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Multicolor star-critical Ramsey numbers and Ramsey-good graphs

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Abstract

This paper seeks to develop the multicolor version of star-critical Ramsey numbers, which serve as a measure of the strength of the corresponding Ramsey numbers. We offer several general theorems, some of which focus on Ramsey-good cases (i.e., cases in which the corresponding Ramsey number is equal to a general lower bound). We also prove some specific cases for small graphs, and conclude with a table of known multicolor star-critical Ramsey numbers.

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1. Introduction

First defined by Hook in [27], star-critical Ramsey numbers seek to measure the strength of the Ramsey number for a given pair of graphs. We focus our investigation on the multicolor analogue of star-critical Ramsey numbers. As is standard, K_n will denote a complete graph of order n and $K_{1,n}$ will denote a star of order $n+1$ (containing exactly n vertices of degree 1 and one vertex of degree n). The Ramsey number $r = r(G_1, G_2, \ldots, G_t)$ is defined to be the least natural number r such that every t-coloring of the edges of K_r contains a monochromatic copy of G_i in color i, for some $1 \leq i \leq t$. The star-critical Ramsey number $r_*(G_1, G_2, \ldots, G_t)$ is defined to be the least

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natural number k such that every t-coloring of the edges of $K_{r-1} \sqcup K_{1,k}$ contains a monochromatic G_i in some color i. Here, $K_{r-1} \sqcup K_{1,k}$ is the graph formed by taking the disjoint union of K_{r-1} with a vertex v, then adding edges between v and exactly k of the vertices in the K_{r-1}. Star-critical Ramsey numbers in the case where $t = 2$ have been extensively studied (e.g., see [24], [27], [28], [29], [33], [34], [36], [42], and [44]).

While star-critical Ramsey numbers determine the minimum number of edges that must be introduced between a vertex and K_{r-1} to establish the Ramsey property, the deleted edge number was introduced in [5] to determine how many edges of a star $K_{1,k}$ must be removed from K_r in order to destroy the Ramsey property. To be precise, first define the k-deleted Ramsey number $D_k(G_1, G_2, \ldots, G_t)$ to be the least natural number p such that every t-coloring of the edges of $K_p - E(K_{1,k})$ contains a monochromatic copy of G_i in some color i. It is easily confirmed that

$$D_0(G_1, G_2, \ldots, G_t) = r(G_1, G_2, \ldots, G_t)$$

and

$$r(G_1, G_2, \ldots, G_t) \leq D_k(G_1, G_2, \ldots, G_t) \leq r(G_1, G_2, \ldots, G_t) + 1,$$

for all k. The deleted edge number $de(G_1, G_2, \ldots, G_t)$ is then defined to be the unique value of k such that

$$D_{k-1}(G_1, G_2, \ldots, G_t) < D_k(G_1, G_2, \ldots, G_t).$$

It follows from these definitions that

$$r_*(G_1, G_2, \ldots, G_t) + de(G_1, G_2, \ldots, G_t) = r(G_1, G_2, \ldots, G_t). \tag{1}$$

When $G_1 = G_2 = \cdots = G_t$, we denote the t-colored Ramsey number by $r^t(G_1)$, the t-colored deleted edge number by $de^t(G_1)$, and the t-colored star-critical Ramsey number by $r_*^t(G_1)$. We extend the definition of the Ramsey number to the 1-color case (and leave off the superscript) by setting $r(G) = |V(G)|$. This extension follows from the observation that G is a subgraph of $K_{|V(G)|}$ and $|V(G)|$ vertices are needed to have a monochromatic G. When G is assumed to be connected, the 1-color deleted edge number can then be defined by $de(G) = |V(G)| - \delta(G)$, where $\delta(G)$ is the minimum degree among the vertices of G. It follows from Equation (1) that $r_*(G) = \delta(G)$.

A well-known lower bound for the Ramsey number $r(G_1, G_2)$ was proved by Burr [7]:

$$r(G_1, G_2) \geq (c(G_1) - 1)(\chi(G_2) - 1) + s(G_2) \quad \text{whenever } c(G_1) \geq s(G_2),$$

where $c(G_1)$ is the order of a maximal connected component in G_1, $\chi(G_2)$ is the chromatic number for G_2, and $s(G_2)$ is the chromatic surplus of G_2 (the least cardinality of a color class among all vertex colorings of G_2 using $\chi(G_2)$ colors). When equality holds, one says that G_1 is G_2-good. This concept was introduced by Burr and Erdős in [8, 9] and was investigated from the perspective of star-critical Ramsey numbers by Zhang, Broersma, and Chen in [44].

In the multicolor setting, assuming that $r(G_1, G_2, \ldots, G_{t-1}) \geq s(G_t)$, it can be shown that

$$r(G_1, G_2, \ldots, G_t) \geq (r(G_1, G_2, \ldots, G_{t-1}) - 1)(\chi(G_t) - 1) + s(G_t).$$

The proof of this inequality follows that of the analogous statement for Gallai-Ramsey hypergraph numbers proved in Theorem 5 of [6]. When equality holds, we say that the multiset

$\mathcal{H} = \{G_1, G_2, \ldots, G_{t-1}\}$ is G_t-good. Section 2 focuses on general multicolor star-critical Ramsey number theorems. In Theorem 2.1, we prove an inequality involving multicolor deleted edge numbers that depends on a multiset being G_t-good. Theorem 2.2 gives bounds for certain star-critical Ramsey numbers involving multiple complete graphs. Theorem 2.3 generalizes the tree-complete graph star-critical Ramsey number evaluated in [29] to multiple complete graphs, and Theorem 2.4 considers certain star-critical Ramsey numbers for multiple stars. While we find it easier to prove (and state) many of these results as deleted edge number results, we provide their analogues in terms of star-critical Ramsey numbers for the sake of being comprehensive.

In Section 3, we prove several explicit values of multicolor star-critical Ramsey numbers for small graphs, including several new 2-color cases. We conclude with a table of known multicolor star-critical Ramsey numbers.

