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Vector weighted Stirling numbers and an application in graph theory

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Abstract

We introduce *vector weighted Stirling numbers*, which are a generalization of ordinary Stirling numbers and restricted Stirling numbers. Some relations between vector weighted Stirling numbers and ordinary Stirling numbers and some of their applications are stated. Moreover, as an application of vector weighted Stirling numbers of the second kind in graph theory, we compute the number of maximal independent sets of different sizes in k-intersection graphs.

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1. Introduction

For any positive integers n and k, the classical *Stirling numbers of the first kind* $\left[{n \atop k}\right]$, *second kind* $\left\{{n \atop k}\right\}$, and *third kind (or Lah numbers)* $\left|{n \atop k}\right|$ enumerate the number of partitions of a set with n elements consisting k disjoint nonempty cycles, sets, and ordered lists, respectively. Stirling numbers also relate the falling factorial $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$, the rising factorial

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$x^{(n)} = x(x+1)\cdots(x+n-1)$, and x^n as follows:

$$x^n = \sum_{k=0}^{n} \left\{ n \atop k \right\} (x)_k, \quad (x)_n = \sum_{k=0}^{n} \left[n \atop k \right] x^k, \quad x^{(n)} = \sum_{k=0}^{n} \left| n \atop k \right| (x)_k.$$

There are different notations for the Stirling numbers in literature, among them $\left[n \atop k \right]$, $\left\{ n \atop k \right\}$, $\left| n \atop k \right|$; $s_1(n,k), s_2(n,k), s_3(n,k)$; and $s(n,k), S(n,k), L(n,k)$ are the most famous notations for Stirling numbers of the first, second, and third kind, respectively. In this paper, we use the first notation to represent the Stirling numbers. It is also well known that Stirling numbers of the first, second, and third kind satisfy recurrence relations.

$$\begin{aligned} \left[n \atop k \right] &= \left[n-1 \atop k-1 \right] + (n-1) \left[n-1 \atop k \right], \\ \left\{ n \atop k \right\} &= \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}, \\ \left| n \atop k \right| &= \left| n-1 \atop k-1 \right| + (n+k-1) \left| n-1 \atop k \right|. \end{aligned}$$

For further information on Stirling numbers, see [6, 17, 16].

There are different generalizations and applications for the Stirling numbers. For example, Broder [3] considered a kind of restricted r-Stirling numbers that subject to the condition that the first r elements must be in distinct cycles or subsets. Carlitz [4] defined and studied a completely different type of generalization of the Stirling numbers and considered the so-called notion of λ-partitions. Choi and Smith [5] considered r-restricted Stirling numbers of the second kind and those partitions that each block contains at most r members. To see other researches on the generalization of Stirling numbers, we refer the reader to [13, 14, 9, 7, 2, 1, 15, 11].

In this paper, we give a new generalization of the Stirling numbers, which we call *vector weighted Stirling numbers*. There are some combinatorial problems in which we need a special kind of partitioning of different objects. In the mentioned partitioning, we need two restrictions, one on the size and the other on the weight of each block such that the weight is a nonnegative integer depending on the size of the block. As an example of these combinatorial problems, we can point to the problem of counting independent vertex sets of different size of a k*-intersection graph*. By a k-intersection graph $\Gamma(n,m,k)$, we mean a graph whose vertices are m-subsets of $\{1,\ldots,n\}$ and two distinct vertices are adjacent if their intersection is not a subset of $\{1,\ldots,k\}$, where n,m, and k are positive integers with $k \le m < n$. One can easily see that the union of some *Johnson* graphs with specified parameters is an induced subgraph of $\Gamma(n,m,k)$; see [10].

The importance of this graph comes from various kinds of examples in many fields of science such as economics, social system modelling, financial matters and so on. For example, consider n people who want to invest in some companies. These people consist of k foreigners and $n-k$ native. Each company has m investors,This situation can be modelled with a k-intersection graph with parameters n,m and k. Each company with m investors is considered as a vertex and two vertices are connected if the corresponding companies have at least one native investor in common. The independence number and click number of $\Gamma_m(\Omega,\Lambda)$ have important interpretations in this situation.

Working on the problem of independent vertex sets of k-intersection graphs leads us to a new generalization of the Stirling numbers as follows. k-intersection graphs were introduced and mentioned extensively in [8].

Section 2 begins with definitions, preliminaries, and some examples for vector weighted Stirling numbers and ends with some related theorems and consequences. In Section 3, we have a quick look on k-intersection graphs and drive a formula for the number of maximal independent vertex sets of $\Gamma(n, m, k)$ with two different approaches.

2. Vector weighted Stirling numbers

We remind that a *weight vector*, denoted by $\vec{w} = (w_1, w_2, \ldots)$, is a vector with possibly infinite positive integer entries. Let us start with the following definitions. Let n and k be two positive integers, and consider partitions of a set of n elements into k nonempty blocks.

