Electronic Journal of Graph Theory and Applications **9** (2) (2021), 377–385

Electronic Journal of Graph Theory and Applications

All missing Ramsey numbers for trees versus the four-page book

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Abstract

For the Ramsey number $r(T_n, B_m)$, where T_n denotes a tree of order n and B_m denotes the m-page book $K_2 + \overline{K_m}$, it is known that $r(T_n, B_m) = 2n - 1$ if $n \geq 3m - 3$. In case of $n < 3m - 3$, $r(T_n, B_m)$ has not been completely evaluated except for $m \leq 3$. Here we determine the missing values of $r(T_n, B_4)$. Our results close one gap in the table of the Ramsey numbers $r(T_n, G)$ for all trees T_n and all connected graphs G of order six.

Keywords: Ramsey number, tree, book, small graph
Mathematics Subject Classification: 05C55, 05D10
DOI: 10.5614/ejgta.2021.9.2.10

1. Introduction

For any connected graph G of order n and any graph H the Ramsey number $r(G, H)$ satisfies

$$r(G, H) \geq (n - 1)(\chi(H) - 1) + 1,$$

where $\chi(H)$ denotes the chromatic number of H. By applying this lower bound, due to Chvátal and Harary [1], to a tree T_n of order n and the m-page book $B_m = K_2 + \overline{K_m}$, we obtain that

$$r(T_n, B_m) \geq 2n - 1. \tag{1}$$

Received: 15 May 2020, Revised: 23 March 2021, Accepted: 13 April 2021.

Erdős, Faudree, Rousseau and Schelp [3] showed that equality holds in (1) for a certain range of n and m, namely

$$r(T_n, B_m) = 2n - 1 \text{ if } n \geq 3m - 3. \tag{2}$$

The case $T_n = S_n$, the star of order n, had already been considered earlier by Rousseau and Sheehan [8] who also proved that, for $n \geq 2$,

$$r(T_n, B_m) \geq \max\left\{(k+2)(n-1)+1, m+2\left\lfloor\frac{m-1}{k+1}\right\rfloor\right\} \text{ with } k = \left\lfloor\frac{m-1}{n-1}\right\rfloor, \tag{3}$$

and that equality holds for $T_n = P_n$, the path of order n. For $T_n \neq P_n$, which implies $n \geq 4$, $r(T_n, B_m)$ is not completely known if $n < 3m - 3$. In [8] it was shown that in case of $n \leq m$ the lower bound (3) also matches the exact value if $n - 1$ divides $m - 1$, in particular if $n = m$. Recently, further results concerning the case $n \leq m$ have been obtained by Zhang, Chen and Zhu [9]. For $m \leq 3$ and $n \geq 4$, $r(T_n, B_m)$ is completely determined by (2) except for $m = 3$ where $4 \leq n \leq 5$. The missing values of $r(T_n, B_3)$ can be found in [2] and [5]. In this paper we focus on the case $m = 4$. By the above mentioned results, the values of $r(T_n, B_4)$ are still missing for $5 \leq n \leq 8$ if $T_n \neq P_n$. Moreover, it is already known that $r(S_5, B_4) = 11$ and $r(S_8, B_4) = 16$ (see [4, 6, 8]). All remaining cases will be settled in this paper. Our results close one gap in the table of the Ramsey numbers $r(T_n, G)$ for all trees T_n and all connected graphs G of order six obtained in [6] and [7].

Some specialized notation will be used. The vertex set of a graph G is denoted by $V(G)$. We write $G' \subseteq G$ if G' is a subgraph of G. For $U \subseteq V(K_n)$, $[U]$ is the subgraph induced by U. A coloring of a graph always means a 2-coloring of its edges with colors red and green. An (F_1, F_2)-coloring is a coloring containing neither a red copy of F_1 nor a green copy of F_2. Given a coloring of K_n, we define the r-degree $d_r(v)$ to be the number of red edges incident to $v \in V(K_n)$. Moreover, $\Delta_r = \max_{v \in V(K_n)} d_r(v)$. The set of vertices joined red to v is denoted by $N_r(v)$. If $U = \{v_1, v_2, \ldots, v_s\} \subseteq V(K_n)$, then we write $N_r(U)$ or $N_r(v_1, v_2, \ldots, v_s)$ instead of $N_r(v_1) \cap N_r(v_2) \cap \ldots \cap N_r(v_s)$. Similarly we define $d_g(v)$, Δ_g, $N_g(v)$, $N_g(U)$ and $N_g(v_1, v_2, \ldots, v_s)$. Furthermore, $[U]_r$ and $[U]_g$ are the red and the green subgraphs induced by U. For disjoint subsets $U_1, U_2 \subseteq V(K_n)$, $q_r(U_1, U_2)$ denotes the number of red edges between U_1 and U_2 and $q_g(U_1, U_2)$ is defined similarly. If U_1 consists of a single vertex v, then we use $q_r(v, U_2)$ and $q_g(v, U_2)$ instead. Moreover, in case of $v \in U$, $q_r(v, U)$ and $q_g(v, U)$ mean $q_r(v, U \setminus \{v\})$ and $q_g(v, U \setminus \{v\})$, respectively. We write $P_k = v_1 v_2 \ldots v_k$ for the path P_k with vertices $v_1, v_2, \ldots, v_k$ and edges $v_i v_{i+1}$ for $i = 1, \ldots, k-1$. Moreover, $(v_1 v_2 \ldots v_k)$ denotes the cycle C_k obtained from $P_k = v_1 v_2 \ldots v_k$ by adding the edge $v_1 v_k$. For $k \geq 2$ and $n \geq k+2$, the broom $B_{n-k,k}$ is defined as a tree of order n obtained by identifying the vertex of degree $n-k$ of a star S_{n-k+1} with an end-vertex of a path P_k.