2. General Multicolor Results

The following theorem can be viewed as a generalization of a weakened version of Theorem 3 in [44].

Theorem 2.1. *Suppose that $G_1, G_2, \ldots, G_t$ are connected graphs such that $\mathcal{H} = \{G_1, G_2, \ldots, G_{t-1}\}$ is a G_t-good multiset satisfying $r(G_1, G_2, \ldots, G_{t-1}) \geq s(G_t)$. Then*

$$de(G_1, G_2, \ldots, G_t) \leq de(G_1, G_2, \ldots, G_{t-1}).$$

Proof. Let $m = r(G_1, G_2, \ldots, G_{t-1})$. We will construct a t-coloring of

$$K_{(m-1)(\chi(G_t)-1)+s(G_t)} - E(K_{1,de(G_1,G_2,\ldots,G_{t-1})})$$

that lacks a monochromatic copy of G_i in color i, for all $1 \leq i \leq t$. Start with a $(t-1)$-coloring of $K_m - E(K_{1,de(G_1,G_2,\ldots,G_{t-1})})$ that lacks a monochromatic copy of G_i in color i for all $1 \leq i \leq t-1$, and call this graph A_1. Let a be the center vertex for the missing star. Let $A_2, A_3, \ldots, A_{\chi(G_t)-1}$ be copies of $A_1 - \{a\}$. Finally, let $A_{\chi(G_t)}$ be formed by taking another copy of $A_1 - \{a\}$ and removing $m - s(G_t)$ vertices. Hence, $A_{\chi(G_t)}$ is a $(t-1)$-colored $K_{s(G_t)-1}$ that lacks a monochromatic copy of G_i in color i for all $1 \leq i \leq t-1$. Form the union

$$\bigcup_{1 \leq j \leq \chi(G_t)} A_j,$$

and interconnect the A_j with edges in color t. By construction, the resulting

$$K_{(m-1)(\chi(G_t)-1)+s(G_t)} - E(K_{1,de(G_1,G_2,\ldots,G_{t-1})})$$

lacks a monochromatic copy of G_i in color i for all $1 \leq i \leq t-1$. To see that it also lacks a monochromatic copy of G_t in color t, we consider two cases. First, if $s(G_t) = 1$, then any subgraph in color t can be vertex colored using $\chi(G_t) - 1$ colors (by assigning colors according to the vertex sets of A_j). Hence, G_t cannot be such a subgraph. Second, if $s(G_t) > 1$, then coloring any t-colored subgraph using $\chi(G_t)$ colors results in a chromatic surplus of at most $s(G_t) - 1$. Once again, we find that G_t cannot be such a subgraph. Thus, the theorem follows. □

When the hypotheses of Theorem 2.1 are met, it follows from Equation (1) that

$$r_*(G_1, G_2, \ldots, G_t) - r(G_1, G_2, \ldots, G_t) \geq r_*(G_1, G_2, \ldots, G_{t-1}) - r(G_1, G_2, \ldots, G_{t-1}).$$

In the case where $t = 2$, the implication becomes

$$de(G_1, G_2) \leq |V(G_1)| - \delta(G_1)$$

(equivalently, $r_*(G_1, G_2) \geq r(G_1, G_2) - |V(G_1)| + \delta(G_1)$).

The fact that $de(K_{n_1}, K_{n_2}) = 1$ was first proved by Erdős and Faudree [18], and it was observed by Cowen [17] that this fact easily extends to the more general multicolor result

$$de(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = 1.$$

Suppose that $G_1, G_2, \ldots, G_s$ are connected graphs such that $r = r(G_1, G_2, \ldots, G_s)$ and $r' = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$. If

$$r(G_1, G_2, \ldots, G_s, K_\ell) = (r-1)(\ell-1)+1$$

for all $\ell \geq 2$, then Omidi and Raeisi (see Theorem 2.1 of [38]) proved that

$$r(G_1, G_2, \ldots, G_s, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = (r-1)(r'-1)+1. \tag{2}$$

This result motivates the following theorem.

Theorem 2.2. *Let $G_1, G_2, \ldots, G_s$ be connected graphs, $r = r(G_1, G_2, \ldots, G_s)$, and $r' = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$. If $r(G_1, G_2, \ldots, G_s, K_\ell) = (r-1)(\ell-1)+1$ for all $\ell \geq 2$, then*

$$de(G_1, G_2, \ldots, G_s, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq de(G_1, G_2, \ldots, G_s).$$

Proof. Consider a t-coloring of $K_{r'} - e$ that lacks a monochromatic copy of K_{n_j} in color $s+j$ for all $1 \leq j \leq t$. The existence of such a coloring follows from

$$de(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = 1.$$

Let a and b be the vertices of the missing edge. Replace each vertex other than a with a copy of K_{r-1} that uses the first s colors and lacks a monochromatic copy of G_i in color i, for all $1 \leq i \leq s$. Edges interconnecting the copies of K_{r-1} with each other and with a are colored according to the edges between the vertices that were replaced. The missing edge between a and b is now a missing $E(K_{1,r-1})$. None of these edges can be given colors $s+1, s+2, \ldots, s+t$ without producing a copy of K_{n_j} in color $s+j$ for some $1 \leq j \leq t$. It is possible to color such edges using colors $1, 2, \ldots, s$, with only $de(G_1, G_2, \ldots, G_s)$ edges having to remain missing. Thus, we have formed an $(s+t)$-colored

$$K_{(r-1)(r'-1)+1} - E(K_{1,de(G_1,G_2,\ldots,G_s)})$$

that lacks G_i in color i for all $1 \leq i \leq s$ and K_{n_j} in color $s+j$, for all $1 \leq j \leq t$. It follows that

$$de(G_1, G_2, \ldots, G_s, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq de(G_1, G_2, \ldots, G_s),$$

completing the proof. □

Assuming the hypotheses stated in Theorem 2.2 and applying Equations (1) and (2), the above implication can be restated as

$$r_*(G_1, G_2, \ldots, G_s, K_{n_1}, K_{n_2}, \ldots K_{n_t}) \geq r_*(G_1, G_2, \ldots, G_s) + (r-1)(r'-2).$$

Next, we consider the case of a tree versus complete graphs. In [29], it was shown that when T_m is a tree of order m,

$$r_*(T_m, K_n) = (m-1)(n-2) + 1.$$

The equivalent result for deleted edge numbers was considered in Theorem 2.1 of [5], but the proof given there contains a mistake in the inductive step. At the present time, the only correct proof that we know of is the proof of Theorem 2.5 in Hook and Isaak's paper [29]. In the next theorem, we extend this result to the multicolor case involving a single tree and multiple complete graphs.