Definition 2.1. Let $\vec{w}$ be a weight vector, and suppose that $P = \{P_1, \ldots, P_k\}$ is a partition of a set of size n into k nonempty unlabeled blocks. Then *weight of partition P with respect to weight vector $\vec{w}$*, denoted by $W(P_{\vec{w}})$, is defined as $\prod_{i=1}^{k} w_{p_i}$, where p_i is the size of P_i.

We say briefly *weight of partition P* when there is no ambiguity. As an example, consider the weight vector $\vec{w} = (1, 2, 3, \ldots)$ and let $P = \{\{1, 2, 8\}, \{3\}, \{4, 5, 6, 7\}\}$ and $P' = \{\{1, 2\}, \{3, 4\}, \{5, 6, 7, 8\}\}$ be two partitions of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into 3 nonempty unlabeled blocks. Then $W(P_{\vec{w}}) = 3 \times 1 \times 4 = 12$ and $W(P'_{\vec{w}}) = 2 \times 2 \times 4 = 16$.

Definition 2.2. Let $\vec{w}$ be a weight vector and let n and k be two positive integers. Then $\left\{ {n \atop k} \right\}_{\vec{w}}$ is the sum of weights of all possible partitions of n distinct objects into k nonempty unlabeled subsets. These numbers are called *vector weighted Stirling numbers of the second kind with weight vector $\vec{w}$*. Similarly, $\left[{n \atop k} \right]_{\vec{w}}$ is the sum of weights of all possible partitions of n distinct objects into k nonempty unlabeled cycles. These numbers are called *vector weighted Stirling numbers of the first kind with weight vector $\vec{w}$*.

In other words, $\left\{ {n \atop k} \right\}_{\vec{w}} = \sum W(P_{\vec{w}})$, where the sum is taken over all possible partitions P of n distinct objects into k nonempty blocks. Now, we state the recurrence relation for the vector weighted Stirling numbers as follows.

Proposition 2.3. *Let n and k be two positive integers and let $\vec{w}$ be a weight vector. Then*

$$\left\{ {n \atop k} \right\}_{\vec{w}} = \sum_{i=1}^{n-k+1} w_i \binom{n-1}{i-1} \left\{ {n-i \atop k-1} \right\}_{\vec{w}}.$$

Proof. Let $[n] = \{1, 2, \ldots, n\}$. We want to count the number of different partitions of $[n]$ into k nonempty blocks with the weight vector $\vec{w}$. Consider the element $1 \in [n]$. Then 1 is in a partition alone in $w_1 \left\{ {n-1 \atop k-1} \right\}_{\vec{w}}$ ways, in a partition with one other number in $w_2 \binom{n-1}{1} \left\{ {n-2 \atop k-1} \right\}_{\vec{w}}$ ways, in a partition with two other numbers in $w_3 \binom{n-1}{2} \left\{ {n-3 \atop k-1} \right\}_{\vec{w}}$ ways, and finally in a partition with $n-k$ other numbers in $w_{n-k+1} \binom{n-1}{n-k} \left\{ {k-1 \atop k-1} \right\}_{\vec{w}}$ ways. Thus the summation of all these possible conditions completes the proof. □

With a similar argument, one can show that a recurrence relation for vector weighted Stirling numbers of the first kind is the following equation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\vec{w}} = \sum_{i=1} w_i (i-1)! \binom{n-1}{i-1} \begin{bmatrix} n-i \\ k-1 \end{bmatrix}_{\vec{w}}.$$

By convention, for positive integers n and k and an arbitrary weight vector $\vec{w}$, we define initial conditions for vector weighted Stirling numbers of the first and second kind as $\left\{ {0 \atop 0} \right\}_{\vec{w}} = \left[{0 \atop 0} \right]_{\vec{w}} = 1$ and $\left\{ {n \atop 0} \right\}_{\vec{w}} = \left\{ {0 \atop k} \right\}_{\vec{w}} = \left[{n \atop 0} \right]_{\vec{w}} = \left[{0 \atop k} \right]_{\vec{w}} = 0$. Using these recurrence relations, we can compute vector weighted Stirling numbers of the first and second kind for an arbitrary weight vector $\vec{w}$. The matrix representation of vector weighted Stirling numbers of the first and second kind for $1 \leq n, k \leq 6$ is shown below. Let $[\left\{ {n \atop k} \right\}_{\vec{w}}]_r$ and $[\left[{n \atop k} \right]_{\vec{w}}]_r$ be square matrices with r rows and r columns whose (i,j) entry is $\left\{ {i \atop j} \right\}_{\vec{w}}$ and $\left[{i \atop j} \right]_{\vec{w}}$, respectively.