2. The missing values of $r(T_n, B_4)$

It follows from (2) that, for any tree T_n,

$$r(T_n, B_4) = 2n - 1 \text{ if } n \geq 9.$$

Moreover, as mentioned above, the lower bound (3) matches the exact value of $r(T_n, B_m)$ for $T_n = P_n$ and, if $n = m$, for any T_n. Using that $T_n = P_n$ if $n \leq 3$ we obtain

$$r(T_2, B_4) = 6, \quad r(T_3, B_4) = 7, \quad r(T_4, B_4) = 10, \tag{4}$$

$$r(P_5, B_4) = 10 \text{ and } r(P_n, B_4) = 2n - 1 \text{ if } n \geq 6. \tag{5}$$

In the remaining case $5 \leq n \leq 8$ and $T_n \neq P_n$ the exact values of $r(T_n, B_4)$ apart from $r(S_5, B_4)$ and $r(S_8, B_4)$ are still missing and will be determined here. By (3) and (1) it is already known that

$$r(T_5, B_4) \geq 10 \text{ and } r(T_n, B_4) \geq 2n - 1 \text{ if } n \geq 6. \tag{6}$$

First we consider $T_n = S_n$. The following theorem, where $r(S_5, B_4)$ and $r(S_8, B_4)$ are contained for the sake of completeness, shows that the values of $r(S_n, B_4)$ with $5 \leq n \leq 8$ differ from the lower bounds in (6) except for $n = 7$.

Theorem 1. *Let* $5 \leq n \leq 8$. *Then*

n	5	6	7	8
$r(S_n, B_4)$	11	12	13	16

Proof. It remains to prove $r(S_6, B_4) = 12$ and $r(S_7, B_4) = 13$. The coloring of K_{11} where $[V]_r = K_3 \cup H$ and H is obtained from a cycle $(v_1 v_2 \dots v_8)$ by adding the edges $v_i v_{(i+2) \bmod 8}$ for $i = 1, \dots, 8$ yields $r(S_6, B_4) \geq 12$, and (6) implies $r(S_7, B_4) \geq 13$. To establish equality, assume that we have an (S_n, B_4)-coloring χ of K_t with $6 \leq n \leq 7$, where $t = 12$ if $n = 6$ and $t = 13$ if $n = 7$. First we derive some properties of χ useful in order to deduce a contradiction from our assumption. Let V denote the vertex set of K_t.

Claim 1. *Let* $v \in V$ *with* $U = N_r(v)$ *and* $W = N_g(v)$, *and let* $w \in W$. *Then*

$$(i)\ q_g(w, W) \leq 3, \quad (ii)\ 7 \leq d_g(v) \leq n + 1, \quad (iii)\ q_g(w, U) \geq 3, \quad (iv)\ q_g(w, W) \geq 1.$$

Proof of Claim 1. (i): If $q_g(w, W) > 3$, then a green B_4 with spine vw occurs, a contradiction.

(ii): $S_n \not\subseteq [V]_r$ forces $\Delta_r \leq n - 2$. Thus, $d_g(v) \geq |V| - 1 - \Delta_r \geq 7$ for every $v \in V$. To prove that $d_g(v) \leq n + 1$ consider a vertex $v^* \in V$ with $d_g(v^*) = \Delta_g$. Let $U^* = N_r(v^*)$ and $W^* = N_g(v^*)$. Clearly, $q_g(w, W^*) + q_r(w, W^*) = |W^*| - 1 = \Delta_g - 1$ for every $w \in W^*$. Using $q_r(w, W^*) \leq \Delta_r \leq n - 2$ and (i) we obtain $\Delta_g - n + 1 \leq q_g(w, W^*) \leq 3$. Hence $\Delta_g \leq n + 2$. Assume that $\Delta_g = n + 2$, i.e. $|W^*| = n + 2$. Then $q_g(w, W^*) = 3$ for every $w \in W^*$. In case of $n = 7$ this implies $[W^*]_g$ to be a 3-regular graph of order 9, a contradiction. In case of $n = 6$, $|V| = 12$ and $|W^*| = n+2$ yield $|U^*| = 3$. Take $w_1, w_2 \in W^*$ with $w_1 w_2$ green. Since $d_g(w_i) \geq 7$ and $q_g(w_i, W^*) = 3$, all edges between $\{w_1, w_2\}$ and U^* have to be green, i.e. $U^* \subseteq N_g(w_1, w_2)$. Moreover, $v^* \in N_g(w_1, w_2)$. But this gives a green B_4 with spine $w_1 w_2$, a contradiction. Thus, $\Delta_g \leq n + 1$, and the proof of (ii) is complete.

(iii) and (iv): By (ii), $d_g(w) \geq 7$. Furthermore, $d_g(w) = q_g(w, W) + q_g(w, U) + 1$. Hence, (iii) is an immediate consequence of (i), and $q_g(w, U) \leq |U| \leq \Delta_r \leq n - 2$ yields (iv).

