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A note on nearly Platonic graphs with connectivity one

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Abstract

A k-regular planar graph G is nearly Platonic when all faces but one are of the same degree while the remaining face is of a different degree. We show that no such graphs with connectivity one can exist. This complements a recent result by Keith, Froncek, and Kreher on non-existence of 2-connected nearly Platonic graphs.

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1. Introduction

A *Platonic graph of type* (k, d) is a k-vertex regular and d-face regular planar graph. It is well known that there exist exactly five Platonic graphs, which can be viewed as skeletons of the five Platonic solids—tetrahedron, cube, dodecahedron, octahedron, and icosahedron, of types $(3, 3), (3, 4), (3, 5), (4, 3)$ and $(5, 3)$, respectively.

There are several classes of vertex-regular planar graphs with all but two faces of one degree and two faces of another degree. Hence, it is an intriguing question whether there exist vertex-

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regular planar graphs with exactly one exceptional face? This question was answered in the negative by Deza, Dutour Sikirič, and Shtogrin [2] with a sketch of a proof, and for 2-connected graphs proved in detail by Keith, Froncek, and Kreher [4].

Theorem 1.1 ([2], [4])**.** *There is no finite, planar, 2-connected, regular graph that has all but one face of one degree and a single face of a different degree.*

We complement the result by offering a detailed case-by-case analysis for the remaining case with connectivity one. The main idea of our proof is the following. If such a graph with connectivity one exists, then there must exist an endblock, that is, a 2-connected graph with all vertices but one of degree k, one vertex of degree $1 < l < k$, all faces but one of degree d_1 and one face of degree $d \neq d_1$. The non-existence of such graphs was claimed by Deza and Dutour Sikirič in [1]. Because we were not satisfied with the proof, a purely combinatorial alternative is presented in this paper.

Our goal is to present an alternative proof of the following:

Theorem 1.2 ([1])**.** *There is no finite, planar, regular graph with connectivity one that has all but one face of one degree and a single face of a different degree.*

The main idea is to look at the blocks of such a potential graph and show that no endblock with required properties can exist.

The paper is organized as follows. In Section 2, we define the relevant notions and prove some basic observations that will be later used in our further proofs.

In Sections 3, 4, and 5 we discuss in details the non-existence of endblocks of types $(3, d)$, $(4, d)$ and $(5, d)$, respectively.

Finally, in Section 6 we use the lemmas proved in the previous sections to present our proof of Theorem 1.2.

2. Basic notions and observations

We start with a formal definition of an endblock.

Definition 2.1. A (k, d_1, d)-*endblock* $B(k, l)$ is a 2-connected planar graph on n vertices with $n-1$ vertices of degree k, one *exceptional vertex* x_1 with $\deg(x_1) = l$ and $1 < l < k$, all faces but one of *common degree* d_1, and the remaining face of degree $d \neq d_1$, where the exceptional vertex x_1 belongs to the face of degree d.

We will often use in our arguments the notion of saturated paths.

Definition 2.2. Let G be a 2-connected planar graph with maximum vertex degree k, common face degree d_1 and outerface $x_1, x_2, \ldots, x_d$ of degree $d \neq d_1$. A vertex $u \neq x_1$ is k-*saturated* (or simply *saturated*) if $\deg(u) = k$ and k-*unsaturated* (or simply *unsaturated*) if $\deg(u) < k$. Similarly, for a given integer $2 \leq l < k$, vertex x_1 is l-*saturated* (or simply *saturated*) if $\deg(x_1) = l$ and l-*unsaturated* (or simply *unsaturated*) if $\deg(x_1) < l$.

Let a path $P = u_1, u_2, \ldots, u_{d_1}$ be an induced subgraph of G such that the graph $G + u_1u_{d_1}$ is still planar and the cycle $C = u_1, u_2, \ldots, u_{d_1}$ is a boundary of a face of degree d_1. If all vertices

u_i for $i = 2, 3, \ldots, d_1 - 1$ are saturated and both u_1 and u_{d_1} are unsaturated, then P is called a *weakly-k-saturated d_1-path*. If at most one of u_1 or u_{d_1} is unsaturated while all other vertices are saturated, then P is called a *strongly-k-saturated d_1-path*. When k and d_1 are fixed, we call these paths simply *weakly saturated* or *strongly saturated*, respectively.

