Electronic Journal of Graph Theory and Applications **10** (1) (2022), 33–50

Electronic Journal of
Graph Theory and Applications

The strong 3-rainbow index of edge-comb product of a path and a connected graph

Zata Yumni Awanis[a], A.N.M. Salman[a], Suhadi Wido Saputro[a]

[a]*Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences,*
Institut Teknologi Bandung, Jalan Ganesa 10, Bandung 40132, Indonesia

zata.yumni@s.itb.ac.id, msalman@math.itb.ac.id, suhadi@math.itb.ac.id

---

Abstract

A tree in an edge-colored connected graph G is a rainbow tree if all of its edges have different colors. Let k be an integer with $2 \leq k \leq n$ and S be a k-subset of $V(G)$. The strong k-rainbow index $srx_k(G)$ of G is the smallest number of colors required in an edge-coloring of G such that every set S in G is connected by a rainbow tree with minimum size. In this paper, we investigate the srx_3 of edge-comb product of a path and a connected graph, denoted by $P_n^o \rhd_{\vec{e}} H$. It is obvious that the natural upper bound for $srx_3(P_n^o \rhd_{\vec{e}} H)$ is $|E(P_n^o \rhd_{\vec{e}} H)|$. Hence, we first provide graphs H with $srx_3(P_n^o \rhd_{\vec{e}} H) = |E(P_n^o \rhd_{\vec{e}} H)|$, then provide a sharper upper bound for $srx_3(P_n^o \rhd_{\vec{e}} H)$ where $srx_3(P_n^o \rhd_{\vec{e}} H) \neq |E(P_n^o \rhd_{\vec{e}} H)|$. We also provide the exact values of $srx_3(P_n^o \rhd_{\vec{e}} H)$ for some graphs H.

Keywords: edge-comb product, rainbow coloring, rainbow Steiner tree, strong 3-rainbow index
Mathematics Subject Classification : 05C05, 05C15, 05C40
DOI: 10.5614/ejgta.2022.10.1.3

---

1. Introduction

Throughout this paper, all graphs are finite, simple, and connected. The terminology and notation refer to Diestel [11]. For simplifying, we define a set $[a, b] = \{x : a \leq x \leq b\}$. Let $G(V, E)$ be an edge-colored graph of order $n \geq 3$. A tree in G is a *rainbow tree* if all of its edges have different colors. Let k be an integer with $k \in [2, n]$. The smallest number of colors required in an edge-coloring of G such that every k-subset S of $V(G)$ is connected by a rainbow tree is

---

Received: 2 September 2020, Revised: 14 July 2021, Accepted: 5 October 2021.

called the *k-rainbow index* $rx_k(G)$ of G. These concepts were first proposed by Chartrand et al. in 2010 [9]. If $k = 2$, then $rx_2(G) = rc(G)$, where $rc(G)$ denotes the *rainbow connection number* of G [8]. Hence, $rc(G) = rx_2(G) \leq rx_3(G) \leq \cdots \leq rx_n(G)$. Caro et al. [6] conjectured that determining the rainbow connection number of graphs is an NP-Hard problem. Chakraborty et al. in [7] then confirmed this conjecture. Therefore, the determination of rainbow connection number is mostly done by limiting the study to certain classes of graphs. The readers can see [8, 12, 14, 15, 16, 18, 19, 20] for more results about the rainbow connection number of graphs.

The concept of k-rainbow index has useful and interesting applications in the security of a communication network. Suppose that every k people are expected to communicate and exchange information securely. To achieve this, we can assign passwords to the line which connects them (which may have other people as intermediaries) so that no passwords are repeated. Since the economic aspect is taken into consideration, the number of passwords that being used are expected to be as minimum as possible. The k-rainbow index represents the smallest number of these distinct passwords.

For two vertices $x, y \in V(G)$, the length of a shortest $x - y$ path in G is called the *distance* between x and y, denoted by $d(x, y)$. The largest distance between two vertices of G is called the *diameter* of G, denoted by $diam(G)$. The *Steiner distance* $d(S)$ of S is the minimum size of a tree containing S. The *k-Steiner diameter* of G, denoted by $sdiam_k(G)$, is the maximum Steiner distance of S among all sets S in G. If $S = \{x, y\}$, then $d(S) = d(x, y)$ and $sdiam_2(G) = diam(G)$. Chartrand et al. [9] stated that for any graph G of order $n \geq 3$ and each integer $k \in [3, n]$, $k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1$. They also determined the rx_k of trees and cycles, where the rx_k of trees is equal to the upper bound for $rx_k(G)$. The first and second authors [4] investigated the rx_3 of amalgamation of some graphs of diameter 2, meanwhile Liu and Hu [17] investigated the rx_3 of three graph product operations, which are strong product, Cartesian product, and lexicographic product. Some other results about rx_k of graphs can be found in [9, 10, 13, 15, 16].

In [3], we proposed a new concept called a strong k-rainbow index. An edge-coloring of G is called a *strong k-rainbow coloring* if every set S in G is connected by rainbow tree of size $d(S)$. Such a tree is called a *rainbow Steiner S-tree*. A rainbow Steiner S-tree is called a *rainbow $x - y$ geodesic* if $S = \{x, y\}$ [8]. The *strong k-rainbow index* of G, denoted by $srx_k(G)$, is the smallest number of colors required such that G admits a strong k-rainbow coloring. Following the definition, $rx_k(G) \leq srx_k(G)$. If $k = 2$, then $srx_2(G) = src(G)$, where $src(G)$ denotes the *strong rainbow connection number* of G [8]. Therefore, $src(G) = srx_2(G) \leq srx_3(G) \leq \cdots \leq srx_n(G)$ for any graph G of order $n \geq 2$. Chartrand et al. [8] stated that $diam(G) \leq rc(G) \leq src(G) \leq |E(G)|$.

It is clearly that for any connected graph G, the strong k-rainbow index is defined for G since every edge-coloring that assigns different colors to the edges of G is a strong k-rainbow coloring. Thus, we have

$$sdiam_k(G) \leq rx_k(G) \leq srx_k(G) \leq |E(G)|. \tag{1}$$

Graph operations have an important rule in making a larger and complex communication network. Hence, we investigated the srx_3 of amalgamation and comb product of some graphs. We also investigated the srx_3 of some certain graphs (see [1, 2, 3]). The following theorems are needed.

Theorem 1.1. *[3] Let T_n be a tree of order $n \geq 3$. Then $srx_3(T_n) = |E(T_n)| = n - 1$.*

Theorem 1.2. *[3] Let L_n be a ladder graph of order $2n$ ($n \geq 3$). Then $srx_3(L_n) = n$.*

Theorem 1.3. *[3] Let $K_{n,n}$ be a regular complete bipartite graph of order $2n$ ($n \geq 3$). Then $srx_3(K_{n,n}) = n$.*

Theorem 1.4. *[3] Let C_n be a cycle of order $n \geq 3$. Then*

$$srx_3(C_n) = \begin{cases} 2, & \text{for } n = 3; \\ n-2, & \text{for } n \in \{4,5,6,8\}; \\ n, & \text{otherwise}. \end{cases}$$

Figure 1 illustrates the strong 3-rainbow colorings of C_n for $n \in [3, 6]$ and $n = 8$.

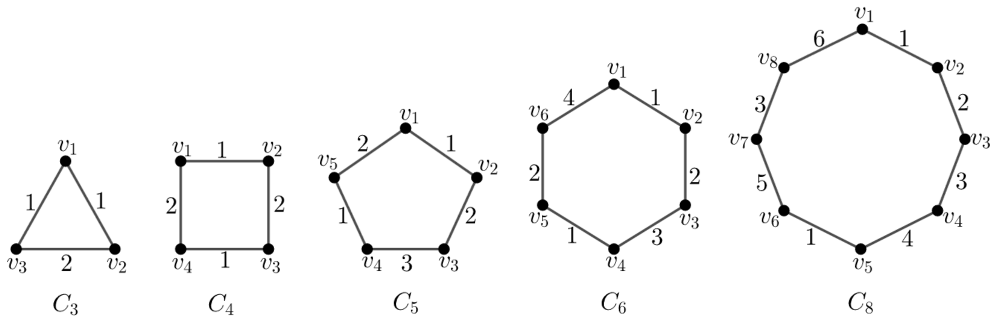


Figure 1. Strong 3-rainbow colorings of C_n for $n \in [3, 6]$ and $n = 8$.

Theorem 1.5. *[1] Let F_n be a fan of order $n + 1$ ($n \geq 3$). Then*

$$srx_3(F_n) = \begin{cases} 3, & \text{for } n = 4; \\ \lceil \frac{n}{2} \rceil, & \text{otherwise}. \end{cases}$$

The following definition of edge-comb product of two graphs is referred to [5]. Given an undirected graph G, an *orientation* of G is an assignment of a direction to every edge of G. Let G and H be two connected graphs. Let O be an orientation of G and $\vec{e}$ be an oriented edge of H. The *edge-comb product* of G and H on $\vec{e}$ (under the orientation O), denoted by $G^o \rhd_{\vec{e}} H$, is a graph formed by taking one copy of G and $|E(G)|$ copies of H and identifying the i-th copy of H at the edge $\vec{e}$ to the i-th edge of G, where the two edges have the same orientation.