Claim 2. *If $v \in V$ exists with $d_r(v) = 4$ where $U = N_r(v)$ and $W = N_g(v)$, then*
(i) $N_g(w_1, w_2) \cap W = N_r(w_1, w_2) \cap U = \emptyset$ *for any $w_1, w_2 \in W$ with w_1w_2 green,*
(ii) $q_g(w, U) = 3$ *for every $w \in W$,* (iii) $[W]_g$ *is 3-regular.*

Proof of Claim 2. (i) and (ii): Consider $w_1, w_2 \in W$ joined green. $B_4 \nsubseteq [V]_g$ forces $|N_g(w_1, w_2)| \leq 3$, and Claim 1(iii) implies $|N_g(w_1, w_2) \cap U| \geq 2$. Since $v \in N_g(w_1, w_2)$, only $N_g(w_1, w_2) \cap W = \emptyset$ and $|N_g(w_1, w_2) \cap U| = 2$ is left. Consequently, $q_g(w_1, U) = q_g(w_2, U) = 3$ and $N_r(w_1, w_2) \cap U = \emptyset$. Additionally we obtain (ii), as every $w \in W$ is incident to at least one green edge in $[W]$ by Claim 1(iv).

(iii): By Claim 1(ii), $d_g(w) \geq 7$ for any $w \in W$. Thus, (ii) and Claim 1(i) yield (iii).

Using Claims 1 and 2 now we deduce a contradiction from our above assumption.

Case 1. $n = 6$. Consider some $v \in V$. Let $W = N_g(v)$. Claim 1(ii) implies $d_g(v) = 7$, i.e. $|W| = 7$, and we obtain $d_r(v) = 4$ from $|V| = 12$. Thus, Claim 2(iii) forces $[W]_g$ to be a 3-regular graph of order 7, a contradiction.

Case 2. $n = 7$. By Claim 1(ii), $7 \leq d_g(v) \leq 8$ for every $v \in V$. Since $|V| = 13$, $[V]_g$ cannot be 7-regular. Consequently, a vertex $v^* \in V$ with $d_g(v^*) = 8$ and $d_r(v^*) = 4$ must occur. Let $U = N_r(v^*)$ and $W = N_g(v^*)$. If $q_r(u, W) = 2$ for every $u \in U$, then $d_g(u) \geq 7$ implies $q_g(u, U) \geq 1$, and we find $u_1, u_2 \in U$ with u_1u_2 green. But this yields a green B_4 since $|N_g(u_1, u_2) \cap W| \geq 4$, a contradiction. As $q_r(U, W) = 8$ by Claim 2(ii), it remains that a vertex $u^* \in U$ with $q_r(u^*, W) \geq 3$ exists. Let $W' \subseteq N_r(u^*) \cap W$ with $|W'| = 3$ and let $W'' = W \setminus W'$. Claim 2(i) forces $[W']$ to be a red K_3. One of the following two subcases must occur.

Case 2.1. $q_g(w^*, W') = 3$ for some $w^* \in W''$. By Claim 2(i), u^*w^* has to be green and Claim 2(ii) yields two further vertices $u', u'' \in U$ joined green to w^*. Since $q_g(u^*, W) \leq 5$ and $d_g(u^*) \geq 7$, at least one of the vertices u' and u'', say u', is joined green to u^*. Moreover, Claim 2(ii) implies that $u'w$ is green for every $w \in W'$. But then $W' \cup \{u^*\} \subseteq N_g(u', w^*)$ and we obtain a green B_4 with spine $u'w^*$, a contradiction.

Case 2.2. $q_g(w, W') \leq 2$ for every $w \in W''$. By Claim 2(iii), $[W]_g$ has to be 3-regular yielding $q_g(W', W'') = 9$. Consequently, since $|W''| = 5$, $q_g(w^*, W') = 1$ for some $w^* \in W''$ and $q_g(w, W') = 2$ for every $w \in W'' \setminus \{w^*\}$. Moreover, the 3-regularity of $[W]_g$ implies $q_g(w^*, W'') = 2$ and $q_g(w, W'') = 1$ for every $w \in W'' \setminus \{w^*\}$. Thus, the two red neighbors of w^* in W'' have to be joined green. Furthermore, they must have at least one common green neighbor in W'. This contradicts Claim 2(i), and we are done. □

It remains to consider the non-star trees $T_n \neq P_n$ with $5 \leq n \leq 8$. The following theorem shows that $r(T_n, B_4)$ matches the bounds given in (6) for all these trees.

