edge e_2 that belongs to the unique triangle in D_{XZ} that contains e_1. Now starting at the last vertex of e_2, take an arbitrary backward edge e_3. Then take the consecutive edge e_4 that belongs to the unique triangle in D_{ZX} that contains e_3. We keep going in this manner and complete the euler tour τ. Observe that this euler tour is lateral on partition $\{X, Z\}$ of W and no lateral torus of $\mathcal{G}$ (on same partition of the tour) is 2-traverse by τ since otherwise there would be two triangles of D_{XZ} (or D_{ZX}) sharing an edge, which by definition cannot happen since D_{XZ} and D_{ZX} are sets of edge-disjoint triangles. Therefore, by Lemma 4.1, τ induces a lateral edge coloring of M^σ.

□

Corollary 5.1. *Every $2n \times 2n$ eulerian bipartite tournament can be decomposed into oriented 4-cycles.*

Proof. Let $\mathcal{G}$ be a $2n \times 2n$ eulerian bipartite tournament. Let M be the biadjacency matrix of the $2n \times 2n$ bipartite graph in which each vertex has n-neighbors and which corresponds to $\mathcal{G}$. Let M^l be a lateral edge coloring constructed from M^σ according to Theorem 5.1. Let $i \in \{1, 2, \ldots, n\}$ and $j \in \{n+1, n+2, \ldots, 2n\}$, observe that the two entries color j in column i and color i in column j of M^l form two opposite corners of a 2×2 submatrix. Moreover, by the requirement A1 in Definition 3.1, the remaining opposite corners of this 2×2 submatrix must both be zero and unique corners for this pair of colors i and j, which corresponds to an oriented 4-cycle of $\mathcal{G}$. Now having paired some entries accordingly let the entry color k in column h not be paired, then by the pairing rule entry color h in column k must not have been paired since otherwise it would have been paired with entry color k in column h. Therefore we can pair all colored entries, which corresponds to edge-disjoint oriented 4-cycles in $\mathcal{G}$.

$\square$

References

[1] J. Bang-Jensen and G. Gutin. Digraphs: Theory, Algorithms and Applications. first ed. *Springer-Verlag, London*, 2000.

[2] R.A. Brualdi and J. Shen. Disjoint Cycles in Eulerian Digraphs and the Diameter of Interchange Graphs. *J. Combi. Theory Ser. B 85* (2002), 189–196.

[3] R. Yuster. Packing 4-Cycles in Eulerian and Bipartite Eulerian Tournaments with an Application to Distances in Interchange Graphs. *Annals of Combi. 9* (2005), 117–124.