Theorem 2.3. *Let $n_i \geq 3$ for all $1 \leq i \leq t$, where $t \geq 1$. Assume that T_m is a tree of order $m \geq 2$ that satisfies $m \leq r(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$. Then*

$$de(T_m, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = m - 1.$$

Proof. Since $de(T_m) = m - 1$ and $r(T_m, K_n) = (m-1)(n-1) + 1$ (see [15]), Theorem 2.2 implies that

$$de(T_m, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq m - 1.$$

To prove the opposite inequality, let $r = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$ and consider a $(t+1)$-coloring of the edges in

$$K_{(m-1)(r-1)+1} - E(K_{1,m-2}).$$

Viewing colors $2, 3, \ldots, t+1$ as a single color and using Hook and Isaak's 2-color result [29], we find that the resulting coloring contains a copy of T_m in the first color or a copy of K_r in the second color. In the first case, we are done, so assume that there is a copy of K_r spanned by edges in colors $2, 3, \ldots, t+1$. Since $r = r(K_{n_1}, K_{n_2}, \ldots, K_{n_t})$, there is a copy of K_{n_i} in color i, for some $2 \leq i \leq t+1$. It follows that

$$de(T_m, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \geq m - 1,$$

completing the proof of the theorem. □

Combining Equations (1) and (2) with Theorem 2.3, it follows that

$$r_*(T_m, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = (m-1)(r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 2) + 1.$$

Now we consider the case of multiple stars. Let $S = \{m_1, m_2, \ldots, m_t\}$, where each $m_i \geq 2$. Define $N = \sum_{1 \leq i \leq t} m_i$ and denote by k the number of elements in S that are even. Burr and Roberts [10] proved

$$r(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = \begin{cases} N - t + 1, & \text{if } k \geq 2 \text{ is even,} \\ N - t + 2, & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 2.4. *If $k = 0$ or k is odd, then*

$$r_*(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = 1.$$

Proof. In the case where $k = 0$ or k is odd,

$$r(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = N - t + 2.$$

Consider a t-colored K_{N-t+2} and observe that removing $N - t$ edges incident with a fixed vertex still leaves one vertex having degree $N - t + 1$. By the pigeonhole principle, this vertex must be incident with at least m_i edges in color i for some i. Hence, all $N - t + 1$ edges incident with a fixed vertex must be removed in order to destroy the Ramey property. It follows that

$$de(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = N - t + 1,$$

and applying (1) and (3) completes the proof. □

In 1974, Harary and Prins [26] defined the Ramsey multiplicity $R(G_1, G_2, \ldots, G_t)$ to be the smallest possible total number of G_1 in color 1, G_2 in color 2, …, G_t in color t in any t-coloring of K_r, where $r = r(G_1, G_2, \ldots, G_t)$. In the case where $G_1, G_2, \ldots, G_t$ are all stars, Jacobson [31] proved that when all $m_i \geq 2$ are integers,

$$R(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = \begin{cases} k/2, & \text{if } k \geq 2 \text{ is even,} \\ N - t + 2, & \text{otherwise.} \end{cases}$$

In particular, when $k = 2$, we find that there exists a t-coloring of K_r that contains a single monochromatic K_{1,m_i} in some color i, and which does not contain a K_{1,m_i+1} in color i. It follows that a single edge in color i can be removed to produce a $K_r - e$ that lacks a monochromatic K_{1,m_i} in color i, for all $1 \leq i \leq t$. Hence, we obtain the following theorem.

Theorem 2.5. *If $m_1, m_2, \ldots, m_t$ are integers greater than 1, exactly two of which are even, then*

$$r_*(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = N - t.$$

In general, we conjecture the following evaluation of $r_*(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t})$.

Conjecture 1. *If $m_1, m_2, \ldots, m_t$ are integers greater than 1, exactly k of which are even, then*

$$r_*(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_t}) = \begin{cases} N - t, & \textit{if } k \geq 2 \textit{ is even,} \\ 1, & \textit{otherwise.} \end{cases}$$

When trying to prove the case where $k > 2$ is even, we must construct a t-coloring of $K_{N-t+1} - e$ that lacks a copy of K_{1,m_i} in color i for all $1 \leq i \leq t$. We can use Theorem 9.1 of Harary's book [25], which states that K_{2n} contains a 1-factorization. Start with K_{N-t}, where $N - t$ is necessarily even. We can color 1-factors, but must switch the colors of some of the edges, as was done in Theorem 3.1 of [5]. This is not very simple to handle in general.

3. Some Small Multicolor Star-Critical Ramsey Numbers

In this section, we give some specific multicolor star-critical Ramsey numbers, including a few new 2-color numbers, when the graphs being considered are small. We denote by $K_n - e$ the graph formed by removing a single edge from K_n and we denote by P_n a path on n vertices. The graph $K_{1,3} + e$ is formed by adding a single edge connecting two of the leaves in $K_{1,3}$.

Theorem 3.1. $r_*(K_4 - e, K_3) = 5$.