Theorem 2. *Let $5 \leq n \leq 8$ and let $T_n \notin \{P_n, S_n\}$. Then*

$$r(T_n, B_4) = 10 \text{ if } n = 5 \text{ and } r(T_n, B_4) = 2n - 1 \text{ if } n \geq 6.$$

Proof. Considering (6) it remains to prove $r(T_5, B_4) \leq 10$ for $T_5 \notin \{P_5, S_5\}$, i.e. $T_5 = B_{2,3}$, and $r(T_n, B_4) \leq 2n-1$ for every $T_n \notin \{P_n, S_n\}$ with $6 \leq n \leq 8$. Assume that we have a $(B_{2,3}, B_4)$-coloring χ of K_{10} or a (T_n, B_4)-coloring χ of K_{2n-1} for some $T_n \notin \{P_n, S_n\}$ with $6 \leq n \leq 8$. To deduce a contradiction from this assumption first we derive some properties of χ. Let V denote the vertex set of the complete graphs K_{10} and K_{2n-1}. $B_4 \nsubseteq [V]_g$ yields

Claim 3. *If $V' \subseteq V$ with $|V'| \geq 2$ and $|N_g(V')| \geq 4$, then $[V']$ is a red complete graph.*

$T_n \nsubseteq [V]_r$ forces $K_n \nsubseteq [V]_r$. Consequently, Claim 3 immediately implies

Claim 4. *If $V' \subseteq V$ and $|V'| \geq n$, then $|N_g(V')| \leq 3$.*

In case of $n \geq 6$ the restriction $K_n \nsubseteq [V]_r$ can be improved.

Claim 5. *If $n \geq 6$, then $K_{n-2} \nsubseteq [V]_r$.*

Proof of Claim 5. Assume to the contrary that $K_{n-2} \subseteq [V]_r$. Let U be the vertex set of a red K_{n-2} and let $W = V \setminus U$. Since $|U| \geq 4$ and $|W| = n+1$, Claim 4 implies $q_r(U, W) \geq 1$. Consider two vertices $u \in U$ and $w \in W$ where uw is red. Let $W' = W \setminus \{w\}$. Again using Claim 4 we obtain that $q_r(U, W') \geq 1$. A red edge $u'w'$ with $u' \in U \setminus \{u\}$ and $w' \in W'$ cannot occur: otherwise, since any non-star tree contains two different vertices adjacent to vertices of degree 1, the red K_{n-2} together with the red edges uw and $u'w'$ would give every $T_n \neq S_n$ in red, a contradiction. It remains that uw' is red for some $w' \in W'$ and that $U \setminus \{u\} \subseteq N_g(W')$. But this contradicts Claim 4 if $n \geq 7$, and in case of $n = 6$, $U \setminus \{u\} = N_g(W')$ is left. Thus, $q_r(w, W') \geq 1$. But then we find any T_6 in red, since every T_6 contains a vertex adjacent to two vertices of degree 1 or a vertex of degree 1 adjacent to a vertex of degree 2. This contradiction completes the proof of Claim 5.

Applying Claims 3 and 5 we obtain an improvement of Claim 4 for $n \geq 6$.

Claim 6. *If $n \geq 6$ and if $V' \subseteq V$ with $|V'| \geq n-2$, then $|N_g(V')| \leq 3$.*

Using Claims 3 to 6 now we deduce a contradiction from the above assumption. Since $T_n \notin \{P_n, S_n\}$, the maximum degree $\Delta(T_n)$ satisfies $3 \leq \Delta(T_n) \leq n-2$. We distinguish the following four cases depending on $\Delta(T_n)$ and use $T_{n,k}$ to denote a tree T_n with $\Delta(T_n) = k$.

Case 1. $\Delta(T_n) = n-2$ where $5 \leq n \leq 8$. There is exactly one tree $T_{n,n-2}$, namely the broom $B_{n-3,3}$. By Theorem 1 and (4), $S_{n-1} \subseteq [V]_r$. Consider a red S_{n-1} in χ with vertex set U and u^* as vertex of degree $n-2$. Let $W = V \setminus U$. Since $|W| \geq n$ and $|U| \geq 4$, Claim 4 yields $q_r(U, W) \geq 1$. If uw is red for some $u \in U \setminus \{u^*\}$ and some $w \in W$, then a red $B_{n-3,3}$ occurs, a contradiction. Otherwise, u^*w is red for some $w \in W$ and $N_g(U \setminus \{u^*\}) = W$. Using Claim 3 we obtain that $[U \setminus \{u^*\}]$ is a red K_{n-2} contradicting Claim 5 for $n \geq 6$. If $n = 5$, then $[U]$ is a red K_4 yielding a red $B_{2,3}$ together with u^*w, a contradiction, and we are done.

Case 2. $\Delta(T_n) = n-3$ where $6 \leq n \leq 8$. There are three trees $T_{n,n-3}$, namely ${T_{n,n-3}}^{(1)}$ and ${T_{n,n-3}}^{(2)}$ obtained from S_{n-2} by adding two vertices of degree 1 joined to the same vertex of degree 1 or to two different vertices of degree 1 of S_{n-2}, respectively, and ${T_{n,n-3}}^{(3)} = B_{n-4,4}$ (for $n = 7$ these three trees $T_{n,n-3}$ are shown in Figure 1). Now we consider a red S_{n-2} in χ with vertex set U and u^* as vertex of degree $n-3$. Let $U \setminus \{u^*\} = \{u_1, \ldots, u_{n-3}\}$ and $W = V \setminus U = \{w_1, \ldots, w_{n+1}\}$. By Claim 5, a green edge, say u_1u_2, must occur in $[U]$. Since

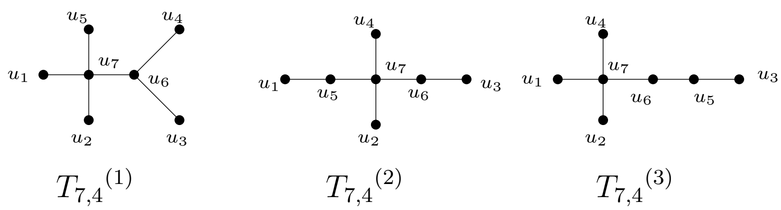

Figure 1. The trees $T_{7,4}$ with vertex labeling.