In this paper, we investigate the strong 3-rainbow index of $P_n^o \rhd_{\vec{e}} H$. In Section 2, we first provide graphs H with $srx_3(P_n^o \rhd_{\vec{e}} H) = |E(P_n^o \rhd_{\vec{e}} H)|$, then we provide a sharper upper bound for $srx_3(P_n^o \rhd_{\vec{e}} H)$. In Section 3, we determine the exact value of $srx_3(P_n^o \rhd_{\vec{e}} H)$ for some graphs H. In Section 4, we give concluding remarks and some open problems for further investigation.

2. Sharp upper bound for $srx_3(P_n^o \triangleright_{\vec{e}} H)$

For two integers $n, m \geq 3$, let P_n^o be a path $P_n = v_1v_2 \dots v_n$ of order n with orientation O, where every edge of P_n has an orientation from v_i to v_{i+1} for each $i \in [1, n-1]$, and H be a connected graph of order m with $V(H) = \{w_1, w_2, \dots, w_m\}$ and $\vec{e} = w_aw_b$ be an oriented edge of H which has an orientation from w_a to w_b. Now, we consider graphs $P_n^o \triangleright_{\vec{e}} H$. For $i \in [1, n-1]$, let the i-th copy of H is denoted by H^i with $V(H^i) = \{v_i^1, v_i^2, \dots, v_i^m\}$ and $E(H^i) = \{v_i^pv_i^q \colon p, q \in [1, m] \text{ and } w_pw_q \in E(H)\}$. We define $V(P_n^o \triangleright_{\vec{e}} H) = \bigcup_{i=1}^{n-1} V(H^i)$ and $E(P_n^o \triangleright_{\vec{e}} H) = \bigcup_{i=1}^{n-1} E(H^i)$, where $v_i^a = v_i$ and $v_i^b = v_{i+1}$ for each $i \in [1, n-1]$.

Let $X \subseteq E(P_n^o \triangleright_{\vec{e}} H)$. For further discussion, if c is a strong 3-rainbow coloring of $P_n^o \triangleright_{\vec{e}} H$, then the set of colors assigned to the edges in X is denoted by $c(X)$. By considering any three vertices of P_n and using Theorem 1.1, we have

$$|c(E(P_n))| = n - 1. \tag{2}$$

According to (1), the natural upper bound for $srx_3(P_n^o \triangleright_{\vec{e}} H)$ is $|E(P_n^o \triangleright_{\vec{e}} H)|$. The following theorem shows that $srx_3(P_n^o \triangleright_{\vec{e}} T_m) = |E(P_n^o \triangleright_{\vec{e}} T_m)|$.

Theorem 2.1. *For two integers $n, m \geq 3$, let P_n and T_m be a path of order n and a tree of order m, respectively. Let $\vec{e}$ be any oriented edge of T_m. Then $srx_3(P_n^o \triangleright_{\vec{e}} T_m) = (m-1)(n-1)$.*

Proof. Note that $P_n^o \triangleright_{\vec{e}} T_m$ is a tree with $|E(P_n^o \triangleright_{\vec{e}} T_m)| = |E(T_m)|(n-1)$, thus $srx_3(P_n^o \triangleright_{\vec{e}} T_m) = |E(P_n^o \triangleright_{\vec{e}} T_m)| = |E(T_m)|(n-1) = (m-1)(n-1)$ by Theorem 1.1. $\square$

Following theorem above, a natural thought arises: Is there any nontrivial graph H of order m besides a tree with $srx_3((P_n^o \triangleright_{\vec{e}} H) = |E(P_n^o \triangleright_{\vec{e}} H)|$? The next theorem provides the characterization of connected graphs H with $srx_3((P_n^o \triangleright_{\vec{e}} H) = |E(P_n^o \triangleright_{\vec{e}} H)|$.

Theorem 2.2. *For two integers $n, m \geq 3$, let P_n and H be a path of order n and a connected graph of order m, respectively. Let $\vec{e}$ be any oriented edge of H. Then H is a tree if and only if $srx_3(P_n^o \triangleright_{\vec{e}} H) = |E(P_n^o \triangleright_{\vec{e}} H)|$.*

Proof. Let H be a tree. Then by using Theorem 2.1, $srx_3(P_n^o \triangleright_{\vec{e}} T_m) = |E(P_n^o \triangleright_{\vec{e}} T_m)|$.

Conversely, let H be a connected graph with $srx_3(P_n^o \triangleright_{\vec{e}} T_m) = |E(P_n^o \triangleright_{\vec{e}} T_m)|$ and not a tree. Hence, graph H contains cycles. Let $g \geq 3$ be the girth of H. For $i \in [1, n-1]$, let C_g^i be a cycle of length g in H^i. Since $n \geq 3$, consider graphs H^1 and H^2. For each $i \in [1, 2]$, relabeling vertices of H^i such that $V(C_g^i) = \{v_i^1, v_i^2, \dots, v_i^g\}$, $E(C_g^i) = \{v_i^pv_i^{p+1} \colon p \in [1, g] \text{ and } v_i^{g+1} = v_i^1\}$, and $d_{H^i}(v_2, v_i^1) \leq d_{H^i}(v_2, v_i^p)$ for all $p \in [2, g]$. For further discussion, let $d_{H^i}(v_2, v_i^p) = l_i^p$ for each $i \in [1, 2]$ and $p \in [1, g]$. Thus, by assumption, we have $l_i^p \in [l_i^1, l_i^1 + p - 1]$ for $p \in [1, \lfloor \frac{g}{2} \rfloor + 1]$ and $l_i^p \in [l_i^1, l_i^1 + g - p + 1]$ for $p \in [\lfloor \frac{g}{2} \rfloor + 2, g]$.

The following two cases show that there is an edge of C_g^i for each $i \in [1, 2]$ which is not contained in a $v_2 - v_i^t$ geodesic for any $t \in [1, g]$.

Case 1. $v_2 \in V(C_g^i)$ for some $i \in [1, 2]$

It means $v_2 = v_i^1$. Thus, we have $l_i^p = p-1$ for $p \in [1, \lfloor \frac{g}{2} \rfloor + 1]$ and $l_i^p = g-p+1$ for $p \in [\lfloor \frac{g}{2} \rfloor + 2, g]$. If g is odd, then $v_i^{\lfloor \frac{g}{2} \rfloor+1} v_i^{\lfloor \frac{g}{2} \rfloor+2}$ is not contained in a $v_2 - v_i^t$ geodesic for any $t \in [1, g]$. If g is even, then there are two $v_2 - v_i^{\lfloor \frac{g}{2} \rfloor+1}$ geodesics, one path contains $v_i^{\lfloor \frac{g}{2} \rfloor} v_i^{\lfloor \frac{g}{2} \rfloor+1}$ and another path contains $v_i^{\lfloor \frac{g}{2} \rfloor+1} v_i^{\lfloor \frac{g}{2} \rfloor+2}$. Hence, we can choose $v_i^{\lfloor \frac{g}{2} \rfloor+1} v_i^{\lfloor \frac{g}{2} \rfloor+2}$ to be an edge that is not contained in a $v_2 - v_i^{\lfloor \frac{g}{2} \rfloor+1}$ geodesic. Furthermore, $v_i^{\lfloor \frac{g}{2} \rfloor+1} v_i^{\lfloor \frac{g}{2} \rfloor+2}$ is not contained in a $v_2 - v_i^t$ geodesic for any $t \in [1, g]$.

Case 2. $v_2 \notin V(C_g^i)$ for some $i \in [1, 2]$

We define several sets as follows.

- For odd g, let $V_i^{1,1}$ be a set of (v_i^p, v_i^q) such that $v_i^p, v_i^q \in V(C_g^i)$ and $l_i^p = l_i^q$ for distinct $p, q \in [1, \lfloor \frac{g}{2} \rfloor + 1]$, and $V_i^{1,2}$ be a set of (v_i^p, v_i^q) such that $v_i^p, v_i^q \in V(C_g^i)$ and $l_i^p = l_i^q$ for distinct $p, q \in \{1\} \cup [\lfloor \frac{g}{2} \rfloor + 2, g]$.

- For even g, let $V_i^{2,1}$ be a set of (v_i^p, v_i^q) such that $v_i^p, v_i^q \in V(C_g^i)$ and $l_i^p = l_i^q$ for distinct $p, q \in [1, \lfloor \frac{g}{2} \rfloor + 1]$, and $V_i^{2,2}$ be a set of (v_i^p, v_i^q) such that $v_i^p, v_i^q \in V(C_g^i)$ and $l_i^p = l_i^q$ for distinct $p, q \in \{1\} \cup [\lfloor \frac{g}{2} \rfloor + 1, g]$.

Note that regardless the parity of g, we have either Subcase 2.1 or 2.2 as follows.