Proof. Since $r(K_4 - e, K_3) = 7$ (see [16]), it follows that $K_4 - e$ is K_3-good. Theorem 2.1 then implies $de(K_4 - e, K_3) \leq 2$ (also, see Figure 1). To obtain the inequality $de(K_4 - e, K_3) \geq 2$, we

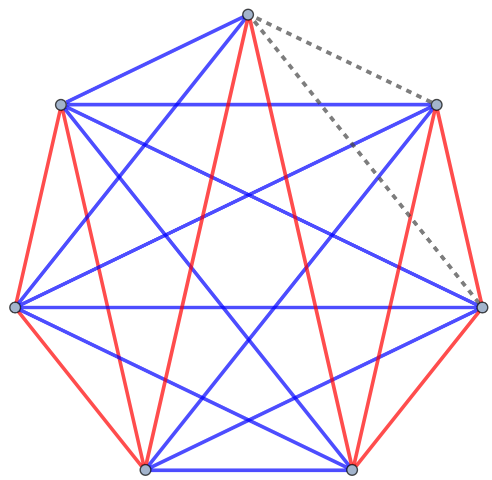

Figure 1. A 2-coloring of $K_7 - E(K_{1,2})$ that lacks a red $K_4 - e$ and a blue K_3 (and hence, a blue $K_{1,3} + e$).

must show that every red/blue coloring of $K_7 - e$ contains a red $K_4 - e$ or a blue K_3. Consider an arbitrary 2-coloring of $K_7 - e$ and let a and b be the vertices of the missing edge. If we remove vertex b, we have a 2-coloring of K_6, which must contain a red K_3 or a blue K_3 since $r(K_3, K_3) = 6$ (see [21]). In the latter case, we are done, so suppose there is a red K_3. We must now consider two cases, based on whether or not a is one of the vertices in the red K_3.

Case 1. Suppose that a is not in the red K_3. Label the vertices in the red K_3 by x, y, z and the other vertices a, b, c, d. If any of a, b, c, d is adjacent via 2 or more red edges to x, y, z, then a red $K_4 - e$ is formed. Otherwise, each of a, b, c, d is adjacent via at least 2 blue edges to $\{x, y, z\}$. If the subgraph induced by $\{a, b, c, d\}$ does not contain a red $K_4 - e$, then at least one edge is blue. Without loss of generality, suppose that ac is blue. Then a and c are each adjacent via at least two blue edges to $\{x, y, z\}$. By the pigeonhole principle, there is a vertex, say x, in which ax and cx are both blue, forcing $\{a, c, x\}$ to form a blue K_3.

Case 2. Suppose that the red K_3 consists of vertices a, x, y and the other vertices are labelled b, c, d, e. If any edge in the subgraph induced by $\{c, d, e\}$ is blue, then we can use an argument similar to the previous case to force the existence of a blue K_3. So, suppose this subgraph is a red K_3, then at least two of the edges bc, bd, and be must be blue (otherwise the subgraph induced by $\{b, c, d, e\}$ is a red $K_4 - e$). Without loss of generality, suppose that bc and bd are blue. If b is

adjacent via red edges to both of x and y, then a red $K_4 - e$ is formed. So, assume one such edge, say bx is blue. If either cx or dx is blue, when including b, we obtain a blue K_3. If they are both red, then the subgraph induced by $\{x, c, d, e\}$ contains a red $K_4 - e$.
In both cases, we find that $K_7 - e$ contains a red $K_4 - e$ or a blue K_3, completing the proof. □

In the case of the pair of graphs, $K_4 - e$ and $K_{1,3} + e$, an interesting phenomenon occurs. Namely, $r(K_4 - e, K_{1,3} + e) = 7$ [16], and it is easily confirmed that $K_4 - e$ is $(K_{1,3} + e)$-good and $K_{1,3} + e$ is $(K_4 - e)$-good. By Theorem 2.1, it follows that

$$de(K_4 - e, K_{1,3} + e) \leq \min\{4 - \delta(K_4 - e), 4 - \delta(K_{1,3} + e)\} = 2.$$

The following theorem shows that this bound is tight.

Theorem 3.2. $r_*(K_4 - e, K_{1,3} + e) = 5.$

Proof. As mentioned above, Theorem 2.1 implies that $de(K_4 - e, K_{1,3} + e) \leq 2$ (also, see Figure 1). Proving the opposite inequality requires showing that every red/blue coloring of $K_7 - e$ contains a red $K_4 - e$ or a blue $K_{1,3} + e$. Consider such a coloring and observe that Theorem 3.1 implies that there is a red $K_4 - e$ or a blue K_3. In the first case, we are done, so assume the latter condition. We obtain two cases, based on whether or not the missing edge is incident with a vertex in the blue K_3. Let a and b be the vertices of the missing edge.
Case 1. Suppose that neither a nor b are contained in the blue K_3. Label the vertices in the blue K_3 by x, y, z and the other vertices by a, b, c, d. If any edge connecting $\{x, y, z\}$ to $\{a, b, c, d\}$ is blue, then a blue $K_{1,3} + e$ is formed. So, assume that all such edges are red. Other than the missing edge, if any edge in the subgraph induced by $\{a, b, c, d\}$ is red, then a red $K_4 - e$ is formed. All such edges must then be blue, forcing a blue $K_{1,3} + e$ as a subgraph.
Case 2. Without loss of generality, suppose that the vertices of the blue K_3 are given by a, x, y and the other vertices are given by b, c, d, e. Similar to the previous case, with the exception of the missing edge, if any edge joining $\{a, x, y\}$ to $\{b, c, d, e\}$ is blue, then a blue $K_{1,3} + e$ is formed. Assume that all such edges are red. Avoiding a red $K_4 - e$ forces the subgraph induced by $\{b, c, d, e\}$ to contain a blue $K_{1,3} + e$.
In both cases, the $K_7 - e$ contains a red $K_4 - e$ or a blue $K_{1,3} + e$, completing the proof of the theorem. □

Theorem 3.3. $r_*(P_4, K_4 - e) = 4.$

Proof. Since $r(P_4, K_4 - e) = 7$ (see [16]), it follows that P_4 is $(K_4 - e)$-good. So, Theorem 2.1 implies that

$$de(P_4, K_4 - e) \leq 3$$

(also, see Figure 2). To prove the other direction, consider a red/blue $K_7 - E(K_{1,2})$. Let vertex a be the center of the missing star, and let vertices b and g be its leaves. Removing vertices a and g results in a red/blue coloring of K_5. Since $r(P_3, K_4 - e) = 5$, this coloring contains a red P_3 or a blue $K_4 - e$. In the latter case, we are done, so assume that there is a red P_3. The location of this P_3 in the original coloring relative to the missing $K_{1,2}$ produces three cases, as illustrated in Figure 3.