$B_4 \not\subseteq [V]_g$, there are at most three common green neighbors of u_1 and u_2 in W, and we may assume that any $w \in W \setminus \{w_{n-1}, w_n, w_{n+1}\}$ is joined red to u_1 or to u_2. This implies that, without loss of generality, u_1w_1 and u_1w_2 are red. Thus, $T_{n,n-3}{}^{(1)}$ is unavoidable in $[V]_r$. If $T_{n,n-3}{}^{(2)} \not\subseteq [V]_r$, then there are only green edges between $\{u_2, \ldots, u_{n-3}\}$ and W. Consequently, all edges from u_1 to $\{w_1, \ldots, w_{n-2}\}$ have to be red. If there are only green edges in $[\{w_1, \ldots, w_{n-2}\}]$, then four vertices from $\{w_1, \ldots, w_{n-2}\}$ and two vertices from $\{u_2, \ldots, u_{n-3}\}$ yield a green $K_6 - e \supseteq B_4$, a contradiction. Hence we may assume that w_1w_2 is red. But this yields a red $T_{n,n-3}{}^{(2)}$ with u_1 as vertex of degree $n-3$, a contradiction. Finally, if $T_{n,n-3}{}^{(3)} \not\subseteq [V]_r$, then in $[W]$ all edges incident to w_1 or to w_2 have to be green yielding a green B_4, a contradiction.

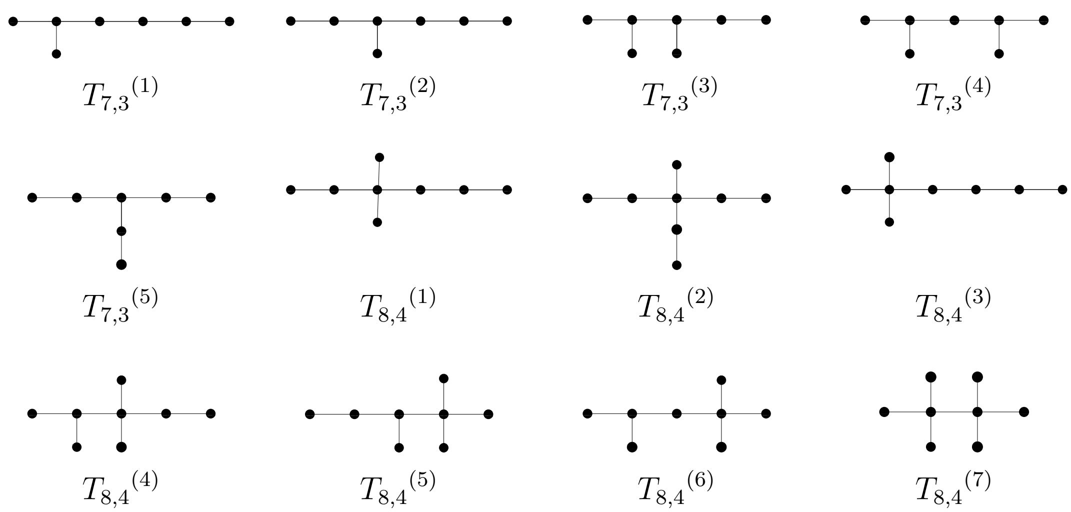

Figure 2. The trees $T_{n,n-3}$ with $7 \leq n \leq 8$.

Case 3. $\Delta(T_n) = n - 4$ where $7 \leq n \leq 8$. The five trees $T_{7,3}$ and the seven trees $T_{8,4}$ are shown in Figure 2. We may use that $T_{6,3}{}^{(1)}$, every $T_{7,4}$ and also P_7 must occur in $[V]_r$ (see Case 2 and (5)). If a red $T_{7,4}$ in χ with $U = V(T_{7,4})$ is considered, then the vertices in U shall be denoted by $u_1, u_2, \ldots, u_7$ as in Figure 1 and W means $V \setminus U$.

• $T_7 \in \{T_{7,3}{}^{(1)}, T_{7,3}{}^{(2)}\}$. Consider a red $P_7 = u_1u_2 \ldots u_7$ in χ. Let $W = V \setminus \{u_1, \ldots, u_7\}$. If $T_{7,3}{}^{(1)} \not\subseteq [V]_r$, then $\{u_2, u_3, u_5, u_6\} \subseteq N_g(W)$ contradicting Claim 6. If $T_{7,3}{}^{(2)} \not\subseteq [V]_r$, then u_3u_5 and all edges between $\{u_3, u_5\}$ and W have to be green contradicting $B_4 \not\subseteq [V]_g$.

• $T_7 = {T_{7,3}}^{(3)}$. Consider a red ${T_{6,3}}^{(1)}$ in χ. Let U be the set of the four vertices of degree 1 of ${T_{6,3}}^{(1)}$ and $W = V \setminus V({T_{6,3}}^{(1)})$. ${T_{7,3}}^{(3)} \not\subseteq [V]_r$ forces $U \subseteq N_g(W)$ contradicting Claim 6.