The following assertions are easy to verify.

Observation 2.3. *Let G be a 2-connected planar graph with maximum vertex degree k, minimum face degree d_1 and outerface $x_1, x_2, \ldots, x_d$ of degree $d \neq d_1$. If a strongly-k-saturated d_1-path $P = u_1, u_2, \ldots, u_{d_1}$ is on a boundary of an inner face of G, then G cannot be completed into a (k, d_1, d)-endblock.*

Proof. If G is a subgraph of a (k, d_1, d)-endblock, then the whole path P must be a part of an inner face of degree d_1, which implies that the remaining edge of that face must be $u_1 u_{d_1}$. However, this edge cannot be added, because at least one of the degrees of u_1 and u_{d_1} would then exceed k, a contradiction. □

Observation 2.4. *Let G be a 2-connected planar graph with maximum vertex degree k, minimum face degree d_1 and outerface $x_1, x_2, \ldots, x_d$ of degree $d \neq d_1$. If a weakly-k-saturated d_1-path $P = u_1, u_2, \ldots, u_{d_1}$ is on a boundary of an inner face of G, then the edge $u_1 u_{d_1}$ must be added in order to complete G into a (k, d_1, d)-endblock.*

Proof. Similarly as above, the whole path P must be a part of an inner face of degree d_1, which implies that the remaining edge of that face must be $u_1 u_{d_1}$. Hence, we must add it to G to complete it into the required endblock. □

Observation 2.5. *Let G be a subgraph of a $(k, 3, d)$-endblock $B(k, l)$ and u, v, w be a triangle such that v is saturated and has no neighbors inside the triangle. Then the triangle u, v, w is a face boundary.*

Proof. By Observation 2.4, the path u, v, w must be a part of a triangular face. Suppose that u has neighbors inside the triangle. Then at least one of them, say u_1, must be on the boundary containing edges $u_1 u$ and uv. Since v has no neighbors inside the triangle, the boundary also contains the edge vw. But then the edges $u_1 u, uv, vw$ bound a face that is longer than a triangle, which is impossible. □

Now we start eliminating certain forbidden configurations. In a (k, d_1, d)-endblock $B(k, l)$ with $x_1, x_2, \ldots, x_d$ as the boundary of the exceptional face, by a *chord* we mean an edge $x_i x_j$ not on the boundary of the exceptional face. That is, if $i < j$ and $x_i x_j$ is a chord, then $j - i \neq 1, d - 1$.

Lemma 2.6. *A $(3, d_1, d)$-endblock $B(3, 2)$ for $d_1 = 4, 5$ does not have a chord.*

Proof. Let the cycle $x_1, x_2, \ldots, x_d$ be the boundary of the exceptional face of this graph and there exists a chord $x_i x_j$ and $j > i$. Then $j - i \geq 3$, otherwise $j = i + 2$ and $x_i x_{i+1} x_j$ is a triangle such that x_i is saturated and has no neighbor inside the triangle. By Observation 2.5, this triangle is the boundary of a face, therefore, x_{i+1} is of degree 2, which is impossible.

Now, we consider the subgraph H induced by all vertices on and inside the cycle $x_i, x_{i+1}, \ldots, x_{j-1}, x_j$. Create an isomorphic copy $\varphi(H) = H'$ of H by assigning $\varphi(v) = v'$ for every $v \in H$. Then amalgamate the edges $x_i x_j$ and $x'_i x'_j$. The resulting graph is a 2-connected, 3-regular planar graph with all faces of degree d_1 except the outerface, which is of degree $2(j-i)$. We proved above that $j - i \geq 3$, which implies that the outerface is of degree $2(j-i) \geq 6$. Thus we have constructed a 2-connected, 3-regular planar graph with one face of degree greater than 5 and all remaining faces of degree $d_1 \leq 5$. This contradicts Theorem 1.1. □

Lemma 2.7. *A $(4,3,d)$-endblock $B(4,l)$, with the cycle $x_1, x_2, \ldots, x_d$ as the boundary of the exceptional face does not have a chord, other than $x_2 x_d$ when $l = 2$.*

Proof. Let the graph have some chords and the chord $x_i x_j$ with $j > i$ be the shortest one. It means that there is no other chord $x_s x_t$ with $0 < t - s < j - i$.