Subcase 2.1. $|V_i^{r,s}| \geq 1$ for some $s \in [1, 2]$

Choose a pair $(v_i^p, v_i^q) \in V_i^{r,s}$ so that $d_{C_g^i}(v_i^p, v_i^q)$ has the smallest value. Thus, we have $d_{C_g^i}(v_i^p, v_i^q)$ is 1 or 2, since there is another pair $(v_i^{p'}, v_i^{q'}) \in V_i^{r,s}$ such that $d_{C_g^i}(v_i^{p'}, v_i^{q'}) < d_{C_g^i}(v_i^p, v_i^q)$ if $d_{C_g^i}(v_i^p, v_i^q) \geq 3$, contradicts the assumption. If $d_{C_g^i}(v_i^p, v_i^q) = 1$, then $v_i^p v_i^q$ is not contained in a $v_2 - v_i^p$ geodesic and a $v_2 - v_i^q$ geodesic. This implies $v_i^p v_i^q$ is also not contained in a $v_2 - v_i^t$ geodesic for any $t \in [1, g]$. If $d_{C_g^i}(v_i^p, v_i^q) = 2$, then there is $v_i^k \in V(C_g^i)$ such that $v_i^p v_i^k, v_i^k v_i^q \in E(C_g^i)$ and $l_i^k = l_i^p + 1$. Hence, there are two $v_2 - v_i^k$ geodesics, one path contains $v_i^p v_i^k$ and another path contains $v_i^k v_i^q$. Similar to Case 1 for even g, edge $v_i^p v_i^k$ can be chosen to be an edge that is not contained in a $v_2 - v_i^t$ geodesic for any $t \in [1, g]$.

Subcase 2.2. $|V_i^{r,s}| = 0$ for all $s \in [1, 2]$

Since $|V_i^{r,s}| = 0$ for all $s \in [1, 2]$, we have $l_i^p = l_i^1 + p - 1$ for $p \in [1, \lfloor \frac{g}{2} \rfloor + 1]$ and $l_i^p = l_i^1 + g - p + 1$ for $p \in [\lfloor \frac{g}{2} \rfloor + 2, g]$. Thus, similar to Case 1, we obtain that $v_i^{\lfloor \frac{g}{2} \rfloor+1} v_i^{\lfloor \frac{g}{2} \rfloor+2}$ is not contained in a $v_2 - v_i^t$ geodesic for any $t \in [1, g]$.

Let S be a 3-subset of $V(P_n^o \rhd_{\vec{e}} H)$. According to Cases 1 and 2, there is an edge $e_i \in E(C_g^i)$ for each $i \in [1, 2]$ such that e_i is not contained in a $v_2 - v_i^t$ geodesic for any $t \in [1, g]$. Therefore, by assigning the color 1 to the edges e_1 and e_2 and the colors $2, 3, \ldots, |E(P_n^o \rhd_{\vec{e}} H)| - 1$ to the remaining $|E(P_n^o \rhd_{\vec{e}} H)| - 2$ edges of $P_n^o \rhd_{\vec{e}} H$, there is a rainbow Steiner S-tree in $P_n^o \rhd_{\vec{e}} H$. Hence, $srx_3(P_n^o \rhd_{\vec{e}} H) \leq |E(P_n^o \rhd_{\vec{e}} H)| - 1$, contradicts the assumption. $\square$

According to Theorem 2.2, graph $P_n^o \rhd_{\vec{e}} T_m$ is the only graph whose srx_3 is equal to its size. The following theorem provides a sharper upper bound for $srx_3(P_n^o \rhd_{\vec{e}} H)$.

Theorem 2.3. *For two integers $n, m \geq 3$, let P_n and H be a path of order n and a connected graph of order m, respectively. Let $\vec{e}$ be any oriented edge of H. Then*

$$srx_3(P_n^o \rhd_{\vec{e}} H) \leq srx_3(H)(n-1).$$

Proof. Let $\vec{e} = w_a w_b$. For $i \in [1, n-1]$, we color all edges of H^i with $srx_3(H)$ colors so that each H^i admits a strong 3-rainbow coloring where $c(E(H^i)) \cap c(E(H^j)) = \emptyset$ for all $j \in [1, n-1]$ with $j \neq i$. According to the definition, graph $P_n^o \rhd_{\vec{e}} H$ can be formed by identifying vertices v_i^b and v_{i+1}^a for each $i \in [1, n-1]$. Since each H^i admits a strong 3-rainbow coloring with $c(E(H^i)) \cap c(E(H^j)) = \emptyset$ for distinct $i, j \in [1, n-1]$, there is a rainbow Steiner S-tree for every 3-subset S of $V(P_n^o \rhd_{\vec{e}} H)$. Thus, $srx_3(P_n^o \rhd_{\vec{e}} H) \leq srx_3(H)(n-1)$. $\square$

Since $srx_3(T_m) = m-1$ by Theorem 1.1, it follows by Theorem 2.1 that $srx_3(P_n^o \rhd_{\vec{e}} T_m)$ is also equal to the upper bound given in Theorem 2.3. Thus, the upper bound is sharp. There are other graphs H such that $srx_3(P_n^o \rhd_{\vec{e}} H) = srx_3(H)(n-1)$. These results are given in Section 3.

3. The strong 3-rainbow index of $P_n^o \rhd_{\vec{e}} H$ for some connected graphs H

Our first two results show that there are two connected graphs H such that $srx_3(P_n^o \rhd_{\vec{e}} H) = srx_3(H)(n-1)$.

For a *ladder* graph L_m of order $2m$ ($m \geq 3$), we define $V(L_m) = \{w_i \colon i \in [1, 2m]\}$ and $E(L_m) = \{w_i w_{i+1} \colon i \in [1, m-1] \cup [m+1, 2m-1]\} \cup \{w_i w_{i+m} \colon i \in [1, m]\}$. The following theorem shows that $srx_3(P_n^o \rhd_{\vec{e}} L_m) = srx_3(L_m)(n-1)$.

Theorem 3.1. *For two integers $n, m \geq 3$, let P_n and L_m be a path of order n and a ladder of order $2m$, respectively. Let $\vec{e}$ be an oriented edge of L_m where $\vec{e} = w_1 w_{m+1}$. Then $srx_3(P_n^o \rhd_{\vec{e}} L_m) = m(n-1)$.*

Proof. Since $srx_3(L_m) = m$ by Theorem 1.2, it follows by Theorem 2.3 that $srx_3(P_n^o \rhd_{\vec{e}} L_m) \leq m(n-1)$. Now, let c be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} L_m$. For $i \in [1, n-1]$, let $X_i = \{v_i v_i^2, v_i^p v_i^{p+1} \colon p \in [2, m-1]\}$. We first verify two properties as follows.

(A1) $c(X_i) \cap c(E(P_n)) = \emptyset$ for each $i \in [1, n-1]$
Suppose that there are $e \in X_i$ for some $i \in [1, n-1]$ and $f \in E(P_n)$ such that $c(e) = c(f)$. Let $e = uv$ and $f = xy$, and assume that $d(v_i, x) < d(v_i, y)$. Observe that edges e and f should be contained in any rainbow Steiner $\{u, v, y\}$-tree, but $c(e) = c(f)$, a contradiction.

(A2) $c(X_i) \cap c(X_j) = \emptyset$ for $i, j \in [1, n-1]$ with $i \neq j$
Suppose that there are $e \in X_i$ and $f \in X_j$ for distinct $i, j \in [1, n-1]$ such that $c(e) = c(f)$. Let $e = uv$ and $f = xy$, and assume that $d(v_j, x) < d(v_j, y)$. By considering $\{u, v, y\}$, we will obtain a contradiction.

Since $|c(X_i)| \geq m-1$ for each $i \in [1, n-1]$, by using (2), (A1), and (A2), $srx_3(P_n^o \rhd_{\vec{e}} L_m) \geq m(n-1)$. $\square$

According to (1), the natural lower bound for $srx_3(P_n^o \rhd_{\vec{e}} H)$ is $sdiam_3(P_n^o \rhd_{\vec{e}} H)$. Consider graphs $P_n^o \rhd_{\vec{e}} L_m$ where $\vec{e} \in E(L_m)$ with $\vec{e} = w_1w_{m+1}$. It is easy to check that $sdiam_3(P_3^o \rhd_{\vec{e}} L_m) = 2m$ and $sdiam_3(P_n^o \rhd_{\vec{e}} L_m) = n + 3m - 4$ for $n \geq 4$. Hence, by Theorem 3.1, $srx_3(P_n^o \rhd_{\vec{e}} L_m) = sdiam_3(P_n^o \rhd_{\vec{e}} L_m)$ for $n \in [3, 4]$.

For further discussion, we define path $v_pv_qv_qv_r = v_pv_qv_r$. For $m \geq 3$, let $K_{m,m}$ be a *regular complete bipartite* graph of order $2m$ with $V(K_{m,m}) = \{w_i \colon i \in [1, 2m]\}$ and $E(K_{m,m}) = \{w_iw_j \colon i \in [1, m], j \in [m + 1, 2m]\}$. The next theorem shows that $srx_3(P_n^o \rhd_{\vec{e}} K_{m,m}) = srx_3(K_{m,m})(n - 1)$.

Theorem 3.2. *For two integers $n, m \geq 3$, let P_n and $K_{m,m}$ be a path of order n and a regular complete bipartite graph of order $2m$, respectively. Let $\vec{e}$ be any oriented edge of $K_{m,m}$. Then $srx_3(P_n^o \rhd_{\vec{e}} K_{m,m}) = m(n - 1)$.*

Proof. Without loss of generality, let $\vec{e} = w_1w_{m+1}$ such that $v_i^1 = v_i$ and $v_i^{m+1} = v_{i+1}$ for each $i \in [1, n - 1]$. By using Theorems 1.3 and 2.3, we have $srx_3(P_n^o \rhd_{\vec{e}} K_{m,m}) \leq m(n - 1)$. Now, let c be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} K_{m,m}$. We first verify two properties as follows.