• $T_7 \in \{{T_{7,3}}^{(4)}, {T_{7,3}}^{(5)}\}$. Consider a red ${T_{7,4}}^{(2)}$ in χ. If ${T_{7,3}}^{(4)} \not\subseteq [V]_r$, then $q_r(u_1, W \cup \{u_3, u_6\}) \leq 1$ and $q_r(u_3, W \cup \{u_1, u_5\}) \leq 1$. Thus, u_1 and u_3 have at least four common green neighbors in W, and $B_4 \not\subseteq [V]_g$ forces u_1u_3 to be red. Consequently, u_3u_5, u_1u_6 and all edges between $\{u_1, u_3\}$ and W have to be green. But then $B_4 \not\subseteq [V]_g$ implies $q_r(u_i, W) \geq 3$ for $i = 5, 6$ yielding a red ${T_{7,3}}^{(4)}$, a contradiction. If ${T_{7,3}}^{(5)} \not\subseteq [V]_r$, then u_2u_4 and all edges between $\{u_2, u_4\}$ and W have to be green contradicting $B_4 \not\subseteq [V]_g$.

• $T_8 \in \{{T_{8,4}}^{(1)}, {T_{8,4}}^{(2)}\}$. Consider a red ${T_{7,4}}^{(2)}$ in χ. If ${T_{8,4}}^{(1)} \not\subseteq [V]_r$, then all edges between $\{u_1, u_3\}$ and W have to be green. Since $B_4 \not\subseteq [V]_g$, u_1u_3 has to be red, and Claim 6 demands at least one red edge from W to $\{u_2, u_4\}$. But this gives a red ${T_{8,4}}^{(1)}$, a contradiction. If ${T_{8,4}}^{(2)} \not\subseteq [V]_r$, then all edges between $\{u_2, u_4\}$ and W have to be green, and this forces u_2u_4 to be red. Consequently, all edges from u_7 to W have to be green, and Claim 6 yields three vertices $w_1, w_2, w_3 \in W$ joined red to u_3. Moreover, $B_4 \not\subseteq [V]_g$ implies $q_r(w_i, W \setminus \{w_1, w_2, w_3\}) \geq 2$ for $1 \leq i \leq 3$. But then we obtain a red ${T_{8,4}}^{(2)}$ with u_3 as vertex of degree 4, a contradiction.

• $T_8 = {T_{8,4}}^{(3)}$. Consider a red ${T_{7,4}}^{(3)}$ in χ. If ${T_{8,4}}^{(3)} \not\subseteq [V]_r$, then all edges from u_3 to W are green and $q_r(u_i, W) \leq 2$ for $i \in \{1, 2, 4\}$. Hence, since $B_4 \not\subseteq [V]_g$, $[\{u_1, u_2, u_3, u_4\}]$ has to be a red K_4, and ${T_{8,4}}^{(3)} \not\subseteq [V]_r$ forces $\{u_5, u_6, u_7\} \subseteq N_g(W)$. But then the eight vertices in W have four common green neighbors, a contradiction to Claim 6.

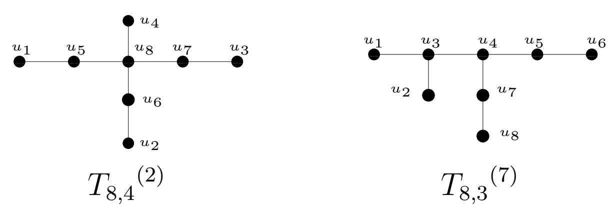

Figure 3. Two trees T_8 with vertex labeling.

• $T_8 = {T_{8,4}}^{(4)}$. From above we already know that ${T_{8,4}}^{(2)} \subseteq [V]_r$. Consider a red ${T_{8,4}}^{(2)}$ in χ where the vertices are denoted as in Figure 3. Let $W = V \setminus \{u_1, \ldots, u_8\}$. If ${T_{8,4}}^{(4)} \not\subseteq [V]_r$, then $\{u_5, u_6, u_7\} \subseteq N_g(W)$ and $q_r(u_4, W) \leq 1$. Thus, we find six vertices in W with four common green neighbors in U, a contradiction to Claim 6.

• $T_8 \in \{{T_{8,4}}^{(5)}, {T_{8,4}}^{(6)}, {T_{8,4}}^{(7)}\}$. Consider a red ${T_{7,4}}^{(1)}$ in χ. If ${T_{8,4}}^{(5)} \not\subseteq [V]_r$, then $q_r(\{u_3, u_4\}, W) = 0$, and $B_4 \not\subseteq [V]_g$ forces u_3u_4 to be red. Consequently, $q_r(u_6, W) = 0$ and $q_r(u_1, W) \leq 2$. But then we find six vertices in W with four common green neighbors in U, a contradiction to Claim 6. If ${T_{8,4}}^{(6)} \not\subseteq [V]_r$, then $q_r(u_i, W) \leq 1$ for $i = 3, 4$ and $q_r(u_i, W) \leq 2$ for $i = 1, 2, 5$. Since $B_4 \not\subseteq [V]_g$, $[\{u_1, u_2, u_3, u_4, u_5\}]$ has to be a red K_5. Moreover, ${T_{8,4}}^{(6)} \not\subseteq [V]_r$ forces $q_r(\{u_6, u_7\}, W) = 0$, and there are six vertices in W with four common green neighbors in U contradicting Claim 6. Finally, if ${T_{8,4}}^{(7)} \not\subseteq [V]_r$, then $q_r(u_6, W) = 0$ and $q_r(u_i, W) \leq 2$ for $i = 1, 2, 5$. Hence $B_4 \not\subseteq [V]_g$ forces $[\{u_1, u_2, u_5, u_6, u_7\}]$ to be a red K_5. Moreover, ${T_{8,4}}^{(7)} \not\subseteq [V]_r$ implies $q_r(u_i, W) = 0$ for $i = 1, 2, 5, 6, 7$ and we find eight vertices in W with five common green neighbors in U, another contradiction to Claim 6.