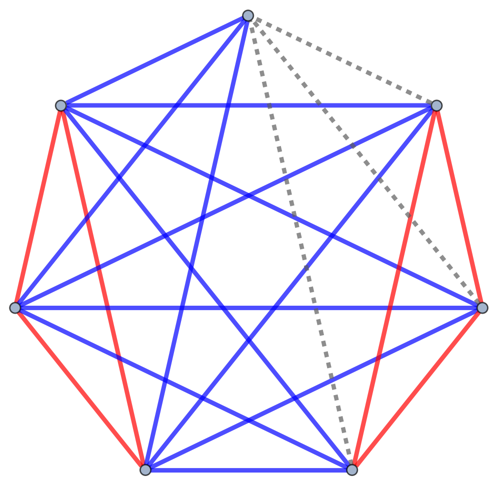

Figure 2. A 2-coloring of $K_7 - E(K_{1,3})$ that lacks a red P_4, and a blue K_3 (and hence, a red $K_{1,3} + e$ and a blue $K_4 - e$).

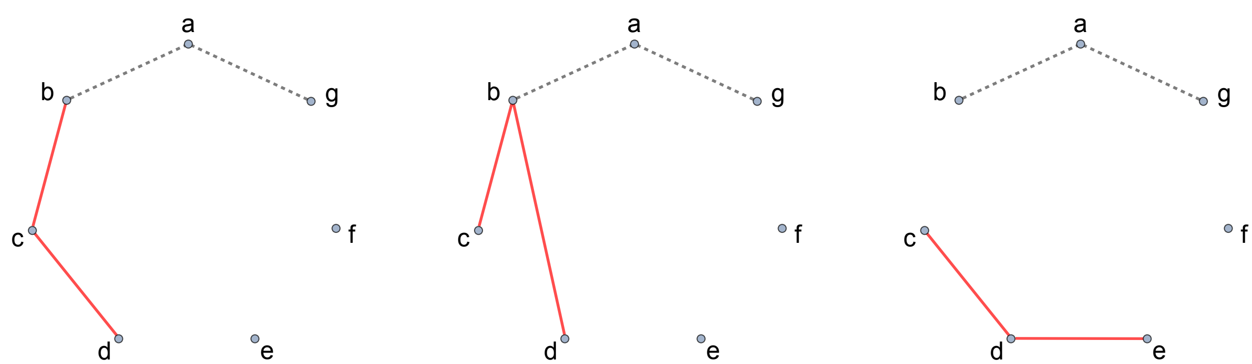

Figure 3. Three cases describing the location of a monochromatic P_3 relative to a missing $K_{1,2}$ in a 2-coloring of $K_7 - E(K_{1,2})$.

Case 1. Consider the case given by the first image in Figure 3. Avoiding a red P_4 forces edges be, bf, bg, de, df, and dg to be blue. If no blue $K_4 - e$ exists, then bd, ef, eg, and fg must be red. If any one of ac, ad, af, or cf are red, then a red P_4 is formed. Otherwise, all four edges are blue and the subgraph induced by $\{a, c, d, f\}$ contains a blue $K_4 - e$.
Case 2. Consider the case given by the second image in Figure 3. Avoiding a red P_4 forces edges ce, cf, de, and df to be blue. If no blue $K_4 - e$ exists, then edge cd must be red. At this point, we have reduced this case to the situation that occurs in Case 1.
Case 3. Consider the case given by the third image in Figure 3. Avoiding a red P_4 forces edges ac, ae, bc, be, cf, cg, ef, and eg to be blue. If no blue $K_4 - e$ exists, then af, ce, and fg must be red. If bg is red, then a red P_4 is formed. Otherwise, the subgraph induced by $\{b, c, e, g\}$ contains a blue $K_4 - e$.
In all three cases, we find that there is either a red P_4 or a blue $K_4 - e$. □

Theorem 3.4. $r_*(K_{1,3} + e, K_3) = 4$.

Proof. In [16], it was shown that $r(K_{1,3} + e, K_3) = 7$, from which it follows that $K_{1,3} + e$ is K_3-good. By Theorem 2.1, it follows that

$$de(K_{1,3} + e, K_3) \leq 3$$

(also, see Figure 2). It remains to be shown that every 2-coloring of $K_7 - E(K_{1,2})$ contains a red $K_{1,3} + e$ or a blue K_3. Consider an arbitrary 2-coloring of $K_7 - E(K_{1,2})$ and let a be the vertex that is incident with the 2 missing edges. Removing vertex a produces a 2-coloring of K_6, which necessarily contains a red K_3 or a blue K_3. Assume the former case, and denote the vertices in the red K_3 by b, c, d. Label the remaining three vertices e, f, g. If any edges connecting $\{b, c, d\}$ with $\{a, e, f, g\}$ are red, then a red $K_{1,3} + e$ is formed. So suppose that all such edges are blue, resulting in three cases (see Figure 4).

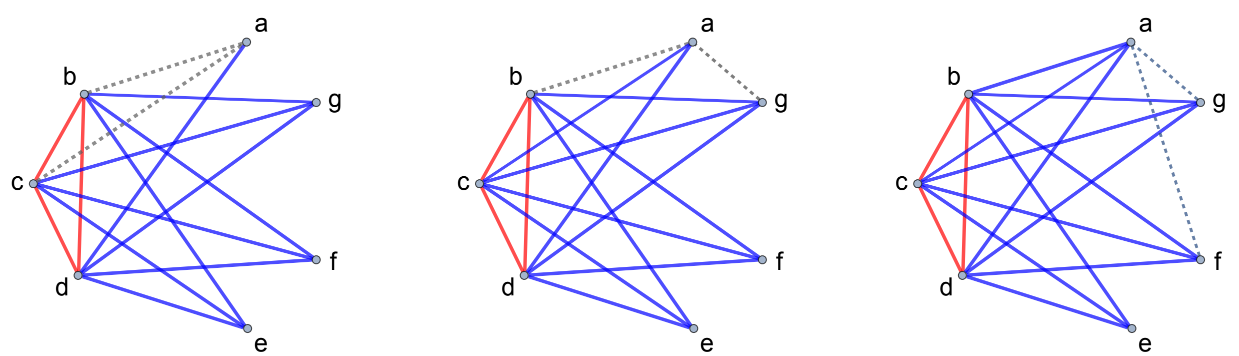

Figure 4. Three cases describing the location of a monochromatic K_3 relative to a missing $K_{1,2}$ in a 2-coloring of $K_7 - E(K_{1,2})$.