(B1) $c(v_iv_i^p) \notin c(E(P_n))$ for each $i \in [1, n - 1]$ and $p \in [m + 2, 2m]$
Suppose that there are $i \in [1, n - 1]$ and $p \in [m + 2, 2m]$ such that $c(v_iv_i^p) \in c(E(P_n))$. Let $e = uv \in E(P_n)$ with $c(v_iv_i^p) = c(e)$, and assume that $d(v_i, u) < d(v_i, v)$. Observe that edges $v_iv_i^p$ and e should be contained in any rainbow Steiner $\{v_i, v_i^p, v\}$-tree, but $c(v_iv_i^p) = c(e)$, a contradiction.

(B2) $c(v_iv_i^p) \neq c(v_jv_j^q)$ for $i, j \in [1, n - 1]$ with $i \neq j$ and $p, q \in [m + 2, 2m]$
Let $i < j$. By considering $\{v_i, v_i^p, v_j^q\}$ for $p, q \in [m + 2, 2m]$, it is clearly that $c(v_iv_i^p) \neq c(v_jv_j^q)$.

Note that $d_{K^i_{m,m}}(v_i) = m$ for $i \in [1, n - 1]$, thus $srx_3(P_n^o \rhd_{\vec{e}} K_{m,m}) \geq m(n - 1)$ by (2), (B1), and (B2). □

According to Theorems 2.1, 3.1, and 3.2, the values of $srx_3(P_n^o \rhd_{\vec{e}} H)$ for some graphs H does not depend on the order of H. Nevertheless, there are some graphs H so that the values of $srx_3(P_n^o \rhd_{\vec{e}} H)$ depends on the order of H. These results are given in Theorems 3.3 and 3.4.

A *fan* graph F_m of order $m + 1$ ($m \geq 3$) is a graph formed by joining a vertex to each vertex of P_m. We define $V(F_m) = \{w_i \colon i \in [1, m+1]\}$ and $E(F_m) = \{w_1w_i \colon i \in [2, m+1]\} \cup \{w_iw_{i+1} \colon i \in [2, m]\}$. For $i \in [2, m + 1]$, vertex w_1 and edge w_1w_i are called the *center vertex* and the *spoke* of F_m, respectively. In [1], we obtained the following lemma which will be used to prove Theorem 3.3.

Lemma 3.1. *[1] Let F_m be a fan of order $m + 1$ ($m \geq 3$) which has a strong 3-rainbow coloring. Then each color is assigned to at most two spokes w_1w_i and w_1w_j where $w_iw_j \in E(F_m)$.*

Theorem 3.3. *For two integers $n, m \geq 3$, let P_n and F_m be a path of order n and a fan of order $m + 1$, respectively. Let $\vec{e}$ be an oriented edge of F_m where $\vec{e} = w_1w_2$. Then*

$$srx_3(P_n^o \rhd_{\vec{e}} F_m) = \begin{cases} 2n - 1, & \text{for } m = 4; \\ \lceil \frac{m}{2} \rceil (n - 1), & \text{otherwise.} \end{cases}$$

Proof. Let c be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} F_m$. Similar to the proof of Theorem 3.2, we have two properties as follows.

(C1) $c(v_iv_i^p) \notin c(E(P_n))$ for each $i \in [1, n-1]$ and $p \in [4, m+1]$

(C2) $c(v_iv_i^p) \neq c(v_jv_j^q)$ for $i, j \in [1, n-1]$ with $i \neq j$ and $p, q \in [4, m+1]$

Now, we distinguish two cases.

Case 1. $m = 4$

Suppose that $srx_3(P_n^o \rhd_{\vec{e}} F_4) \leq 2n-2$. Let $c : E(P_n^o \rhd_{\vec{e}} F_4) \to [1, 2n-2]$ be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} F_4$. By using (2), Lemma 3.1, (C1), and (C2), we need at least $2n-2$ different colors assigned to the edges of P_n and spokes of F_4^i for all $i \in [1, n-1]$. Without loss of generality, let $c(v_iv_{i+1}) = c(v_iv_i^3) = i$ and $c(v_iv_i^4) = c(v_iv_i^5) = i + n - 1$ for $i \in [1, n-1]$. Now, observe that identifying vertex v_2 in a rainbow Steiner $\{v_2, v_1^4, v_1^5\}$-tree and a rainbow $v_2 - v_n$ geodesic will obtain a rainbow Steiner $\{v_1^4, v_1^5, v_n\}$-tree. Similarly, identifying vertex v_2 in a rainbow Steiner $\{v_2, v_1^4, v_1^5\}$-tree and a rainbow $v_2 - v_i^5$ geodesic for all $i \in [2, n-1]$ will obtain a rainbow Steiner $\{v_1^4, v_1^5, v_i^5\}$-tree. Since these rainbow Steiner trees must contain edge $v_1^4v_1^5$, we have $c(v_1^4v_1^5) \notin \{c(v_iv_{i+1}), c(v_iv_i^5)\}$ for all $i \in [2, n-1]$. This means we have two colors, which are 1 and n, to be assigned to the three edges in a Steiner tree containing $\{v_2, v_1^4, v_1^5\}$, which is impossible. Thus, $srx_3(P_n^o \rhd_{\vec{e}} F_4) \geq 2n-1$.

Next, we show that $srx_3(P_n^o \rhd_{\vec{e}} F_4) \leq 2n-1$. For each $i \in [1, n-1]$, define $c(v_iv_{i+1}) = c(v_iv_i^3) = c(v_i^3v_i^4) = i$, $c(v_iv_i^4) = c(v_iv_i^5) = c(v_{i+1}v_i^3) = i + n - 1$, and $c(v_i^4v_i^5) = 2n-1$. Let S be a 3-subset of $V(P_n^o \rhd_{\vec{e}} F_4)$. If $S \subseteq V(F_4^i)$ for some $i \in [1, n-1]$, then it is not hard to find a rainbow Steiner S-tree. Hence, there are two possible sets S as follows. First, we consider case when two vertices of S belong to the same fan F_4^i for some $i \in [1, n-1]$. Let $y \in V(F_4^j)$ for $j \in [1, n-1]$ with $j \neq i$. For $i < j$, let P be a $v_{i+1} - v_j$ geodesic. Then there is a rainbow Steiner S-tree as given in Table 1. The proof for $i > j$ is similar to the case for $i < j$.

Table 1. A rainbow Steiner S-tree of $P_n^o \rhd_{\vec{e}} F_4$ for $i < j$.

Set S	Condition	A rainbow Steiner S-tree
$\{v_i, v_{i+1}, y\}$	$p = 1, q = 2$	$v_iv_{i+1} \cup P \cup v_jy$
$\{v_i, v_i^q, y\}$	$p = 1, q = 3$	$v_iv_i^3v_{i+1} \cup P \cup v_jy$
	$p = 1, q \in [4, 5]$	$v_i^qv_iv_{i+1} \cup P \cup v_jy$
$\{v_{i+1}, v_i^q, y\}$	$p = 2, q = 3$	$v_i^3v_{i+1} \cup P \cup v_jy$
	$p = 2, q \in [4, 5]$	$v_i^qv_iv_{i+1} \cup P \cup v_jy$
$\{v_i^p, v_i^q, y\}$	$p, q \in [3, 5], p < q$	$v_i^qv_i^{q-1}v_i^pv_i^3v_{i+1} \cup P \cup v_jy$

Next, we consider case when each vertex of S belongs to three different fans F_4^i, F_4^j, and F_4^k for $i, j, k \in 1[, n-1]$. Without loss of generality, let $i < j < k$. Let $S = \{v_i^p, v_j^q, z\}$ where $z \in V(F_4^k)$. Let P be a $v_{i+1} - v_k$ geodesic, $P_i^1 = v_{i+1}v_i^3v_i^p$ if $p \in [3, 4]$, $P_i^2 = v_{i+1}v_iv_i^5$ if $p = 5$, $P_j^1 = v_{j+1}v_j^3$ if $q = 3$, $P_j^2 = v_jv_j^q$ if $q \in [4, 5]$, and $P_k = v_kz$. Then the tree $T = P \cup P_i^a \cup P_j^b \cup P_k$

with $a, b \in [1,2]$ is a rainbow Steiner S-tree, where the values of a and b depend on the values of p and q, respectively. Note that the case when S contains at least one vertex of P_n has been proven.

An illustration of a strong 3-rainbow coloring of $P_5^o \rhd_{\vec{e}} F_4$ is given in Figure 2.

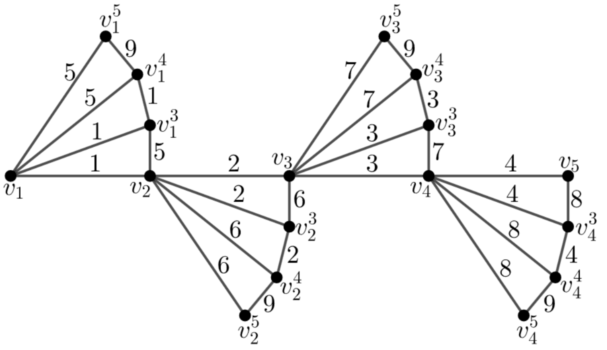


Figure 2. A strong 3-rainbow coloring of $P_5^o \rhd_{\vec{e}} F_4$.