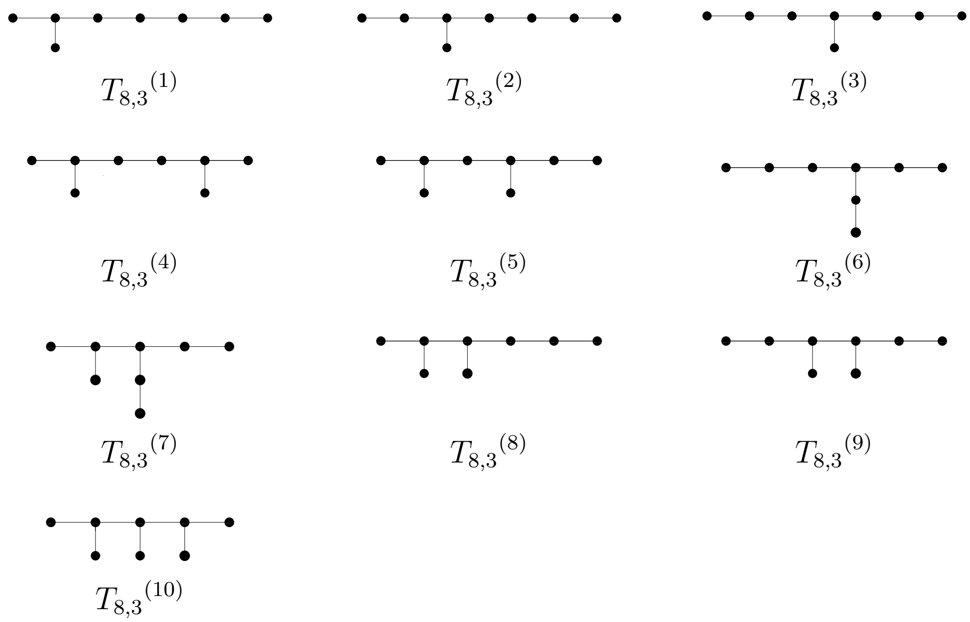

Figure 4. The trees $T_{8,3}$.

Case 4: $\Delta(T_n) = n - 5$ where $n = 8$. The ten trees $T_{8,3}$ are shown in Figure 4. We may use that P_8 and $T_{7,3}{}^{(i)}$ for $i \in \{3, 4, 5\}$ occur in $[V]_r$ (see (5) and Case 3). If a red $T_{7,3}{}^{(i)}$ in χ with $U = V(T_{7,3}{}^{(i)})$ is considered, then the vertices in U shall be denoted as in Figure 5 and W means $V \setminus U$.

• $T_8 \in \{T_{8,3}{}^{(1)}, T_{8,3}{}^{(2)}, T_{8,3}{}^{(3)}\}$. Consider a red $P_8 = u_1u_2 \dots u_8$ in χ. Let $W = V \setminus \{u_1, \dots, u_8\}$. If $T_{8,3}{}^{(1)} \not\subseteq [V]_r$, then $\{u_2, u_3, u_6, u_7\} \subseteq N_g(W)$, and $T_{8,3}{}^{(2)} \not\subseteq [V]_r$ forces that $\{u_3, u_4, u_5, u_6\} \subseteq N_g(W)$, both cases contradicting Claim 6. If $T_{8,3}{}^{(3)} \not\subseteq [V]_r$, then u_3u_5, u_4u_6 and all edges between W and $\{u_4, u_5\}$ are green. Hence, by Claim 6, $q_r(\{u_1, u_8\}, W) \geq 1$, and we may assume that u_1w^* for some $w^* \in W$ is red. But this forces all edges from u_3 to $W \setminus \{w^*\}$to be green yielding a green B_4, a contradiction.

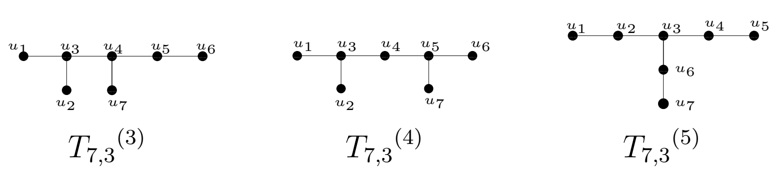

Figure 5. Some trees $T_{7,3}$ with vertex labeling.