$$[\begin{Bmatrix} n \\ k \end{Bmatrix}_{\vec{w}}]_6 = \begin{bmatrix} w_1 & 0 & 0 & 0 & 0 & 0 \\ w_2 & w_1^2 & 0 & 0 & 0 & 0 \\ w_3 & 3w_1w_2 & w_1^3 & 0 & 0 & 0 \\ w_4 & 4w_1w_3+3w_2^2 & 6w_1^2w_2 & w_1^4 & 0 & 0 \\ w_5 & 5w_1w_4+10w_2w_3 & 10w_1^2w_3+15w_1w_2^2 & 10w_1^3w_2 & w_1^5 & 0 \\ w_6 & 6w_1w_5+15w_2w_4+10w_3^2 & 15w_1^2w_4+60w_1w_2w_3+15w_2^3 & 20w_1^3w_3+45w_1^2w_2^2 & 15w_1^4w_2 & w_1^6 \end{bmatrix},$$

$$[\begin{bmatrix} n \\ k \end{bmatrix}_{\vec{w}}]_6 = \begin{bmatrix} w_1 & 0 & 0 & 0 & 0 & 0 \\ w_2 & w_1^2 & 0 & 0 & 0 & 0 \\ 2w_3 & 3w_1w_2 & w_1^3 & 0 & 0 & 0 \\ 6w_4 & 8w_1w_3+3w_2^2 & 6w_1^2w_2 & w_1^4 & 0 & 0 \\ 24w_5 & 30w_1w_4+20w_2w_3 & 20w_1^2w_3+15w_1w_2^2 & 10w_1^3w_2 & w_1^5 & 0 \\ 120w_6 & 144w_1w_5+90w_2w_4+40w_3^2 & 90w_1^2w_4+120w_1w_2w_3+15w_2^3 & 40w_1^3w_3+45w_1^2w_2^2 & 15w_1^4w_2 & w_1^6 \end{bmatrix}.$$

It is clear that when the weight vector is $\vec{v} = (1, 1, \ldots)$, then vector weighted Stirling numbers of the first and second kind are the following ordinary Stirling numbers of the first and second kind:

$$[\begin{Bmatrix} n \\ k \end{Bmatrix}_{\vec{v}}]_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix}, \qquad [\begin{bmatrix} n \\ k \end{bmatrix}_{\vec{v}}]_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 6 & 11 & 6 & 1 & 0 & 0 \\ 24 & 50 & 35 & 10 & 1 & 0 \\ 120 & 274 & 225 & 85 & 15 & 1 \end{bmatrix}.$$

Example 2.4. Let m be a positive square free integer with n prime factors. The number of ways that we can write m as the multiplication of k positive integers $m = m_1 \cdots m_k$ such that each m_i has even prime factors is $\left\{ {n \atop k} \right\}_{\vec{v}}$, where $\vec{v} = (0, 1, 0, 1, \ldots)$.

Recall that for positive integers $n, n_1, \ldots, n_k$ that $n = n_1 + \cdots + n_k$, the multinomial coefficient $\binom{n}{n_1, \ldots, n_k}$ is equal to $\frac{n!}{n_1! \cdots n_k!}$. In the following theorem, we have an explicit formula for vector weighted Stirling numbers of the second kind.

Theorem 2.5. *Let k and n be two positive integers with $n \geq k$. Let $\vec{w}$ be a weight vector whose a and A are the minimum and maximum nonzero indices, respectively. Then*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{\vec{w}} = \sum_{\substack{\sum_{i=a}^{A} x_i = k \\ \sum_{i=a}^{A} i x_i = n}} \frac{\prod w_i^{x_i}}{k!} \binom{k}{x_a, \ldots, x_A} \binom{n}{\underbrace{a, \ldots, a}_{x_a}, \ldots, \underbrace{A, \ldots, A}_{x_A}}.$$

Proof. Let x_i stands for the number of blocks with i balls. Then, we are going to find the sum taken over all possible sets of $\{x_a, x_{a+1}, \dots, x_A\}$ such that $\sum_{i=a}^{A} x_i = k$ and $\sum_{i=a}^{A} ix_i = n$. For fix $x_a, \dots, x_A$, first we count the number of all possible partitions with x_i blocks of size i. We choose ax_a members of n and then put them in x_a blocks of size a. The number of such choices is equal to

$$\binom{n}{ax_a}\frac{1}{x_a!}\binom{ax_a}{a}\binom{a(x_a-1)}{a}\cdots\binom{a}{a} = \binom{n}{ax_a}\frac{1}{x_a!}\frac{(ax_a)!}{(a!)^{x_a}}.$$

Next, we choose $(a+1)x_{a+1}$ members of the remaining $n - ax_a$ members and then put them in x_{a+1} blocks of size $a+1$. The number of them is equal to

$$\binom{n-ax_a}{(a+1)x_{a+1}}\frac{1}{x_{a+1}!}\binom{(a+1)x_{a+1}}{a+1}\cdots\binom{a+1}{a+1} = \binom{n-ax_a}{(a+1)x_{a+1}}\frac{1}{x_{a+1}!}\frac{(a+1)x_{a+1}!}{(a+1)!^{x_{a+1}}}.$$