Case 2. $m = 3$ or $m \geq 5$

Since $srx_3(F_m) = \lceil \frac{m}{2} \rceil$ by Theorem 1.5, it follows by Theorem 2.3 that $srx_3(P_n^o \rhd_{\vec{e}} F_m) \leq \lceil \frac{m}{2} \rceil (n-1)$. Now, let c be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} F_m$. By using (2), Lemma 3.1, (C1), and (C2), we have $srx_3(P_n^o \rhd_{\vec{e}} F_m) \geq \lceil \frac{m}{2} \rceil (n-1)$. $\square$

Following Theorem 3.3, we obtain that $srx_3(P_n^o \rhd_{\vec{e}} F_m)$ for $m = 3$ or $m \geq 5$ is equal to the upper bound given in Theorem 2.3.

Now, we consider graphs $P_n^o \rhd_{\vec{e}} C_m$ where $\vec{e}$ is any oriented edge of C_m. For $m \geq 3$, let $V(C_m) = \{w_i \colon i \in [1,m]\}$ and $E(C_m) = \{w_i w_{i+1} \colon i \in [1,m] \text{ and } w_{m+1} = w_1\}$. Our next result provides the exact value of $srx_3(P_n^o \rhd_{\vec{e}} C_m)$.

Theorem 3.4. *For two integers $n \geq 3$ and $m \geq 4$, let P_n and C_m be a path of order n and a cycle of order m, respectively. Let $\vec{e}$ be any oriented edge of C_m. Then*

$$srx_3(P_n^o \rhd_{\vec{e}} C_m) = \begin{cases} 2n-2, & \text{for } m = 4; \\ 2n + \lfloor \frac{n}{2} \rfloor - 2, & \text{for } m = 5; \\ (m-3)(n-1)+1, & \text{for } m \in \{6,8\}; \\ (m-1)(n-1)+1, & \text{for odd } m \geq 7; \\ (m-2)(n-1)+3, & \text{for even } m \geq 10. \end{cases}$$

Proof. Without loss of generality, let $\vec{e} = w_1 w_2$ such that $v_i^1 = v_i$ and $v_i^2 = v_{i+1}$ for each $i \in [1, n-1]$. We consider several cases.

Case 1. m is odd

We distinguish two subcases.

Subcase 1.1. $m = 5$

Suppose that $srx_3(P_n^o \rhd_{\vec{e}} C_5) \leq 2n + \lfloor \frac{n}{2} \rfloor - 3$. Let $c : E(P_n^o \rhd_{\vec{e}} C_5) \to [1, 2n + \lfloor \frac{n}{2} \rfloor - 3]$ be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} C_5$. Observe that $c(v_i v_i^5) \notin c(E(P_n))$ and $c(v_i v_i^5) \neq c(v_j v_j^5)$ for $i, j \in [1, n-1]$ with $i \neq j$. Hence, by using (2), we need at least $2n - 2$ different colors assigned to all edges $v_i v_{i+1}$ and $v_i v_i^5$ for $i \in [1, n-1]$, implying that we have at most $\lfloor \frac{n}{2} \rfloor - 1$ colors left. Let X be the set of these $\lfloor \frac{n}{2} \rfloor - 1$ colors. Now, we consider edges $v_{i+1} v_i^3$, $v_i^3 v_i^4$, and $v_i^4 v_i^5$ for all $i \in [1, n-1]$. By considering $\{v_{i+1}, v_i^3, v_j\}$ and $\{v_i^p, v_i^{p+1}, v_j\}$ for all $j \in [1, n-1]$, $j \neq i$, $j \neq i+1$, and $p \in [3, 4]$, we obtain that these three edges can not be assigned with colors from $c(E(P_n) \setminus \{v_i v_{i+1}\})$. Furthermore, by considering $\{v_{i+1}, v_i^3, v_j^5\}$ and $\{v_i^p, v_i^{p+1}, v_j^5\}$ for all $j \in [1, n-1]$, $j \neq i$, and $p \in [3, 4]$, these three edges also can not be assigned with $c(v_j v_j^5)$. This implies $\{c(v_{i+1} v_i^3), c(v_i^3 v_i^4), c(v_i^4 v_i^5)\} \subseteq \{c(v_i v_{i+1}), c(v_i v_i^5)\} \cup X$ for all $i \in [1, n-1]$. Since every two adjacent edges in C_5^i must have different colors, this forces

$$\begin{aligned} c(v_{i+1} v_i^3) \in \{c(v_i v_i^5)\} \cup X, c(v_i^3 v_i^4) \in \{c(v_i v_{i+1}), c(v_i v_i^5)\} \cup X, \text{ and} \\ c(v_i^4 v_i^5) \in \{c(v_i v_{i+1})\} \cup X, \end{aligned} \tag{3}$$

and at least one edge of edges $v_{i+1} v_i^3$, $v_i^3 v_i^4$, or $v_i^4 v_i^5$ should be assigned with colors from X. This condition implies there are two possible proofs that might happen. Before we proceed further, we consider the following two properties.

(D1) $\{c(v_{i+1} v_i^3), c(v_i^3 v_i^4)\} \cap \{c(v_{j+1} v_j^3), c(v_j^3 v_j^4)\} = \emptyset$ for $i, j \in [1, n-1]$ with $i \neq j$
Without loss of generality, let $i < j$. By considering $\{v_i^4, v_{j+1}, v_j^3\}$ and $\{v_i^4, v_j^3, v_j^4\}$, we have $\{c(v_{i+1} v_i^3), c(v_i^3 v_i^4)\} \cap \{c(v_{j+1} v_j^3), c(v_j^3 v_j^4)\} = \emptyset$.

(D2) $c(v_i^4 v_i^5) \neq c(v_j^4 v_j^5)$ for $i, j \in [1, n-1]$ with $i \neq j$
Without loss of generality, let $i < j$. By considering $\{v_i^4, v_i^5, v_j^4\}$, we have $c(v_i^4 v_i^5) \neq c(v_j^4 v_j^5)$.

Now, we consider these two possible proofs, which are: (i) $\{c(v_{i+1} v_i^3), c(v_i^3 v_i^4)\} \subseteq X$ for some $i \in [1, n-1]$; and (ii) $c(v_i^4 v_i^5) \in X$ for some $i \in [1, n-1]$. If the first case happens, then observe that there are at most $\lfloor \frac{n}{2} \rfloor - 1$ pairs of two edges $\{v_{i+1} v_i^3, v_i^3 v_i^4\}$ for some $i \in [1, n-1]$ such that $\{c(v_{i+1} v_i^3), c(v_i^3 v_i^4)\} \subseteq X$. Hence, by using (D1), there are at least $n - \lfloor \frac{n}{2} \rfloor$ pairs of two edges $\{v_{j+1} v_j^3, v_j^3 v_j^4\}$ for all $j \in [1, n-1]$ with $j \neq i$ such that $\{c(v_{j+1} v_j^3), c(v_j^3 v_j^4)\} \nsubseteq X$. This forces $c(v_{j+1} v_j^3) = c(v_j v_j^5)$, $c(v_j^3 v_j^4) = c(v_j v_{j+1})$, and $c(v_j^4 v_j^5) \in X$ by (3). However, by using (D2), we need at least $n - \lfloor \frac{n}{2} \rfloor$ different colors assigned to the edges $v_j^4 v_j^5$ for all $j \in [1, n-1]$ with $j \neq i$, which is impossible since $|X| \leq \lfloor \frac{n}{2} \rfloor - 1$. A similar argument applies if the second case happens.

For the upper bound, we first define an edge-coloring c of $P_n^o \rhd_{\vec{e}} C_5$ using $2n + \lfloor \frac{n}{2} \rfloor - 2$ colors as follows.

1. For each $i \in [1, n-1]$, define $c(v_i v_{i+1}) = i$ and $c(v_{i+1} v_i^3) = c(v_i v_i^5) = i + n - 1$.
2. For each $i \in [1, \lfloor \frac{n}{2} \rfloor]$, define $c(v_i^4 v_i^5) = i + 2(n-1)$ and $c(v_i^3 v_i^4) = c(v_i v_{i+1})$.
3. For each $i \in [\lfloor \frac{n}{2} \rfloor + 1, n-1]$, define $c(v_i^3 v_i^4) = i - \lfloor \frac{n}{2} \rfloor + 2(n-1)$ and $c(v_i^4 v_i^5) = c(v_i v_{i+1})$.

Now, let S be a 3-subset of $V(P_n^o \rhd_{\vec{e}} C_5)$. Since the edge-coloring c assigns 3 different colors to all edges of C_5^i and has the same coloring pattern as given in Figure 1, there is a rainbow Steiner S-tree if $S \subseteq V(C_5^i)$ for some $i \in [1, n-1]$. Hence, we distinguish two cases.

First, we consider $S = \{v_i^p, v_i^q, v_j^r\}$ for distinct $i, j \in [1, n-1]$. For $i < j$, let P' be a $v_{i+1} - v_j$ geodesic, $P_j^1 = v_j v_{j+1} v_j^3$ if $r = 3$, and $P_j^2 = v_j v_j^5 v_j^r$ if $r \in [4, 5]$. Let $P_1 = P' \cup P_j^b$ with $b \in [1, 2]$. Then there is a rainbow Steiner S-tree as given in Table 2. Meanwhile for $i > j$, let P'' be a $v_{j+1} - v_i$ geodesic, $P_j^3 = v_j^r v_j^3 v_{j+1}$ if $r \in [3, 4]$, and $P_j^4 = v_j^5 v_j v_{j+1}$ if $r = 5$. Let $P_2 = P_j^b \cup P''$ with $b \in [3, 4]$. Then there is a rainbow Steiner S-tree as given in Table 2. Note that the value of b for each case depends on the value of $r \in [3, 5]$.