Regardless of which case we are in, if any edges in the subgraph induced by $\{a, e, f, g\}$ are blue, then a blue K_3 is formed. The only other possibility is that all such edges (other than those removed) are red. In all three cases, we obtain a red $K_{1,3} + e$ as a subgraph. It follows that

$$de(K_{1,3} + e, K_3) \geq 3,$$

completing the proof. $\square$

Current literature indicates that the star-critical Ramsey number $r_*(K_3 - e, K_n - e)$ is known and can be found in [27]. As this document is not readily available, and because the upper bounds to $de(K_3 - e, K_n - e)$ follow from Theorem 2.1, we offer a complete proof.

Theorem 3.5. *For all $n \geq 4$,*

$$r_*(K_3 - e, K_n - e) = 2n - 1.$$

Proof. It is easily shown that $R(K_3 - e, K_n - e) = 2n - 3$ (see Section 3.1 of [39]), from which we see that $K_3 - e$ is $(K_n - e)$-good. It follows from Theorem 2.1 that

$$de(K_3 - e, K_n - e) \leq 2$$

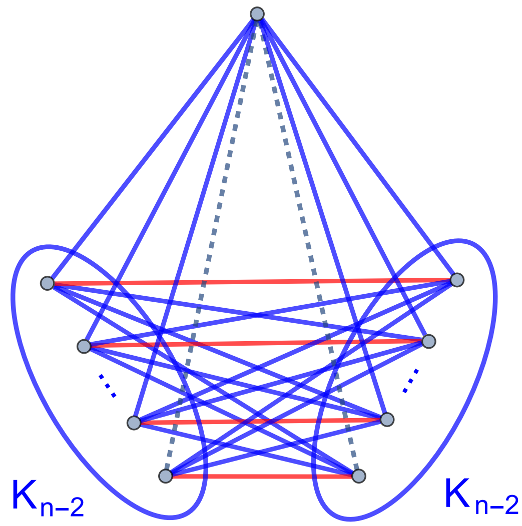

Figure 5. A 2-coloring of $K_{2n-3} - E(K_{1,2})$ that lacks a red $K_3 - e$ and a blue $K_n - e$.

(also, see Figure 5). It remains to be proved that every red/blue coloring of the edges of $K_{2n-3} - e$ contains a red $K_3 - e$ or a blue $K_n - e$. Consider a two- coloring of $K_{2n-3} - e$ and let the missing edge be between vertices a and b. Remove vertex a and assume that the resulting K_{2n-4} lacks a red copy of $K_3 - e$ and a blue copy of $K_n - e$ (otherwise we are done). Notice that the red edges must form a matching M. Let m be the size of this matching. Certainly $m \leq n-2$. A blue copy of $K_n - e$ could only be formed by taking, at most, one vertex from $m-1$ matchings, two vertices from the remaining matching, and all vertices that are not incident with a red edge. In order to avoid this, we need

$$1 + m + (2n - 4 - 2m) < n \quad \Longrightarrow \quad m > n - 3.$$

So, a red/blue coloring of a K_{2n-4} that lacks a red copy of $K_3 - e$ and a blue copy of $K_n - e$ must contain a red matching of size $n-2$. Now consider vertex a. If a is incident with any red edges, a red $K_3 - e$ is formed. If all $2n-5$ edges incident with a are blue, then a must be adjacent to at least one vertex in each matching, labeled $x_1, x_2, ..., x_{n-2}$. This only accounts for $n-2$ edges, so there is certainly a vertex y such that ay is blue and $x_k y$ is red for some $1 \leq k \leq n-2$. Then the subgraph induced by $\{a, y, x_1, x_2, ..., x_{n-2}\}$ forms a blue $K_n - e$. □

Note that $K_3 - e = P_3 = K_{1,2}$.

Theorem 3.6. *If $t \geq 1$ is odd, then*

$$r_*^t(P_3) = 1.$$

Proof. It is known that when t is odd, $r^t(P_3) = t + 2$ (see [30]). Consider a t-coloring of K_{t+1} that lacks a monochromatic P_3. Such a coloring has every vertex incident with exactly one edge in each of the t colors. Adding in an additional vertex and assigning any color to an edge joining this vertex with the K_{t+1} necessarily produces a monochromatic P_3. $\square$

Finally, we conclude this section by considering the star-critical numbers for multiple copies of P_3 versus complete graphs. Using the observation that P_3 is a star, Jacobson proved in [32] that

$$r(\underbrace{P_3, P_3 \ldots, P_3}_{s\ terms}, K_\ell) = (r^s(P_3) - 1)(\ell - 1) + 1$$

for all $\ell \geq 1$. It follows that the multiset consisting of s copies of P_3 is K_ℓ-good. Thus, Theorem 2.2 implies that

$$de(\underbrace{P_3, P_3 \ldots, P_3}_{s\ terms}, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq de^s(P_3).$$

When s is odd, Theorem 3.6 implies that

$$de(\underbrace{P_3, P_3 \ldots, P_3}_{s\ terms}, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) \leq s + 1.$$

When $s = 2$, we have the following theorem.