Continuing the same process for other x_i's, we will have the following number of partitions:

$$\begin{aligned}
&\left(\prod_{i=a}^{A}\binom{n-\sum_{j=a}^{i-1} jx_j}{ix_i}\right)\left(\frac{1}{x_a!}\cdots\frac{1}{x_A!}\right)\left(\frac{n!}{a!^{x_a}\cdots A!^{x_A}}\right)\\
&= \frac{n!}{(ax_a)!\cdots(Ax_A)!}\left(\frac{1}{x_a!}\cdots\frac{1}{x_A!}\right)\left(\frac{(ax_a)!\cdots(Ax_A)!}{a!^{x_a}\cdots A!^{x_A}}\right)\\
&= \frac{1}{k!}\binom{k}{x_a,\dots,x_A}\binom{n}{\underbrace{a,\dots,a}_{x_a},\dots,\underbrace{A,\dots,A}_{x_A}}.
\end{aligned} \tag{2.1}$$

Since these partitions have x_a blocks of size a, x_{a+1} blocks of size $a+1$, and so on, the weight of partition is $w_a^{x_a} w_{a+1}^{x_{a+1}} \cdots w_A^{x_A}$. Now, it is clear that $\left\{ {n \atop k} \right\}_{\vec{w}}$ is the summation of (2.1) over all possible sets like $\{x_a, x_{a+1}, \dots, x_A\}$ such that $\sum_{i=a}^{A} x_i = k$ and $\sum_{i=a}^{A} ix_i = n$. Thus, the number of all possible partitions is obtained.

□

Example 2.6. Consider n dice with different colors. Now we compute the number of ways that one can put these n dice in k nonempty boxes such that each box contains at most m dice.

Consider a box that contains r dice. Since each die has six sides, so there are 6^r different ways for this box. Let $\vec{w} = (6, 6^2, 6^3, \dots, 6^m, 0, 0, \dots)$. Then the number of all possible ways is exactly $\left\{ {n \atop k} \right\}_{\vec{w}}$.

As we mentioned above, for positive integers n and k, different kinds of Stirling numbers (the first, second and third) are precisely the number of partitions of an n-set into k unlabeled cycles, subsets, and lists, respectively. If one puts some restrictions on blocks, then we will have *restricted Stirling numbers*. Different restrictions have been mentioned in some papers. Most of these restrictions are special cases of vector weighted Stirling numbers for some special weight vectors. For instance, Comtet [6] introduced r-associated Stirling numbers of the second kind, that is, those partitions that the size of each block is at least r, which is a vector weighted partitioning with weight vector $\vec{w} = \{w_1, w_2, \dots\}$ where $w_i = 1$ if $i \geq r$ and $w_i = 0$ if $i < r$. Choi and Smith [5] considered r-restricted Stirling numbers of the second kind, that is, the number of those partitions that the size of each block is at most r, which is $\left\{ {n \atop k} \right\}_{\vec{w}}$ for the weight vector

$\vec{w} = \{w_1, w_2, \ldots\}$ where $w_i = 1$ if $i \leq r$ and $w_i = 0$ if $i > r$. Recently, Engbergs et al. [7] considered the R-restricted Stirling number of the second kind, that is, the number of partitions of $[n]$ into k nonempty subsets such that the cardinality of each subset is restricted to lie in R. That is, $\left\{ {n \atop k} \right\}_{\vec{w}}$ for the weight vector $\vec{w} = \{w_1, w_2, \ldots\}$ where $w_i = 1$ if $i \in R$ and $w_i = 0$ if $i \notin R$.

At the end of this section, let us compute $\left\{ {n \atop k} \right\}_{\vec{w}}$ for some special cases. It is clear that if $k = 1$, then $\left\{ {\mathbf{n} \atop \mathbf{1}} \right\}_{\vec{w}} = w_n$. If $k = 2$, then for the weight vector $\vec{w} = (w_1, w_2, \ldots)$, we have

$$\left\{ {\mathbf{n} \atop \mathbf{2}} \right\}_{\vec{w}} = \sum_{i=1}^{[\frac{n}{2}]} \binom{n}{i} w_i w_{n-i}.$$

Since we have only two blocks and we can divide the balls into two parts that one of them contains i balls and the other has $n - i$ balls. Blocks are nonempty and unlabeled, therefore the summation is over $1 \leq i \leq [\frac{n}{2}]$.