Table 2. A rainbow Steiner S-tree of $P_n^o \triangleright_{\vec{e}} C_5$.

Set S	Condition	A rainbow Steiner S-tree
$\{v_i, v_{i+1}, v_j^r\}$	$i < j, p = 1, q = 2$	$v_i v_{i+1} \cup P_1$
	$i > j, p = 1, q = 2$	$P_2 \cup v_i v_{i+1}$
$\{v_i, v_i^q, v_j^r\}$	$i < j, p = 1, q = 3$	$v_i v_{i+1} v_i^3 \cup P_1$
	$i < j, p = 1, q \in [4, 5]$	$v_i^q v_i^5 v_i v_{i+1} \cup P_1$
	$i > j, p = 1, q = 3$	$P_2 \cup v_i v_{i+1} v_i^3$
	$i > j, p = 1, q \in [4, 5]$	$P_2 \cup v_i v_i^5 v_i^q$
$\{v_{i+1}, v_i^q, v_j^r\}$	$i < j, p = 2, q \in [3, 4]$	$v_i^q v_i^3 v_{i+1} \cup P_1$
	$i < j, p = 2, q = 5$	$v_i^5 v_i v_{i+1} \cup P_1$
	$i > j, p = 2, q \in [3, 4]$	$P_2 \cup v_i v_{i+1} v_i^3 v_i^q$
	$i > j, p = 2, q = 5$	$P_2 \cup v_{i+1} v_i v_i^5$
$\{v_i^p, v_i^q, v_j^r\}$	$i < j, p, q \in [3, 5], p < q$	$v_i^q v_i^{q-1} v_i^p v_i^3 v_{i+1} \cup P_1$
	$i > j, p, q \in [3, 5], p < q$	$P_2 \cup v_i v_i^5 v_i^q v_i^{p+1} v_i^p$

Next, we consider $S = \{v_i^p, v_j^q, v_k^r\}$ for distinct $i, j, k \in [1, n-1]$. Without loss of generality, let $i < j < k$. Let P be a $v_{i+1} - v_k$ geodesic, $P_i^1 = v_{i+1} v_i^3 v_i^p$ if $p \in [3, 4]$, $P_i^2 = v_{i+1} v_i v_i^5$ if $p = 5$, $P_j^1 = v_{j+1} v_j^3$ if $j \in [2, \lfloor \frac{n}{2} \rfloor]$ and $q = 3$, $P_j^2 = v_j v_j^5 v_j^q$ if $j \in [2, \lfloor \frac{n}{2} \rfloor]$ and $q \in [4, 5]$, $P_j^3 = v_{j+1} v_j^3 v_j^q$ if $j \in [\lfloor \frac{n}{2} \rfloor + 1, n-2]$ and $q \in [3, 4]$, $P_j^4 = v_j v_j^5$ if $j \in [\lfloor \frac{n}{2} \rfloor + 1, n-2]$ and $q = 5$, $P_k^1 = v_k v_{k+1} v_k^3$ if $r = 3$, and $P_k^2 = v_k v_k^5 v_k^r$ if $r \in [4, 5]$. Then the tree $T = P \cup P_i^a \cup P_j^b \cup P_k^c$ with $a, c \in [1, 2]$ and $b \in [1, 4]$ is a rainbow Steiner S-tree, where the values of a and c depend on the values of p and r, respectively, and the value of b depends on the values of j and q.

Note that the case when S contains at least one vertex of P_n has been proven for each of the above cases. Figure 3 illustrates a strong 3-rainbow coloring of $P_5^o \triangleright_{\vec{e}} C_5$.

Subcase 1.2. $m \geq 7$

Suppose that $srx_3(P_n^o \triangleright_{\vec{e}} C_m) \leq (m-1)(n-1)$. Let $c : E(P_n^o \triangleright_{\vec{e}} C_m) \to [1, (m-1)(n-1)]$ be a strong 3-rainbow coloring of $P_n^o \triangleright_{\vec{e}} C_m$. According to Theorem 1.4, all edges of C_m^i should be assigned with different colors. Hence, by considering $\{v_i^p, v_j, v_j^q\}$ and $\{v_i^{\lceil \frac{m}{2} \rceil + 1}, v_i^{\lceil \frac{m}{2} \rceil + 2}, v_j^q\}$ for $i, j \in [1, n-1]$ with $i < j$, $p \in \{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 2\}$, and $q \in [\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1]$, we need at least $(m-1)(n-1)$ different colors assigned to the edges of $P_n^o \triangleright_{\vec{e}} C_m$ except edges $v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil + 1}$ for all $i \in [1, n-1]$. This means we have used all colors. Now, we consider edge $v_1^{\lceil \frac{m}{2} \rceil} v_1^{\lceil \frac{m}{2} \rceil + 1}$. By using

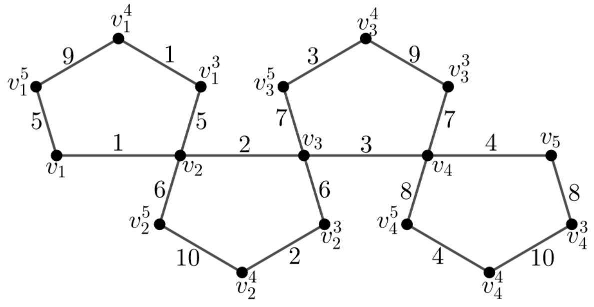


Figure 3. A strong 3-rainbow coloring of $P_5^o \rhd_{\vec{e}} C_5$.

Theorem 1.4 and considering $\{v_1^{\lfloor \frac{m}{2} \rfloor}, v_1^{\lfloor \frac{m}{2} \rfloor+1}, v_i^p\}$ for all $i \in [2, n-1]$ and $p \in [\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil + 1]$, we need one new different color assigned to the edge $v_1^{\lceil \frac{m}{2} \rceil} v_1^{\lceil \frac{m}{2} \rceil+1}$, which is impossible.

For the upper bound, we first define an edge-coloring c of $P_n^o \rhd_{\vec{e}} C_m$ using $(m-1)(n-1)+1$ colors as follows.

1. Assign the colors $1, 2, \ldots, (m-1)(n-1)$ to all edges of $P_n^o \rhd_{\vec{e}} C_m$ except edges $v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil+1}$ for all $i \in [1, n-1]$.
2. Define $c(v_1^{\lceil \frac{m}{2} \rceil} v_1^{\lceil \frac{m}{2} \rceil+1}) = (m-1)(n-1)+1$ and $c(v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil+1}) = c(v_{i-1}^{\lceil \frac{m}{2} \rceil+1} v_{i-1}^{\lceil \frac{m}{2} \rceil+2})$ for each $i \in [2, n-1]$.

Now, let S be a 3-subset of $V(P_n^o \rhd_{\vec{e}} C_m)$. Observe that edges of $P_n^o \rhd_{\vec{e}} C_m$ have different colors except edges $v_i^{\lceil \frac{m}{2} \rceil+1} v_i^{\lceil \frac{m}{2} \rceil+2}$ for $i \in [1, n-2]$ and $v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil+1}$ for $i \in [2, n-1]$, that is $c(v_i^{\lceil \frac{m}{2} \rceil} v_i^{\lceil \frac{m}{2} \rceil+1}) = c(v_{i-1}^{\lceil \frac{m}{2} \rceil+1} v_{i-1}^{\lceil \frac{m}{2} \rceil+2})$ for all $i \in [2, n-1]$. This means if $S \subseteq V(C_m^i)$ for some $i \in [1, n-1]$, then there is a rainbow Steiner S-tree since all edges of C_m^i have different colors. Hence, without loss of generality, we distinguish two cases.

First, we consider $S = \{v_i^p, v_i^q, v_j^r\}$ for distinct $i, j \in [1, n-1]$. For $i < j$, observe that there are a rainbow $v_{i+1} - v_j$ geodesic T_1 in P_n, a rainbow Steiner $\{v_{i+1}, v_i^p, v_i^q\}$-tree T_2 in C_m^i, and a rainbow $v_j - v_j^r$ geodesic T_3 in C_m^j, so that $c(E(T_a)) \cap c(E(T_b)) = \emptyset$ for all distinct $a, b \in [1, 3]$. Then the tree $T = T_1 \cup T_2 \cup T_3$ is a rainbow Steiner S-tree. The proof for $i > j$ is similar to the case for $i < j$.

Next, we consider $S = \{v_i^p, v_j^q, v_k^r\}$ for distinct $i, j, k \in [1, n-1]$. Without loss of generality, let $i < j < k$. Note that there are a rainbow $v_{i+1} - v_k$ geodesic T_1 in P_n, a rainbow $v_{i+1} - v_i^p$ geodesic T_2 in C_m^i, a rainbow $v_j - v_j^q$ geodesic T_3 in C_m^j, a rainbow $v_{j+1} - v_j^q$ geodesic T_4 in C_m^j, and a rainbow $v_k - v_k^r$ geodesic T_5 in C_m^k, so that $c(E(T_a)) \cap c(E(T_b)) = \emptyset$ for all distinct $a, b \in [1, 5]$. Then the tree $T = T_1 \cup T_2 \cup T_3 \cup T_5$ or $T = T_1 \cup T_2 \cup T_4 \cup T_5$ is a rainbow Steiner S-tree.