Theorem 3.7. *For all* $1 \leq i \leq t$ *and* $n_i \geq 1$*,*

$$de(P_3, P_3, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) = 1.$$

Proof. It is easily confirmed that $r(P_3, P_3) = 3$ and $de(P_3, P_3) = 1$, from which Theorem 2.2 implies the statement of the theorem. $\square$

Using Equations (1) and (2), Theorem 3.7 implies that

$$\begin{aligned} r_*(P_3, P_3, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) &= r(P_3, P_3, K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 1 \\ &= (r(P_3, P_3) - 1)(r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 1) \\ &= 2(r(K_{n_1}, K_{n_2}, \ldots, K_{n_t}) - 1) \\ &= 2r_*(K_{n_1}, K_{n_2} \ldots, K_{n_t}). \end{aligned}$$

In particular, $r_*(P_3, P_3, K_\ell) = 2\ell - 2$ for all $\ell \geq 1$.

4. Conclusion

In this section, we compile the known values of $r_*(G_1, G_2, \ldots, G_t)$. Besides the partial results contained in Theorems 2.4 and 2.5, Table 1 is intended to provide the current known multicolor star-critical Ramsey numbers, along with the corresponding Ramsey numbers and relevant citations. The only graph in Table 1 that we have not yet defined is the fan F_m, defined to be the join of K_1 and mK_2, where mK_2 consists of m disjoint copies of K_2.

Table 1. Known star-critical Ramsey numbers, along with their corresponding Ramsey numbers.

$r(G_1, G_2, \ldots, G_t)$	$r_*(G_1, G_2, \ldots, G_t)$
$r(T_m, K_n) = (m-1)(n-1)+1$ [15]	$(m-1)(n-2)+1$ [29]
$r(C_m, K_3) = 2m-1$ for $m > 3$ [11]	$m+1$ [44]
$r(C_4, K_4) = 10$ [43]	9 [34]
$r(C_m, K_4) = 3m-2$ for $m \geq 5$ [43]	$2m$ [34]
$r(C_m, K_5) = 4m-3$ for $m \geq 5$ [2]	$3m-1$ [33]
$r(F_m, K_3) = 4m+1$ for $m \geq 2$ [35]	$2m+2$ [36]
$r(F_m, K_4) = 6m+1$ [41]	$4m+2$ [24]
$r(K_4 - e, K_3) = 7$ [39]	5 (Theorem 3.1)
$r(K_4 - e, K_{1,3} + e) = 7$ [16]	5 (Theorem 3.2)
$r(P_4, K_4 - e) = 7$ [16]	4 (Theorem 3.3)
$r(K_{1,3} + e, K_3) = 7$ [16]	4 (Theorem 3.4)
$r(K_3 - e, K_n - e) = 2n-3$ for $n \geq 4$ [39]	$2n-5$ [27], (Theorem 3.5)
$r^t(P_3) = t+2$ for $t \geq 1$ odd [30]	1 (Theorem 3.6)
$r(P_3, P_3, K_\ell) = 2\ell - 1$ [32]	$2\ell - 2$ (Theorem 3.7)

Much of the recent research on star-critical Ramsey numbers has focused on the evaluation of $r_*(C_m, K_n)$. This is due to the fact that the complete evaluation of $r(C_m, K_n)$ is a well known open problem. The fact that

$$r(C_m, K_n) \geq (m-1)(n-1)+1$$

follows from Lemma 4 of [16]. In 1973, the work of Bondy and Erdős [3] led to the conjecture (see [19]) that

$$r(C_m, K_n) = (m-1)(n-1)+1$$

whenever $m \geq n \geq 3$, except for the case $m = n = 3$. At the present time, this conjecture has been shown to be true when $m \geq 4n+2$ [37], when $m > n = 3$ [11], when $m \geq n = 4$ [43], when $m \geq n = 5$ [2], when $m \geq n = 6$ [40], when $m \geq n = 7$ [13], as well as a few other special cases.

The recent paper by Wang, Li, and Li [42] introduced variations of the concept of a star-critical Ramsey number defined by removing graphs other than just stars from complete graphs. In this sense, their generalizations demonstrate that the deleted edge number is a little more natural than the star-critical Ramsey number. One direction for future inquiry that seems to follow from such generalizations is to define a deleted cycle number, where cycles of various lengths are removed to destroy the Ramsey property.

Finally, we encourage the reader to consider star-critical versions of other Ramsey-type numbers. In particular, Gallai-Ramsey numbers (see [20], [22],and [23]) and bipartite Ramsey numbers (see [1] and [14]) can also be destroyed by the removal of edges incident with a fixed vertex. We reserve such investigations for future research.

References

[1] L. Beineke and A. Schwenk, On a bipartite form of the Ramsey problem, In: Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), Congressus Numerantium, No. XV, *Utilitas Math.* 1976, 17–22.

[2] B. Bollobás, C. Jayawardene, J. Yang, Y. Huang, C. Rousseau, and K. Zhang, On a conjecture involving cycle-complete graph Ramsey numbers, *Australas. J. Combin.* **22** (2000), 63–71. http://ajc.maths.uq.edu.au/pdf/22/ocr-ajc-v22-p63.pdf

[3] J. Bondy and P. Erdős, Ramsey number for cycles in graphs, *J. Combin. Theory Ser. B* **14** (1973) 46–54.

[4] L. Boza, M. Cera, P. García-Vázquez, and M. Revuelta, On the Ramsey numbers for stars versus complete graphs, *Eur. J. Combin.* **31** (2010), 1680–1688.

[5] M. Budden and E. DeJonge, Destroying the Ramsey property by the removal of edges, *Australas. J. Combin.* **74** (2019), 476–485. http://ajc.maths.uq.edu.au/pdf/74/ajc_v74_p476.pdf

[6] M. Budden, J. Hiller, and A. Penland, Constructive methods in Gallai-Ramsey theory for hypergraphs, *Integers* **20A** (2020), #A4. http://math.colgate.edu/%7Eintegers/uproc4/uproc4.pdf

[7] S. Burr, Ramsey numbers involving graphs with long suspended paths, *J. London Math. Soc.* **24(2)** (1981), 405–413.