Moreover, in the case when $\vec{w} = (w_1, w_2, 0, 0, \ldots)$ for nonzero positive integers w_1, w_2, we have

$$\left\{ {\mathbf{n} \atop \mathbf{k}} \right\}_{\vec{w}} = \begin{cases} \dfrac{w_1^{2k-n} w_2^{n-k}}{(n-k)!} \prod_{i=0}^{n-k-1} \binom{n-2i}{2}, & k \leq n \leq 2k, \\ 0 & otherwise. \end{cases}$$

Because, this is the case that the size of each block can be either 1 or 2. Consider $n \leq 2k$, then $n - k$ blocks contain 2 balls and other blocks have just one ball. Thus the statement is clear.

3. Application in graph theory

In this section, we give an application of vector weighted Stirling numbers in graph theory. First, let us remind some main concepts of graph theory, which are necessary for this section. A *simple graph* is a graph that is without any loop and multiple edges. The *diameter* of a simple graph is the length of the largest path between any arbitrary two distinct vertices. A *Hamiltonian graph* is a graph possessing a Hamiltonian cycle and a Hamiltonian cycle is a cycle that visits each vertex exactly once. An *independent set of vertices* in a graph is a set of vertices such that for every two vertices in that set, there is no edge connecting the two vertices. The cardinality of the largest independent vertex set (that is, the size of a maximum independent vertex set) is called the *independence number of a graph*. For more information on graph theory, see [18]. It is a graph associated with a finite set Ω of size n whose vertex set is the power set of Ω and two subsets A and B of Ω are adjacent if and only if $A \cap B \neq \emptyset$.

A special kind of intersection graph which was recently introduced in [8] is *k-intersection graph*. It is denoted by $\Gamma(n, m, k)$ and defined as follows. Let Ω be a set of size n and let Λ be a subset of Ω of size k. The vertex set of $\Gamma(n, m, k)$ is the set of all subsets of Ω of size m, namely, m-subset of Ω. Two distinct m-subsets A and B are adjacent if and only if $A \cap B \nsubseteq \Lambda$. It is shown that a k-intersection graph is nonempty if and only if $n \geq m$ and that it is connected if and only if $m > k$. Moreover, when $\Gamma(n, m, k)$ is connected, then its diameter is less than or equal to 2 and $\Gamma(n, m, k)$ is complete if and only if $2(m - k) > n - k$. For more results on k-intersection graphs, we refer the reader to [8].

Let $\Gamma(n, m, k)$ be a k-intersection graph. For a given positive integer ℓ, we are going to find the number of different ways of choosing maximal independent vertex sets of size ℓ in $\Gamma(n, m, k)$. We note that a maximal independent set of vertices is a set of vertices that all of them are independent and one cannot add any other vertex to them such that they stay independent. Let $\mathbf{I}_\ell(G)$ be the set of all different ℓ-maximal independent sets in G and put $I_\ell(G) = |\mathbf{I}_\ell(G)|$. As we mentioned earlier in this section, for a graph G, the size of the largest independent vertex set is called *independence number of* G and is shown by $\alpha(G)$.

In this paper, we always assume that $n > m$ and without loss of generality, let $\Omega = \{1, \dots, k, \dots, n\}$ and let $\Lambda = \{1, \dots, k\}$. When there is no ambiguity, we shortly write Γ instead of $\Gamma(n, m, k)$.

The following proposition computes the independence number of $\Gamma(n, m, k)$.

Proposition 3.1. *The independence number of* $\Gamma(n, m, k)$ *is* $[\frac{n-k}{m-k}]$.

Proof. Consider all vertices of $\Gamma(n, m, k)$ that contain $\{1, \dots, k\}$ and other members of them are disjoint, that is, all m-subsets of the form $V = \{1, 2, \dots, k, v_{k+1}, \dots, v_m\}$ with disjoint $\{v_{k+1}, \dots, v_m\}$. The size of the set consisting of these vertices is $[\frac{n-k}{m-k}]$. Clearly, it is an independent set and there is no greater independent set of vertices of $\Gamma(n, m, k)$. $\square$

Proposition 3.2. *The necessary and sufficient condition that* $\Gamma(n, m, k)$ *has at least one* ℓ*-maximal independent set is*

$$k(1 - \ell) \leqslant n - m\ell < m.$$

Proof. It is clear that the necessary and sufficient condition to have an ℓ-independent set of vertices is $(m - k)\ell \leq n - k$. Moreover, to have at least one ℓ-maximal independent set, it is necessary that $n - k - m\ell < m - k$. Therefore the necessary and sufficient condition for this property is $k(1 - \ell) \leqslant n - m\ell < m$. $\square$

As an example, let $G = \Gamma(6, 3, 2)$. For the sake of brevity, we may write each vertex as abc, instead of $\{a, b, c\}$. It is clear that G has 20 vertices, 144 edges, and $\alpha(G) = 4$. So $V(G) = \{345, 346, 356, 456, 134, 234, 135, 235, 136, 236, 145, 245, 146, 246, 156, 256, 123, 124, 125, 126\}$.