Note that the case when S contains at least one vertex of P_n has been proven for each of the above cases. Figure 4 illustrates a strong 3-rainbow coloring of $P_5^o \rhd_{\vec{e}} C_7$.

Case 2. m is even

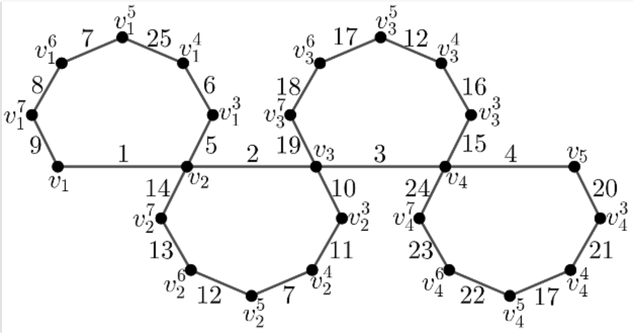


Figure 4. A strong 3-rainbow coloring of $P_5^o \rhd_{\vec{e}} C_7$.

We first consider the following properties. Let c be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} C_m$. For $i \in [1, n-1]$, let $X_i = E(C_m^i) \setminus \{v_iv_{i+1}, v_i^{\frac{m}{2}}v_i^{\frac{m}{2}+1}, v_i^{\frac{m}{2}+1}v_i^{\frac{m}{2}+2}\}$.

(E1) $c(X_i) \cap c(E(P_n)) = \emptyset$ for each $i \in [1, n-1]$
By considering $\{v_i, v_i^p, v_j\}$ for $j \in [1, n-1]$, $j \neq i$, and $p \in \{\frac{m}{2}, \frac{m}{2}+2\}$, it is clearly that $c(X_i) \cap c(E(P_n)) = \emptyset$.

(E2) $c(X_i) \cap c(X_j) = \emptyset$ for $i, j \in [1, n-1]$ with $i \neq j$
By considering $\{v_i, v_i^p, v_j^q\}$ for $i < j$ and $p, q \in \{\frac{m}{2}, \frac{m}{2}+2\}$, we obtain that $c(X_i) \cap c(X_j) = \emptyset$.

(E3) For each $i \in [1, n-1]$, $|c(X_i)| \geq 1$ for $m = 4$, $|c(X_i)| \geq m-4$ for $m \in \{6, 8\}$, and $|c(X_i)| \geq m-3$ for $m \geq 10$
For $m = 4$, it is clearly that $|c(X_i)| \geq 1$. For $m = 6$, since $c(v_i^5v_i^6) \neq (v_iv_i^6)$ for each $i \in [1, n-1]$, we have $|c(X_i)| \geq 2$. For $m = 8$, suppose that $|c(X_i)| \leq 3$. Observe that edges $v_i^6v_i^7$, $v_i^7v_i^8$, and $v_iv_i^8$ should be assigned with different colors, which means every color in $c(X_i)$ should be used to color these edges. Next, we consider edges $v_{i+1}v_i^3$ and $v_i^3v_i^4$. By considering $\{v_{i+1}, v_i^4, v_i^8\}$, $\{v_i, v_i^3, v_i^7\}$, and $\{v_i^3, v_i^5, v_i^7\}$, we obtain that $c(v_{i+1}v_i^3) \notin \{c(v_i^7v_i^8), c(v_iv_i^8)\}$ and $c(v_i^3v_i^4) \notin \{c(v_i^6v_i^7), c(v_iv_i^8)\}$. This forces $c(v_{i+1}v_i^3) = c(v_i^6v_i^7)$ and $c(v_i^3v_i^4) = c(v_i^7v_i^8)$. However, there is no rainbow Steiner $\{v_{i+1}, v_i^4, v_i^7\}$-tree, a contradiction. For $m \geq 10$, it is clearly that $|c(X_i)| \geq m-3$ by Theorem 1.4.

Now, we distinguish three subcases.

Subcase 2.1. $m = 4$

By using (2), (E1), (E2), and (E3), we have $srx_3(P_n^o \rhd_{\vec{e}} C_4) \geq 2n-2$. Furthermore, since $srx_3(C_4) = 2$ by Theorem 1.4, it follows by Theorem 2.3 that $srx_3(P_n^o \rhd_{\vec{e}} C_4) \leq 2(n-1) = 2n-2$. An illustration of a strong 3-rainbow coloring of $P_5^o \rhd_{\vec{e}} C_4$ is given in Figure 5.

Subcase 2.2. $m \in \{6, 8\}$

Suppose that $srx_3(P_n^o \rhd_{\vec{e}} C_m) \leq (m-3)(n-1)$. Let $c : E(P_n^o \rhd_{\vec{e}} C_m) \to [1, (m-3)(n-1)]$ be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} C_m$. By using (2), (E1), (E2), and (E3), we need at least $(m-3)(n-1)$ different colors assigned to all edges of $P_n^o \rhd_{\vec{e}} C_m$ except edges $v_i^{\frac{m}{2}}v_i^{\frac{m}{2}+1}$ and $v_i^{\frac{m}{2}+1}v_i^{\frac{m}{2}+2}$ for $i \in [1, n-1]$. Now, we consider $\{v_1^p, v_1^{p+1}, v_i^q\}$ for all $i \in [2, n-1]$, $p \in [\frac{m}{2}, \frac{m}{2}+1]$,

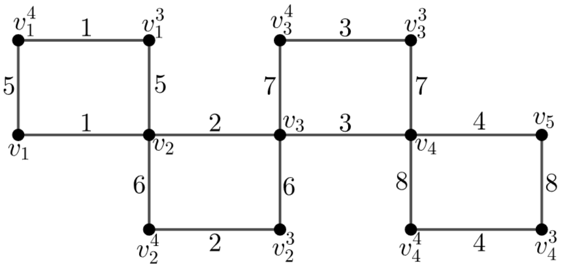


Figure 5. A strong 3-rainbow coloring of $P_5^o \rhd_{\vec{e}} C_4$.

and $q \in \{\frac{m}{2}, \frac{m}{2}+2\}$. This forces $\{c(v_1^{\frac{m}{2}} v_1^{\frac{m}{2}+1}), c(v_1^{\frac{m}{2}+1} v_1^{\frac{m}{2}+2})\} \subseteq \{c(v_1v_2)\} \cup c(X_1)$, implying that all edges of C_m^1 are assigned with $m-3$ different colors, contradicts Theorem 1.4.

For the upper bound, we first define an edge-coloring c of $P_n^o \rhd_{\vec{e}} C_m$ using $(m-3)(n-1)+1$ colors as follows. For $m=6$ and $i \in [1, n-1]$, define $c(v_iv_{i+1}) = i$, $c(v_{i+1}v_i^3) = c(v_i^5v_i^6) = i+n-1$, $c(v_i^3v_i^4) = c(v_iv_i^6) = i+2(n-1)$, and $c(v_i^4v_i^5) = 3(n-1)+1$. Meanwhile for $m=8$ and $i \in [1, n-1]$, define $c(v_iv_{i+1}) = i$, $c(v_{i+1}v_i^3) = c(v_i^6v_i^7) = i+n-1$, $c(v_i^4v_i^5) = c(v_iv_i^8) = i+2(n-1)$, $c(v_i^5v_i^6) = 3(n-1)+1$, and assign the colors $3(n-1)+2, 3(n-1)+3, \ldots, 5(n-1)+1$ to the remaining $2(n-1)$ edges of $P_n^o \rhd_{\vec{e}} C_8$.

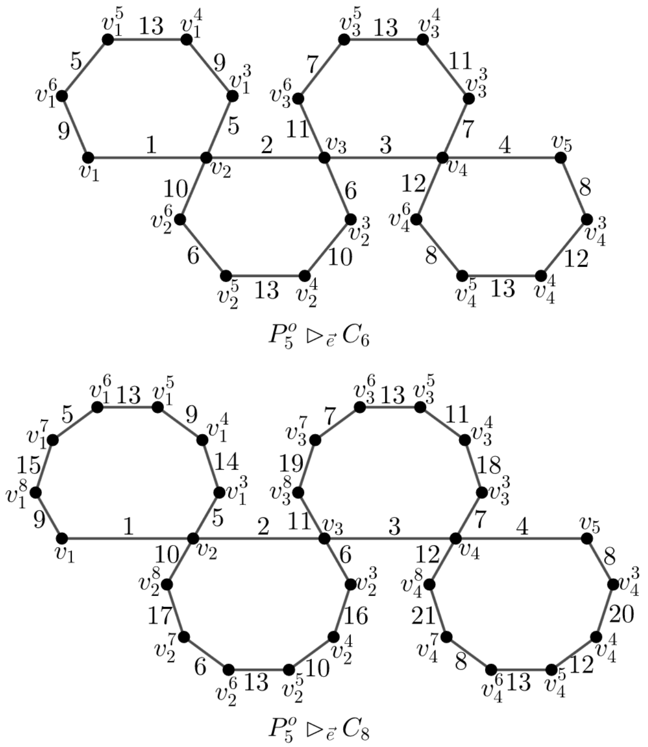
Now, let S be a 3-subset of $V(P_n^o \rhd_{\vec{e}} C_m)$. Similar to the proof of Subcase 1.1, we distinguish two cases. First, we consider $S = \{v_i^p, v_i^q, v_j^r\}$ for distinct $i, j \in [1, n-1]$. For $i < j$, let P be a $v_{i+1} - v_j$ geodesic, $P_j^1 = v_jv_{j+1}v_j^3v_j^{r-1}v_j^r$ if $r \in [3, \frac{m}{2}+1]$, and $P_j^2 = v_jv_j^mv_j^{r+1}v_j^r$ if $r \in [\frac{m}{2}+2, m]$. Let $P_1 = P \cup P_j^b$ with $b \in [1,2]$. Then there is a rainbow Steiner S-tree as given in Table 3, where the value of b depends on the value of $r \in [3, m]$. The proof for $i > j$ is similar to the case for $i < j$.