If $i = 1$, then $l = 3$. In this case the graph has only one vertex of an odd degree, which is impossible.

Now let $i > 1$ and y_i be the fourth neighbor of x_i. If y_i is on or inside of the cycle $x_1, x_2, \ldots, x_i, x_j, x_{j+1} \ldots, x_d$, then the path x_j, x_i, x_{i+1} is a weakly-4-saturated 3-path and we must have the triangular face x_j, x_i, x_{i+1}. If $j - (i+1) = 1$, then $\deg(x_{i+1}) = 2$, which is impossible and so $j - (i+1) \geq 2$ and $x_{i+1} x_j$ is a chord shorter than $x_i x_j$, a contradiction. Hence, y_i must be inside of the cycle $x_i, x_{i+1}, \ldots, x_j$. By symmetry, the fourth neighbor y_j of x_j must be inside that cycle as well. But then we see that the path x_{i-1}, x_i, x_j is a weakly saturated 3-path and by Observation 2.4, we must have the edge $x_{i-1} x_j$.

If $j < d$, then x_j has neighbors $x_{i-1}, x_i, y_j, x_{j-1}, x_{j+1}$ and is of degree at least 5, a nonsense. Therefore, $j = d$ and we must have $x_{i-1} = x_1$, which concludes the proof. □

Lemma 2.8. *A $(5,3,d)$-endblock $B(5,l)$, with the cycle $x_1, x_2, \ldots, x_d$ as the boundary of the exceptional face does not have a chord other than $x_2 x_d$ when $l = 2$.*

Proof. Let the graph have some chords and the chord $x_i x_j$ with $j - i > 1$ be the shortest one. It means that there is no other chord $x_s x_t$ with $0 < t - s < j - i$.

We denote by C the cycle $x_i, x_{i+1}, \ldots, x_j$ and by C' the cycle $x_j, x_{j+1}, \ldots, x_d, x_1, \ldots, x_i$.

First, we consider the case $i \neq 1$ and so $\deg(x_i) = \deg(x_j) = 5$. Call y_i^1 and y_i^2 the neighbors of x_i other than x_{i-1}, x_{i+1}, x_j and those of x_j other than x_{j-1}, x_{j+1}(or x_1),x_i similarly y_j^1 and y_j^2.

We will discuss several cases based on placement of the vertices y_s^t within cycles C and C'.

If both y_i^1, y_i^2 are within C', then the path x_j, x_i, x_{i+1} is a weakly-5-saturated 3-path and by Observations 2.4 and 2.5, we must have the triangular face x_i, x_{i+1}, x_j. Since $j - (i+1) < j - i$ the edge $x_{i+1} x_j$ is not a chord. But then x_{i+1} would have three other neighbors inside that face, which is impossible.

Similarly to the previous case, if both y_j^1, y_j^2 are within C', then the path x_i, x_j, x_{j-1} is a weakly-5-saturated 3-path and by Observations 2.4 and 2.5, we must have the triangular face $x_i, , x_j x_{j-1}$. Since $(j-1) - i < j - i$ the edge $x_i x_{j-1}$ is not a chord. But then x_{j-1} would have three other neighbors inside that face, which is impossible.

If one of x_i, x_j has both remaining neighbors inside C, say x_i, then the path x_{i-1}, x_i, x_j is weakly 5-saturated path and we must have edge $x_{i-1} x_j$ completing the triangle. We observe that

x_{i-1} cannot have any neighbors inside this triangle. If $j = d$, then it follows that $i - 1 = 1$ and $l = 2$ and we are done. If $j < d$, then by the previous case, x_j has one neighbor other than x_{j-1} inside C and the graph bounded by the cycle $x_{i-1}, x_i, \ldots, x_j, x_{i-1}$ has x_{i-1} of degree 2 and x_j of degree 4. Hence, we can take two copies and amalgamate them to obtain a 5-regular graph with the outer face of degree more than 3 and all other faces triangular. However, such a graph does not exist, so this case is impossible.