For $\ell = 2, 3, 4$ we have

$$\begin{aligned}
\mathbf{I}_4(G) &= \{(123, 124, 125, 126)\} \\
\mathbf{I}_3(G) &= \{(123, 124, 156), (123, 124, 256), (123, 125, 146), (123, 125, 246), \\
&\quad (123, 126, 145), (123, 126, 245), (124, 125, 134), (124, 125, 234), \\
&\quad (124, 126, 135), (124, 126, 235), (125, 126, 134), (125, 126, 234)\} \\
\mathbf{I}_2(G) &= \{(134, 156), (134, 256), (234, 156), (234, 256), (135, 146), (135, 246), \\
&\quad (235, 146), (235, 246), (136, 145), (136, 245), (236, 145), (236, 245), \\
&\quad (123, 456), (124, 356), (125, 346), (126, 345)\}.
\end{aligned}$$

Therefore $I_4(G) = 1,\ I_3(G) = 12,\ I_2(G) = 16$, and $I_i(G) = 0$ for $i \neq 2, 3, 4$.

In the rest of this section, we have two different approaches to find $I_\ell(\Gamma(n, m, k))$. First, we define a new graph $\tilde{\Gamma}(n, m, k)$ and compute $I_\ell(\tilde{\Gamma}(n, m, k))$. Then we give a formula for

$I_\ell(\Gamma(n,m,k))$ by means of the relation between $\Gamma(n,m,k)$ and $\tilde{\Gamma}(n,m,k)$. The second approach, which is shorter and easier, uses weighted Stirling numbers of the second kind to demonstrate a recurrence relation for $I_\ell(\Gamma(n,m,k))$.

Let V_1 and V_2 be two different vertices of $\Gamma(n,m,k)$ such that $V_1 \cap \{k+1,\ldots,n\} = V_2 \cap \{k+1,\ldots,n\}$. Then these two vertices have exactly the same manner in $\Gamma(n,m,k)$, that is, for any vertex V of $\Gamma(n,m,k)$, V_1 is adjacent to V if and only if V_2 is adjacent to V. We define an equivalence relation on vertices of $\Gamma(n,m,k)$ as follows.

Two vertices V_1 and V_2 are equivalent if and only if $V_1 \cap \{k+1,\ldots,n\} = V_2 \cap \{k+1,\ldots,n\}$, which we show it by $V_1 \sim V_2$. One can easily check that $\sim$ is an equivalence relation on the set of vertices of $\Gamma(n,m,k)$.

Definition 3.3. For a given k-intersection graph $\Gamma(n,m,k)$, we define a new graph $\tilde{\Gamma}(n,m,k)$ in such a way that the vertices of $\tilde{\Gamma}(n,m,k)$ are equivalent classes of $\Gamma(n,m,k)$ under $\sim$. Two different vertices of $\tilde{\Gamma}(n,m,k)$, say $[V_1]$ and $[V_2]$ are adjacent in $\tilde{\Gamma}(n,m,k)$ if and only if V_1 and V_2 are adjacent in $\Gamma(n,m,k)$.

The number of vertices and edges of graph $\tilde{\Gamma}(n,m,k)$ is much less than the vertices and edges of $\Gamma(n,m,k)$. In fact, since all vertices in an equivalence class have the same manner in relation to other vertices of $\Gamma(n,m,k)$, we can put one representative of each class in $\tilde{\Gamma}(n,m,k)$. The following proposition gives a recurrence relation for $I_\ell(\tilde{\Gamma}(n,m,k))$.

Proposition 3.4. *For positive integers n, m, k, and ℓ with $n \geq m > k$ and $k(1-\ell) \leqslant n-m\ell < m$, we have*

$$I_\ell(\tilde{\Gamma}(n,m,k)) = \sum_{i=m-k}^{\min\{m,n-k\}} \sum_{j=1}^{\ell} \frac{(n-k)!}{(i!)^j (n-k-ij)!} I_{\ell-j}(\tilde{\Gamma}(n-ij-1,m,k-1))$$

with the initial assumption

$$I_0(\tilde{\Gamma}(n,m,k)) = \begin{cases} 1, & n < m, \\ 0, & n \geq m, \end{cases}$$

and

$$I_\ell(\tilde{\Gamma}(n,m,0)) = \begin{cases} \dfrac{n!}{\ell!(m!)^\ell (n-ml)!}, & [\frac{n}{m}] = \ell, \\ 0 & otherwise. \end{cases}$$

Proof. Consider a vertex $V = \{v_1,\ldots,v_m\}$ of $\Gamma(n,m,k)$, where $v_i \in \{1,\ldots,n\}$. The members of each vertex can be divided into two parts. First, the *main* part that contains those elements belonging to $\{k+1,\ldots,n\}$ and second, the *null* part that contains those elements belonging to $\{1,\ldots,k\}$. The reason for these names is that members that belong to $\{1,\ldots,k\}$ have no effect on the adjacency of vertices. Note that under the equivalence relation $\sim$, the size of the null part of all members of an equivalence class is the same as each other. The size of the null part and the main part of each vertex of $\tilde{\Gamma}$ is equal to the size of the null part and the main part of the representative element of that class, respectively.