[8] S. Burr and P. Erdős, Generalized Ramsey numbers involving subdivision graphs, and related problems in graph theory, *Ann. Discrete Math.* **9** (1980), 37–42.

[9] S. Burr and P. Erdős, Generalizations of a Ramsey-theoretic result of Chvátal, *J. Graph Theory* **7** (1983), 39–51.

[10] S. Burr and J. Roberts, On Ramsey numbers for stars, *Utilitas Math.* **4** (1973), 217–220.

[11] G. Chartrand and S. Schuster, On the existence of specified cycles in complementary graphs, *Bull. Amer. Math. Soc.* **77(6)** (1971), 995–998.

[12] G. Chartrand and S. Schuster, On a variation of the Ramsey number, *Trans. Amer. Math. Soc.* **173** (1972), 353–362.

[13] Y. Chen, T. Cheng, and Y. Zhang, The Ramsey numbers $R(C_m, K_7)$ and $R(C_7, K_8)$, *European J. Combin.* **29(5)** (2008), 1337–1352.

[14] M. Christou, C. Iliopoulos, and M. Miller, Bipartite Ramsey numbers involving stars, stripes and trees, *Electron. J. Graph Theory Appl. 1(2)* (2013), 89–99. https://ejgta.org/index.php/ejgta/article/view/26

[15] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** (1977), 93.

[16] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs III. Small off-diagonal numbers, *Pacific J. Math.* **41** (1972), 335–345.

[17] R. Cowen, Deleting edges from Ramsey minimal examples, *Amer. Math. Monthly* **122** (2015), 681–683.

[18] P. Erdős and R. Faudree, Size Ramsey functions, In: "Sets, Graphs and Numbers," *Colloq. Math. Soc. Janos Bolyai* **60** (1992), 219–238.

[19] P. Erdős, R. Faudree, C. Rousseau, and R. Schelp, On cycle-complete graph Ramsey numbers, *J. Graph Theory* **2** (1978), 53–64.

[20] R. Faudree, R. Gould, M. Jacobson, and C. Magnant, Ramsey numbers in rainbow triangle free colorings, *Australas. J. Combin.* **46** (2010), 269–284. `http://ajc.maths.uq.edu.au/pdf/46/ajc_v46_p269.pdf`

[21] R. Greenwood and A. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955), 1–7.

[22] A. Gyárfás, G. Sárközy, A. Sebő, and S. Selkow, Ramsey-type results for Gallai colorings, *J. Graph Theory* **64** (2010), 233–243.

[23] A. Gyárfás and G. Simonyi, Edge colorings of complete graphs without tricolored triangles, *J. Graph Theory* **46(3)** (2004), 211–216.

[24] S. Haghi and H. Maimani, A. Seify, Star-critical Ramsey numbers of F_n versus K_4, *Discrete Appl. Math.* **217** (2017), 203–209.

[25] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Inc., 1969.

[26] F. Harary and G. Prins, Generalized Ramsey theory for graphs, IV: the Ramsey multiplicity of a graph, *Networks* **4** (1974), 163–173.

[27] J. Hook, *The Classification of Critical Graphs and Star-Critical Ramsey Numbers*, Ph.D. Thesis, Lehigh University, 2010.

[28] J. Hook, Critical graphs for $R(P_n, P_m)$ and the star-critical Ramsey number for paths, *Discuss. Math. Graph Theory* **35** (2015), 689–701.

[29] J. Hook and G. Isaak, Star-critical Ramsey numbers, *Discrete Appl. Math.* **159** (2011), 328–334.

[30] R. Irving, Generalised Ramsey numbers for small graphs, *Discrete Math.* **9** (1974) 251– 264.

[31] M. Jacobson, On the Ramsey multiplicity for stars, *Discrete Math.* **42** (1982), 63–66.

[32] M. Jacobson, On the Ramsey number for stars and a complete graph, *Ars Combin.* **17** (1984), 167–172.

[33] C. Jayawardene, Star-critical Ramsey numbers for cycles versus the complete graph on 5 vertices, preprint.
https://arxiv.org/abs/1901.04802

[34] C. Jayawardene, D. Narváez, and S. Radziszowski, Star-critical Ramsey numbers for cycles versus K_4, *Discuss. Math. Graph Theory* **41(2)** (2021), 381–390.

[35] Y. Li and C. Rousseau, Fan-complete Ramsey numbers, *J. Graph Theory* **23(4)** (1996), 413–420.

[36] Z. Li and Y. Li, Some star-critical Ramsey numbers, *Discrete Appl. Math.* **181** (2015), 301–305.

[37] V. Nikiforov, The cycle-complete graph Ramsey numbers, *Combin. Probab. Comput.* **14(3)** (2005), 349–370.

[38] G. Omidi and G. Raeisi, A note on the Ramsey number of stars-complete graphs, *Eur. J. Combin.* **32** (2011), 598–599.

[39] S. Radziszowski, Small Ramsey numbers - revision 16, *Electron. J. Combin.* **DS1.16** (2021), 1–116.
https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS1/pdf

[40] I. Schiermeyer, All cycle-complete graph Ramsey numbers $r(C_m, K_6)$, *J. Graph Theory* **44(4)** (2003), 251–260.

[41] Surahmat, E. Baskoro, and H. Broersma, The Ramsey numbers of fans versus K_4, *Bull. Inst. Combin. Appl.* **43** (2005), 96–102.

[42] Y. Wang, Y. Li, and Y. Li, Redundant edges in Ramsey graphs, preprint.
https://arxiv.org/pdf/1712.03226.pdf

[43] J. Yang, Y. Huang, and K. Zhang, The value of the Ramsey number $R(C_n, K_4)$ is $3(n-1)+1$ $(n \geq 4)$, *Australas. J. Combin.* **20** (1999), 205–206.
http://ajc.maths.uq.edu.au/pdf/20/ocr-ajc-v20-p205.pdf

[44] Y. Zhang, H. Broersma, and Y. Chen, On star-critical and upper size Ramsey numbers, *Discrete Appl. Math.* **202** (2016), 174–180.