The only remaining case is that x_i and x_j have exactly one neighbor within both C and C'. In this case, we can again obtain a contradiction in a similar manner as in Lemma 2.7. Denote by H the induced subgraph of G consisting of all vertices on or within C and create an isomorphic copy H'. Then amalgamate x_i with x'_i and x_j with x'_j. The resulting graph is a 2-connected 5-regular graph with the outerface of degree $2(j - i) > 3$ and all other faces of degree 3. Such a graph cannot exist by Theorem 1.1.

Finally, we consider the case $i = 1$, that is, the graph has a chord x_1x_j with $j \neq 2, d$. If $l = 3$, then x_1 has no neighbor within C and so by Observation 2.4, x_2x_j is an edge and the graph has a shorter chord than x_1x_j, which is impossible.

For $l = 4$, the vertex x_1 has the fourth neighbor $y_1^1 \notin \{x_2, x_3, \ldots, x_j\} \cup \{x_d\}$.

If y_1^1 is not the inside of C, then as in the previous case, the graph has a shorter chord x_2x_j, a contradiction. Thus, y_1^1 is within C. By applying Observation 2.4 on the weakly-4-saturated 3-path x_d, x_1, x_j, we deduce that the triangle x_1, x_d, x_j is the boundary of a triangular face of the graph.

We have $1 < j < d$. If x_jx_d is a chord, we find the shortest chord $x_{i'}x_{j'}$ such that $j \leq i' < j' \leq d$ and repeat the case $i' \neq 1$ from the first part of the proof.

If x_jx_d is not a chord, by the first part of this proof, and so $j = d - 1$ and $\deg(x_d) = 2$, which is impossible.

We have exhausted all possibilities and the proof is complete. □

Lemma 2.9. *Let t be the number of vertices of the (k, d_1, d)-endblock $B(k, l)$ not on the boundary of the outer exceptional face. Then the values of t are as follows:*

k	d_1	l	t
3	3	2	$(5-d)/3$
3	4	2	3
3	5	2	$d+7$
4	3	2	2
5	3	2	$d+3$
5	3	3	$d+4$
5	3	4	$d+5$

Proof. Denote the order of the graph by n, the number of its edges by m and the number of faces by f, thus the sum of the vertex degrees will be $k(n-1)+l$, which is twice the number of edges. By Euler's formula, the number of faces is

$$f = m + 2 - n = \frac{k(n-1)+l}{2} + 2 - n.$$

Also since the sum of the face degrees is twice the number of edges, we have

$$(f-1)d_1 + d = 2m = k(n-1) + l.$$

Solve for n, we have

$$n = \frac{(2-d_1)(k-l) + 2d_1 + 2d}{2k + 2d_1 - kd_1}.$$

Recall that $t = n - d$, so when we plug in the corresponding values of k, d_1, and l, we obtain our desired values of t as a function of d. □

3. Type $(3, d_1)$

Lemma 3.1. *A $(3, 3, d)$-endblock $B(3, 2)$ does not exist for any d.*

Proof. By Lemma 2.9, we must have $d = 2$ or $d = 5$, otherwise t is not a non-negative integer. Recall that the number of vertices is $d+t$. If $d = 2$, then $t = 1$ and the graph has 3 vertices in total. Hence, we cannot have vertices of degree 3. When $d = 5$, then $t = 0$ and the graph has 5 vertices in total. By applying Observation 2.4 on the weakly-2-saturated 3-path x_5, x_1, x_2 we conclude that x_2x_5 is an edge of the graph. Now, the path x_5, x_2, x_3 is a strongly-3-saturated 3-path. Hence, the graph has a face with the length greater than 3 and G cannot be completed into $B(3, 2)$. □

Lemma 3.2. *A $(3, 4, d)$-endblock $B(3, 2)$ does not exist for any d.*

Proof. Recall that by t we denote the number of vertices of $B(3, 2)$ inside of the cycle bounding the face of degree d, that is, all vertices other than $x_1, x_2, \ldots, x_d$. It follows from Lemma 2.9 that $t = 3$.