For each vertex of Γ, like V, the size of the main part varies between $m-k$ and $\min\{m, n-k\}$. Let L be an ℓ-maximal independent set of $\tilde{\Gamma}(n,m,k)$ and let i be the minimum size of main part of vertices of L. An ℓ-maximal independent set may contain j vertices with main part size equaling i, where $1 \leq j \leq \ell$. It can be easily seen that the number of different ways to choose j sets of $\tilde{\Gamma}(n,m,k)$ with i main members, is equal to $\dfrac{(n-m)!}{(i!)^j(n-k-ij)!}$.

Finally, since the size of the main part of other members of L is greater than i, the number of different ways of choosing other $\ell - j$ members of L is equal to the number of different $(\ell - j)$-maximal independent sets of graph $\tilde{\Gamma}(n-ij-1, m, k-1)$ and the proof is completed.

□

Example 3.5. Consider the graph $H = \tilde{\Gamma}(6,3,2)$. Then, H has 14 vertices and 66 edges. Thus

$$V(H) = \{[345], [346], [356], [456], [134], [135], [136], [145], [146], [156], [123], [124], [125], [126]\}.$$

For $\ell = 2, 3, 4$, $\mathbf{I}_\ell(H)$ is nonempty and we have $I_4(H) = 1,\ I_3(H) = 6$, and $I_2(H) = 7$ as follows:

$$\begin{aligned}
\mathbf{I}_4(H) &= \{([123], [124], [125], [126])\},\\
\mathbf{I}_3(H) &= \{([123], [124], [156]), ([123], [125], [146]), ([123], [126], [145]),\\
&\quad ([124], [125], [134]), ([124], [126], [135]), ([125], [126], [134])\},\\
\mathbf{I}_2(H) &= \{([134], [156]), ([135], [146]), ([136], [145]), ([123], [456]),\\
&\quad ([124], [356]), ([125], [346]), ([126], [345])\}.
\end{aligned}$$

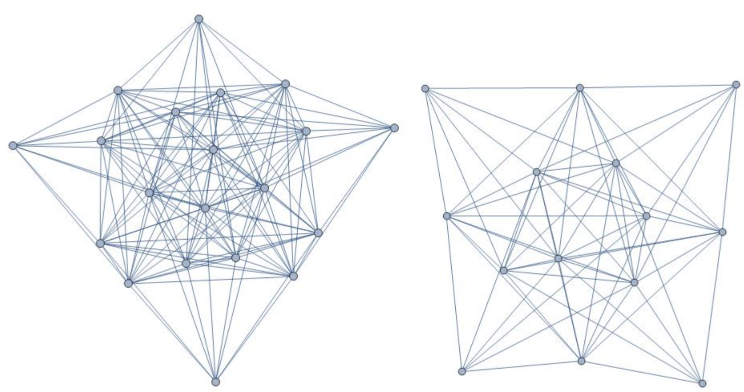

Figure 1. $\Gamma(6,3,2)$ and $\tilde{\Gamma}(6,3,2)$

Now, we define the map $f : V(\Gamma(n,m,k)) \to \mathbb{N}$ such that $f(V) = |[V]|$ for every vertex V, where $|[V]|$ is the size of the equivalence class of V under the equivalence relation $\sim$. By using the function f, we can give a formula for $I_\ell(\Gamma(n,m,k))$ in the following theorem.

Theorem 3.6. *We have*

$$I_\ell(\Gamma(n,m,k)) = \sum_{L \in \boldsymbol{I}_\ell(\tilde{\Gamma})} \prod_{v_i \in L} f(v_i).$$

Proof. Assume that $L = \{[V_1], \ldots, [V_\ell]\}$ is an arbitrary ℓ-maximal independent set of $\tilde{\Gamma}(n, m, k)$. Then one can easily check that

$$\{[V_1], \ldots, [V_\ell]\} \in \mathbf{I}_\ell(\tilde{\Gamma}(n, m, k)) \Leftrightarrow \{V_1, \ldots, V_\ell\} \in \mathbf{I}_\ell(\Gamma(n, m, k)).$$

More generally, if we replace $V_1, V_2, \ldots, V_\ell$ by $U_1, U_2, \ldots, U_\ell$, where $U_1 \in [V_1], \ldots, U_\ell \in [V_\ell]$, then we have the same property, again. Since the size of equivalence classes $[V_1], [V_2], \ldots, [V_\ell]$ are $f(V_1), f(V_2), \ldots, f(V_\ell)$, respectively, we can get the result directly. □

As we mentioned at the beginning of this section, we can compute $I_\ell(\Gamma(n, m, k))$ in two different ways. The first way is using the graph $\tilde{\Gamma}(n, m, k)$, which we stated in Theorem 3.6. Now, the second way is counting l-maximal independent sets of $\Gamma(n, m, k)$ directly without using the graph $\tilde{\Gamma}(n, m, k)$. The following theorem gives the new formula for $I_\ell(\Gamma(n, m, k))$ by means of vector weighted partitioning.