• $T_8 \in \{T_{8,3}{}^{(4)}, T_{8,3}{}^{(5)}\}$. Consider a red $T_{7,3}{}^{(4)}$ in χ. If $T_{8,3}{}^{(4)} \not\subseteq [V]_r$, then $q_r(u_i, W) \leq 1$ for $i = 1, 2, 6, 7$, and $B_4 \not\subseteq [V]_g$ implies that $[\{u_1, u_2, u_6, u_7\}]$ is a red K_4. By Claim 6, $q_r(\{u_1, u_2, u_6, u_7\}, W) \geq 1$, and we may assume that u_1w^* for some $w^* \in W$ is red. But this yields a red $T_{8,3}{}^{(4)}$, a contradiction. If $T_{8,3}{}^{(5)} \not\subseteq [V]_r$, then $\{u_1, u_2, u_6, u_7\} \subseteq N_g(W)$ contradicting Claim 6.

• $T_8 = T_{8,3}{}^{(6)}$. Consider a red $T_{7,3}{}^{(5)}$ in χ. If $T_{8,3}{}^{(6)} \not\subseteq [V]_r$, then $\{u_1, u_5, u_7\} \subseteq N_g(W)$. Since $B_4 \not\subseteq [V]_g$, $[\{u_1, u_5, u_7\}]$ is a red K_3, and Claim 6 forces $q_r(u_2, W) \geq 1$. But then we find a red

${T_{8,3}}^{(6)}$ with u_5 as vertex of degree 3, a contradiction.

• $T_8 = {T_{8,3}}^{(7)}$. Consider a red ${T_{7,3}}^{(3)}$ in χ. If ${T_{8,3}}^{(7)} \nsubseteq [V]_r$, then $q_r(u_7, W) = 0$. Moreover, $B_4 \nsubseteq [V]_g$ implies that $q_r(w, W) \geq 4$ for every $w \in W$. Hence, as ${T_{8,3}}^{(7)} \nsubseteq [V]_r$, $q_r(u_4, W) = 0$ and $q_r(u_3, W) \leq 1$. But then Claim 6 forces $q_r(u_i, W) \geq 2$ for $i = 1, 2$, and this yields a red ${T_{8,3}}^{(7)}$, a contradiction.

• $T_8 \in \{{T_{8,3}}^{(8)}, {T_{8,3}}^{(9)}, {T_{8,3}}^{(10)}\}$. From above we already know that ${T_{8,3}}^{(7)} \subseteq [V]_r$. Consider a red ${T_{8,3}}^{(7)}$ in χ where the vertices are denoted as in Figure 3. Let $W = V \setminus \{u_1, \ldots, u_8\}$. If ${T_{8,3}}^{(8)} \nsubseteq [V]_r$, then u_6u_8 and all edges between $\{u_6, u_8\}$ and W have to be green. But this yields a green B_4, a contradiction. If ${T_{8,3}}^{(9)} \nsubseteq [V]_r$, then $q_r(\{u_1, u_2\}, W) = 0$, and $B_4 \nsubseteq [V]_g$ implies that u_1u_2 is red. Hence $q_r(u_3, W) = 0$, and, by Claim 6, a red edge u_4w^* with $w^* \in W$ must occur. Moreover, $B_4 \nsubseteq [V]_g$ implies $q_r(w, W) \geq 3$ for every $w \in W$. But then we find a red ${T_{8,3}}^{(9)}$ in $[\{u_4, u_5, u_6, u_7, w^*, w_1, w_2, w_3\}]$ where w_1 and w_2 are red neighbors of w^* in W and w_3 is a red neighbor of w_2 in W different from w^* and w_1, a contradiction. Finally, if ${T_{8,3}}^{(10)} \nsubseteq [V]_r$, then $q_r(\{u_5, u_7\}, W) = 0$. Hence $B_4 \nsubseteq [V]_g$ implies that u_5u_7 is red. Consequently, $q_r(u_4, W) = 0$, and, by Claim 6, $q_r(u_1, W) \geq 2$. But this yields a red ${T_{8,3}}^{(10)}$ in $[\{u_1, u_2, u_3, u_4, u_5, u_7, w_1, w_2\}]$ where w_1 and w_2 are red neighbors of u_1 in W, a contradiction, and the proof of Theorem 2 is complete. □

References

[1] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs III: Small off-diagonal numbers, *Pacific J. Math.* **4**1 (1972), 335–345.

[2] M. Clancy, Some small Ramsey numbers, *J. Graph Theory* **1** (1977), 89–91.

[3] P. Erdős, R.J. Faudree, C.C. Rousseau and R.H. Schelp, The book-tree Ramsey numbers, *Sci. Ser. A Math. Sci. (N.S)* **1** (1988), 111–117.

[4] G. Hua, S. Hongxue and L. Xiangyang, Ramsey numbers $r(K_{1,4}, G)$ for all three-partite graphs G of order six, *J. Southeast Univ. (English Ed.)* **2**0 (2004), 378–380.

[5] G.R.T. Hendry, Ramsey numbers for graphs with five vertices, *J. Graph Theory* **1**3 (1989), 245–248.

[6] R. Lortz and I. Mengersen, On the Ramsey numbers for stars versus connected graphs of order six, *Australas. J. Combin.* **7**3 (2019), 1–24.

[7] R. Lortz and I. Mengersen, On the Ramsey numbers for non-star trees versus connected graphs of order six, *Discuss. Math. Graph Theory*, to appear.

[8] C.C. Rousseau and J. Sheehan, A class of Ramsey problems involving trees, *J. London Math. Soc.*(2) **1**8 (1978), 392–396.

[9] L. Zhang, K. Chen and D. Zhu, Some tree-book Ramsey numbers, *Ars Combin.* **1**30 (2017), 97–102.