We denote the internal vertices by y_1, y_2 and y_3. Since $d_1 = 4$, there are at most two edges y_iy_j, which implies that there are at least five edges y_ix_j. As there is no chord by Lemma 2.6, each $x_i, i \neq 1$ has exactly one neighbor y_j and hence $d \geq 6$. Because x_1 is of degree 2, it is a saturated vertex. Let x_2y_1 be an edge. Then y_1, x_2, x_1, x_d is a weakly-3-saturated 4-path, and we must have the edge y_1x_d.

If the third neighbor of x_3 is y_1, then we have a triangular face, which is impossible. Assume that x_3 is adjacent to y_2. Then y_1, x_2, x_3, y_2 is a weakly-3-saturated 4-path, and we must have the edge y_1y_2. Now, the path y_2, y_1, x_d, x_{d-1} is a weakly-3-saturated 4-path, and we must have the edge y_2x_{d-1}. Since $d \geq 6$, we have $d - 1 \neq 4$ and so $x_4 \neq x_{d-1}$. Then x_4, x_3, y_2, x_{d-1} is a strongly-3-saturated 4-path, and $B(3, 2)$ cannot be completed. □

Lemma 3.3. *A $(3, 5, d)$-endblock $B(3, 2)$ does not exist for any d.*

Proof. Assume that the cycle $x_1, x_2, \ldots, x_d$ is the boundary of the outerface and $\deg(x_1) = 2$. By Lemma 2.6 the graph has no chord and so each x_i, except x_1, is adjacent to an interior vertex, say y_{i-1}. All y_i's are distinct, otherwise as we see in Figure 1(right), if $i < j$ and two vertices x_i and x_j have a common interior neighbor, say y, then we have two cycles $x_i, x_{i+1}, \ldots, x_{j-1}, x_j, y$ and $x_1, x_2, \ldots, x_i, x_j, x_{j+1}, \ldots, x_d$ and we consider the cycle that the third neighbor of y is not in. If this cycle is a triangle then the graph has an interior triangular face or a cut-vertex, which

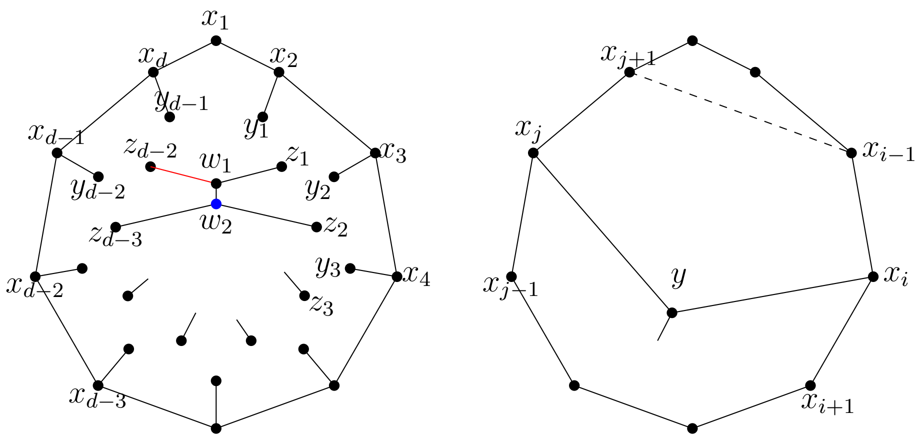

Figure 1

is impossible. If this cycle is a square or pentagon then the graph has a cut-vertex, which is impossible. If the length of this cycle is greater than 5 then one of two paths $x_{i-1}, x_i, y, x_j, x_{j+1}$ or $x_{i+1}, x_i, y, x_j, x_{j-1}$ is a weakly-3-saturated 5-path and the graph has a chord $x_{i-1}x_{j+1}$ or $x_{i+1}x_{j-1}$, respectively, which is impossible.

The path $y_1, x_2, x_1, x_d, y_{d-1}$ is a weakly-3-saturated 5-path and so y_1y_{d-1} is an edge.