Theorem 3.7. *For positive integers $n,\ m$, and k, let $\Gamma(n, m, k)$ be a k-intersection graph. Then, for a given positive integer ℓ, we have*

$$I_\ell(\Gamma(n, m, k)) = \sum_{i=0}^{s} \binom{n-k}{i} \left\{ \begin{matrix} n-k-i \\ \ell \end{matrix} \right\}_{\vec{w}},$$

where $s = m - k - 1$ and $\vec{w} = (\underbrace{0, \ldots, 0}_{m-k-1}, \binom{k}{k}, \ldots, \binom{k}{1}, \binom{k}{0}, 0, 0, \ldots)$.

Proof. Let $L = \{V_1, \ldots, V_\ell\}$ be a maximal independent set of ℓ vertices, where each V_i is an m-subset of $\{1, \ldots, n\}$. Let $S = \cup_{j=1}^{\ell} V_j \setminus \{1, \ldots, k\}$, and consider $i =\mid \{k+1, \ldots, n\} \setminus S \mid$. Since L is a maximal independent set of vertices of Γ, therefore i ranges from 0 to $m - k - 1 = s$. To count the number of different ℓ-maximal independent sets for fixed i, we choose i numbers from $\{k+1, \ldots, n\}$ to be aside in $\binom{n-k}{i}$ different ways and then we partition the remains into ℓ nonempty subsets in some restrictions. Each subset contains at least $m - k$ and at most m members such that by adding some members of $\{1, \ldots, k\}$ it become an m-subset of $\{1, \ldots, n\}$. On the other hand, whereas for each subset of size r of the above partition, there are $\binom{k}{m-r}$ vertices that their behavior is as the same as each other, so a partition of size r should be counted $\binom{k}{m-r}$ times. Therefore the answer is a vector weighted partitioning with weight vector $\vec{w} = (0, \ldots, 0, \binom{k}{k}, \ldots, \binom{k}{1}, \binom{k}{0}, 0, 0, \ldots)$. Since i runs from 0 to $m - k - 1$, we have a sum over i. □

Example 3.8. Let $G = \Gamma(6, 3, 2)$. The number of 2-maximal independent sets of G is $I_\ell(\Gamma(6, 3, 2)) = \binom{4}{0} \left\{ \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \right\}_{\vec{w}}$ where $\vec{w} = (1, 2, 1, 0, 0, \ldots)$.

$$\begin{aligned}
\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\}_{\vec{w}} &= \sum_{i=1}^{3} w_i \binom{3}{i-1} \left\{ \begin{matrix} 4-i \\ 1 \end{matrix} \right\}_{\vec{w}} \\
&= w_1 \binom{3}{0} \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\}_{\vec{w}} + w_2 \binom{3}{1} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\}_{\vec{w}} + w_3 \binom{3}{2} \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}_{\vec{w}} \\
&= 1 + 12 + 3 \\
&= 16.
\end{aligned}$$

Finally, we remind that for a graph G, the polynomial $I(x) = \sum_{\ell=0}^{\alpha(G)} a_\ell x^\ell$ is called *independence polynomial of* G, where a_ℓ is the number of independent vertex sets of cardinality ℓ in the graph G (see [12] for more details). For a k-intersection graph, $I_\ell(\Gamma)$ is the number of maximal independent vertex sets of size ℓ as we computed $I_\ell(\Gamma)$ in Theorems 3.6 and 3.7. In a similar way, we can compute the coefficients of x_ℓ in the independence polynomial of $\Gamma(n, m, k)$ by using vector weighted Stirling numbers of the second kind as follows.

Proposition 3.9. *For positive integers n, m and k, let $\sum_{\ell=0}^{\alpha(G)} a_\ell x^\ell$ be the independence polynomial of $\Gamma(n, m, k)$. Then*

$$a_\ell = \sum_{e=\ell(m-k)}^{n-k} \binom{n-k}{e} \left\{ {e \atop \ell} \right\}_{\vec{w}},$$

where $\vec{w} = (\underbrace{0, \ldots, 0}_{m-k-1}, \binom{k}{k}, \ldots, \binom{k}{1}, \binom{k}{0}, 0, 0, \ldots)$.

The proof is very similar to the proof of Theorem 3.7 and we omit it here. One should note that the desired ℓ-independent sets are not necessarily maximal.

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