For any $1 \leq i \leq d-1$, the path $y_i, x_{i+1}, x_{i+2}, y_{i+1}$ is a weakly-3-saturated 4-path and so two vertices y_i and y_{i+1} have a common adjacent z_i to construct a pentagonal face. These vertices are distinct. If $z_i = z_{i+1}$ for some i, then $\deg(y_{i+1}) = 2$ and if $z_i = z_j$, where $j - i > 1$, then $\deg(z_i) \geq 4$, which is impossible. If $d = 3$, then z_1 is a cut-vertex, which is impossible and so $d \geq 4$.

If $d \geq 4$, then the path $z_{d-2}, y_{d-1}, y_1, z_1$ is a weakly-3-saturated 4-path and two vertices z_{d-2} and z_1 must have a common neighbor w_1 to obtain a pentagonal face. We have $w_1 \neq z_i$, $2 \leq i \leq d-3$, otherwise $\deg(w_1) \geq 4$, which is impossible. If $d = 4$, then w_1 is a cut-vertex, which is impossible and so $d \geq 5$. If $d \geq 5$, then the path w_1, z_1, y_2, z_2 is a weakly-3-saturated 4-path and two vertices w_1 and z_2 must have a common neighbor w_2 to obtain a pentagonal face (see Figure 1 (left)). We have $w_2 \neq z_i$, $3 \leq i \leq d-4$, otherwise $\deg(w_2) \geq 4$, which is impossible. If $d = 5$, then w_2 is a cut-vertex and so $d \geq 6$. If $d = 6$, then the cycle $w_2z_2y_3z_3$ is the boundary of a square face, which is impossible and so $d \geq 7$.

If $d \geq 7$, then the path $y_{d-3}, z_{d-3}, w_2, z_2, y_3$ is a strongly-3-saturated 5-path and we have an interior face with the length greater than 5, which is impossible. □

4. Type $(4, 3)$

For the two remaining cases, we use dual graphs. Recall that the *dual graph* G^D of a planar graph G with vertex, edge, and face sets $V(G), E(G), F(G)$, respectively, has $V(G^D) = F(G)$, $F(G^D) = V(G)$ and and edge $e = f_1f_2 \in V(G^D)$ if and only if the faces f_1 and f_2 share an edge in G. In general, G^D can be a multigraph with loops. In our case, we only look at dual graphs of blocks, hence no loops will arise. Concerning multiple edges, we can only have one double edge when $l = 2$.

Lemma 4.1. *A* $(4, 3, d)$*-endblock* $B(4, l)$ *does not exist for any* l *and* d*.*

Proof. We cannot have $l = 3$, as in that case the endblock would have a single vertex of an odd degree, a nonsense. Thus, we have $l = 2$.

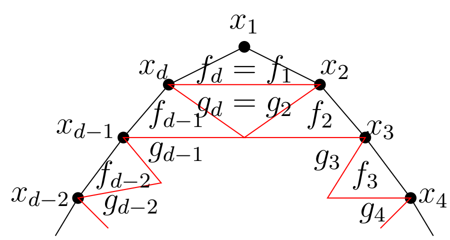

Figure 2: $B(4, 2)$

First we show that $d \geq 4$. Suppose $d = 3$. Then the outerface is a triangle x_1, x_2, x_3. Vertex x_1 is saturated, hence the inner face containing x_1 is the triangle x_1, x_2, x_3. By Lemma 2.9, we have $t = 2$, hence there are exactly two other vertices y_1 and y_2, each of degree 4. But then y_1 would have to be adjacent to x_2, x_3, y_2 and also to x_1, which is impossible.

Now we denote the outerface of $B(4, 2)$ by f and an inner face containing edge $x_i x_{i+1}$ by f_i for $i = 1, 2, \ldots, d$. Further, for $i = 2, 3, \ldots, d$ the inner face containing x_i but not sharing an edge with the outerface will be denoted by g_i (see Figure 2).

Let D be the dual graph of $B(4, 2)$. Because $d \geq 4$, we have $\deg_D(f) \geq 4$. Notice that we have double edge $f f_1$ (see Figure 3).

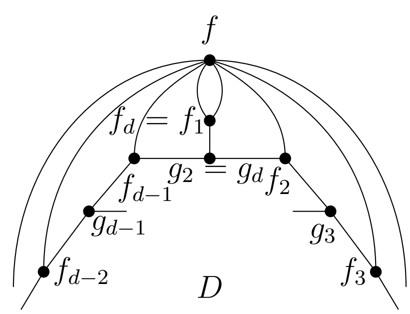

Figure 3: The dual graph of $B(4, 2)$

Let $d = 3q + r$ where $0 \leq r \leq 2$. We have $d \geq 4$, hence $q \geq 1$. We now construct a new graph D' with $\Delta(D') = 3$ as follows. We split vertex f into vertices $f^0, f^1, \ldots, f^q$ and each f^i will be incident with edges $f^i f_{3i+1}, f^i f_{3i+2}, f^i f_{3i+3}$ except possibly for f^q, which may be of degree zero, one, or two (see Figure 4).

The outer boundary is now $f^0, f_3, g_4, f_4, f^1, f_6, \ldots, f_1$. If f^q is of degree zero, then we remove it and obtain an outerface $f^0, f_3, g_4, f_4, f^1, f_6, \ldots, f_{d-2}, f_{q-1}, f_1$ of length at least six. But then we

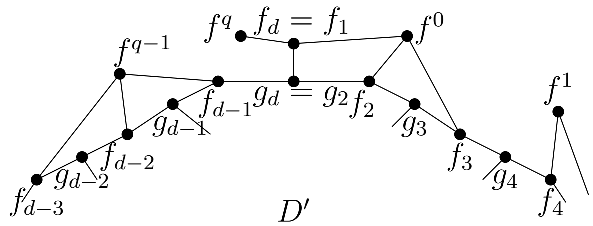

Figure 4: $r = 1$

have a 2-connected, 3-vertex regular nearly Platonic graph with one face of size at least 6 and all other faces of size 4, which does not exist by Theorem 1.1.

If f^q is of degree one, as in Figure 4, we remove it and obtain an outerface $f^0, f_3, g_4, f_4, f^1, f_6, \ldots, f_{q-1}, f_{d-1}, g_2, f_1$ where f_1 is now of degree two and we have a $(3, 4, d')$-endblock $B(3, 2)$, which does not exist by Lemma 3.2.

Finally, when f^q is of degree two, then the boundary is $f^0, f_3, g_4, f_4, f^1, f_6, \ldots, g_{d-1}, f_{d-1}, f_q, f_1$ where f^q is now of degree two and then again we have a $(3, 4, d')$-endblock $B(3, 2)$, which does not exist by Lemma 3.2. □

5. Type $(5, 3)$

Lemma 5.1. *A $(5, 3, d)$-endblock $B(5, l)$ does not exist for any l and d.*

Proof. We again use the dual graph technique to prove the claim. We start with an observation that the case $d = 3$ is impossible. If we have such a graph G with vertex x_1 of degree l where $3 \leq l \leq 4$, all other vertices of degree 5, inner faces triangular, and $d = 3$, then the outerface is also a triangle. But then the dual graph G^D is a 2-connected, cubic graph with one face of degree $l \neq 5$, and all remaining faces of degree 5, which is impossible by Theorem 1.1.

If $l = 2$, then the path x_3, x_1, x_2 is weakly saturated, and we must have the edge x_2x_3 completing the boundary of the inner triangular face. But then the remaining neighbors of x_2 and x_3 are outside of the cycle x_1, x_2, x_3, that is, within the outerface, which is impossible.

We use the same notation as in the previous proof, with the exception that for $i = 1, 2, \ldots, d$ the two inner faces containing x_i but not sharing an edge with the outerface will be denoted by g_i and h_i (see Figure 5).

The case $l = 2$ is essentially the same as for the $(4, 3, d)$-endblock $B(4, 2)$ and we omit it.

When $l = 3$, the graphs $B(5, 3)$ and its dual graph are shown in Figures 5 and 6, respectively. After splitting vertex f the outerface of D' is $f^0, f_3, g_4, h_4, f_4, f^1, \ldots, f_1$ of length at least six.

When f_q is of degree zero, the boundary is $f^0, f_3, g_4, h_4, f_4, f^1, \ldots, f^{q-1}, f_d, f_1$ and we have a 2-connected, 3-vertex regular nearly Platonic graph with one face of size at least 7 and all other faces of size 5, and such a graph does not exist by Theorem 1.1. When f^q is of degree one, by omitting f^q, the boundary is $f^0, f_3, g_4, h_4, f_4, f^1, \ldots, g_d, h_d, f_d, f_1$ and $\deg(f_d) = 2$, then the outer boundary is of length at least 6, and when f^q is of degree two, the boundary is $f^0, f_3, g_4, h_4, f_4, f^1, \ldots, f_{d-1}, f^q, f_d, f_1$ and $\deg(f^q) = 2$, then the outer boundary is of length

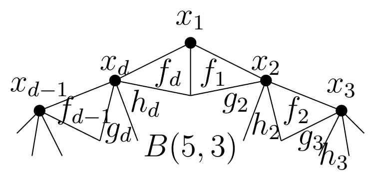

Figure 5: $l = 3$

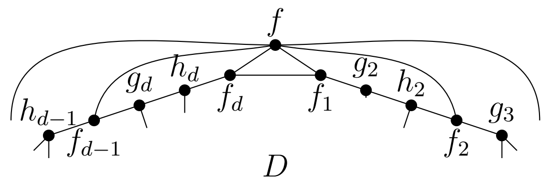

Figure 6: The dual graph of $B(5,3)$

at least 7. In the both cases, we have a $(5,3,d')$-endblock $B(5,3)$, which does not exist by Lemma 3.2.

When $l = 4$, then the outerface in D' is $f^0, f_3, g_4, h_4, f_4, f^1, \ldots, g_1, f_1$ of length at least seven.

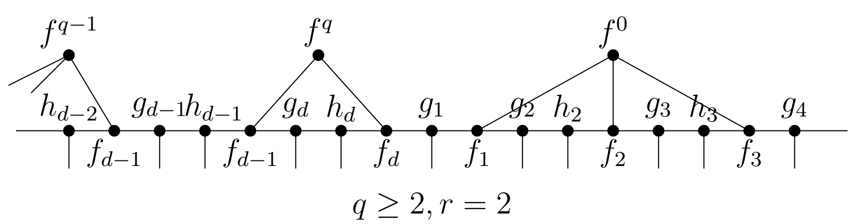

Figure 7: D'

Now similarly as in the previous proof, when f_q is of degree zero, the boundary is $f^0, f_3, g_4, h_4, f_4, f^1, \ldots, f^q, f_d, g_1, f_1$ and we have a 2-connected, 3-vertex regular nearly Platonic graph with one face of size at least 8 and all other faces of size 5, and such a graph does not exist by Theorem 1.1. When f^q is of degree one, by omitting f^q, the boundary is $f^0, f_3, g_4, h_4, f_4, f^1, \ldots, g_d, h_d, f_d, g_1, f_1$ and $\deg(f_d) = 2$, then the outer boundary is of length at least 7, and when f^q is of degree two, the boundary is $f^0, f_3, g_4, h_4, f_4, f^1, \ldots, f^q, f_d, g_1, f_1$ and $\deg(f^q) = 2$, then the outer boundary is of length at least 9 (see Figure 7). In the both cases, we have a $(5,4,d')$-endblock $B(5,4)$, which does not exist by Lemma 3.2. □

6. Main result

Now we are ready to prove our main result.

Theorem 6.1. *There is no (k, d_1, d)-endblock for any admissible triple (k, d_1, d).*

Proof. Follows directly from Lemmas 3.1–3.3, 4.1, and 5.1. □

An alternative proof of the result presented by Deza and Dutour Sikirič in [1] now follows immediately.

Theorem 1.2. *There is no finite, planar, regular graph with connectivity one that has all but one face of one degree and a single face of a different degree.*

Proof. It is well known that every graph with connectivity one and minimum degree at least three has at least two endblocks, that is, 2-connected graphs with minimum degree at least two. If there was a graph defined in the Theorem, it would have to contain a (k, d_1, d)-endblock for some admissible triple (k, d_1, d). However, such an endblock does not exist by Theorem 6.1. This proves the claim. □

References

References

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