Table 3. A rainbow Steiner S-tree of $P_n^o \rhd_{\vec{e}} C_m$ for $m \in \{6, 8\}$ and $i < j$.

Set S	Condition	A rainbow Steiner S-tree
$\{v_i, v_{i+1}, v_j^r\}$	$p=1, q=2$	$v_iv_{i+1} \cup P_1$
$\{v_i, v_i^q, v_j^r\}$	$p=1, q \in [3, \frac{m}{2}+1]$	$v_i^qv_i^{q-1}v_i^3v_{i+1}v_i \cup P_1$
	$p=1, q \in [\frac{m}{2}+2, m]$	$v_i^qv_i^{q+1}v_i^mv_iv_{i+1} \cup P_1$
$\{v_{i+1}, v_i^q, v_j^r\}$	$p=2, q \in [3, \frac{m}{2}+1]$	$v_i^qv_i^{q-1}v_i^3v_{i+1} \cup P_1$
	$p=2, q \in [\frac{m}{2}+2, m]$	$v_i^qv_i^{q+1}v_i^mv_iv_{i+1} \cup P_1$
$\{v_i^p, v_i^q, v_j^r\}$	$p, q \in [3, \frac{m}{2}+2], p < q$	$v_i^qv_i^{q-1}..v_i^p..v_i^3v_{i+1} \cup P_1$
	$p, q \in [\frac{m}{2}+2, m], p < q$	$v_i^pv_i^{p+1}v_i^qv_i^mv_iv_{i+1} \cup P_1$
	$m=6, p=3, q=6$	$v_i^6v_iv_{i+1}v_i^3 \cup P_1$
	$m=6, p=4, q=6$	$v_i^4v_i^5v_i^6v_iv_{i+1} \cup P_1$
	$m=8, p \in [3,4], q \in [7,8]$	$v_i^qv_i^8v_iv_{i+1}v_i^3v_i^p \cup P_1$
	$m=8, p=5, q \in [7,8]$	$v_i^5v_i^6v_i^7v_i^8v_iv_{i+1} \cup P_1$

Next, without loss of generality, we consider $S = \{v_i^p, v_j^q, v_k^r\}$ for $i, j, k \in [1, n-1]$ with $i < j < k$. Let P be a $v_{i+1} - v_k$ geodesic, $P_i^1 = v_{i+1}v_i^3v_i^{p-1}v_i^p$ if $p \in [3, \frac{m}{2}+1]$, $P_i^2 = v_{i+1}v_iv_i^mv_i^{p+1}v_i^p$ if $p \in [\frac{m}{2}+2, m]$, $P_j^1 = v_{j+1}v_j^3v_j^{q-1}v_j^q$ if $q \in [3, \frac{m}{2}+1]$, $P_j^2 = v_jv_j^mv_j^{q+1}v_j^q$ if $q \in [\frac{m}{2}+2, m]$, $P_k^1 = v_kv_{k+1}v_k^3v_k^{r-1}v_k^r$ if $r \in [3, \frac{m}{2}+1]$, and $P_k^2 = v_kv_k^mv_k^{r+1}v_k^r$ if $r \in [\frac{m}{2}+2, m]$. Then the tree $T = P \cup P_i^a \cup P_j^b \cup P_k^c$ with $a, b, c \in [1, 2]$ is a rainbow Steiner S-tree, where the values of a, b, and c depend on the values of p, q, and r, respectively.

Note that the case when S contains at least one vertex of P_n has been proven for each of the above cases. Figure 6 illustrates the strong 3-rainbow colorings of $P_5^o \rhd_{\vec{e}} C_m$ for $m \in \{6, 8\}$.



$P_5^o \rhd_{\vec{e}} C_6$

$P_5^o \rhd_{\vec{e}} C_8$

Figure 6. Strong 3-rainbow colorings of $P_5^o \rhd_{\vec{e}} C_m$ for $m \in \{6, 8\}$.

Subcase 2.3. $m \geq 10$

Suppose that $srx_3(P_n^o \rhd_{\vec{e}} C_m) \leq (m-2)(n-1)+2$. Let $c : E(P_n^o \rhd_{\vec{e}} C_m) \to [1, (m-2)(n-1)+2]$ be a strong 3-rainbow coloring of $P_n^o \rhd_{\vec{e}} C_m$. By using (2), (E1), (E2), and (E3), we need at least $(m-2)(n-1)$ different colors assigned to the edges of $P_n^o \rhd_{\vec{e}} C_m$ except edges $v_i^{\frac{m}{2}}v_i^{\frac{m}{2}+1}$ and $v_i^{\frac{m}{2}+1}v_i^{\frac{m}{2}+2}$ for $i \in [1, n-1]$. This means we have at most two colors left, say 1 and 2. Next, we consider edges $v_1^{\frac{m}{2}}v_1^{\frac{m}{2}+1}$ and $v_1^{\frac{m}{2}+1}v_1^{\frac{m}{2}+2}$. Note that by Theorem 1.4, edges of C_m^1 should be assigned with different colors. Hence, by considering $\{v_1^p, v_1^{p+1}, v_i^q\}$ for all $i \in [2, n-1]$, $p \in [\frac{m}{2}, \frac{m}{2}+1]$, and $q \in \{\frac{m}{2}, \frac{m}{2}+2\}$, we obtain that edges $v_1^{\frac{m}{2}}v_1^{\frac{m}{2}+1}$ and $v_1^{\frac{m}{2}+1}v_1^{\frac{m}{2}+2}$ can not be assigned with colors from $c(E(P_n))$ and $c(X_i)$ for all $i \in [1, n-1]$. This forces $\{c(v_1^{\frac{m}{2}} v_1^{\frac{m}{2}+1}), c(v_1^{\frac{m}{2}+1} v_1^{\frac{m}{2}+2})\} \subseteq \{1,2\}$. Without loss of generality, let $c(v_1^{\frac{m}{2}} v_1^{\frac{m}{2}+1}) = 1$ and $c(v_1^{\frac{m}{2}+1} v_1^{\frac{m}{2}+2}) = 2$. Similarly, we obtain that edges $v_2^{\frac{m}{2}} v_2^{\frac{m}{2}+1}$ and $v_2^{\frac{m}{2}+1} v_2^{\frac{m}{2}+2}$ can not be assigned with colors from $c(E(P_n) \setminus \{v_1v_2\})$ and $c(X_i)$ for all $i \in [2, n-1]$, implying that $\{c(v_2^{\frac{m}{2}} v_2^{\frac{m}{2}+1}), c(v_2^{\frac{m}{2}+1} v_2^{\frac{m}{2}+2})\} \subseteq \{c(v_1v_2), 1, 2\} \cup c(X_1)$. However, by considering $\{v_2^{\frac{m}{2}}, v_2^{\frac{m}{2}+2}, v_1^p\}$ for $p \in \{\frac{m}{2}+1, \frac{m}{2}+3\}$, this forces $\{c(v_2^{\frac{m}{2}} v_2^{\frac{m}{2}+1}), c(v_2^{\frac{m}{2}+1} v_2^{\frac{m}{2}+2})\} \subseteq \{2, c(v_1^{\frac{m}{2}+2} v_1^{\frac{m}{2}+3})\}$. But, there is no rainbow Steiner $\{v_2^{\frac{m}{2}}, v_2^{\frac{m}{2}+2}, v_1^{\frac{m}{2}+2}\}$-tree, a contradiction.

For the upper bound, we first define an edge-coloring c of $P_n^o \triangleright_{\vec{e}} C_m$ using $(m-2)(n-1)+3$ colors as follows.

1. Assign the colors $1, 2, \ldots, (m-2)(n-1)$ to the edges of $P_n^o \triangleright_{\vec{e}} C_m$ except edges $v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}$ and $v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}$ for all $i \in [1, n-1]$.

2. Define $c(v_1^{\frac{m}{2}} v_1^{\frac{m}{2}+1}) = (m-2)(n-1)+1$, $c(v_1^{\frac{m}{2}+1} v_1^{\frac{m}{2}+2}) = (m-2)(n-1)+2$, and $c(v_2^{\frac{m}{2}} v_2^{\frac{m}{2}+1}) = (m-2)(n-1)+3$.

3. For each $i \in [2, n-1]$, define $c(v_i^{\frac{m}{2}+1} v_i^{\frac{m}{2}+2}) = c(v_{i-1}^{\frac{m}{2}+2} v_{i-1}^{\frac{m}{2}+3})$.

4. For each $i \in [3, n-1]$, define $c(v_i^{\frac{m}{2}} v_i^{\frac{m}{2}+1}) = c(v_{i-2}^{\frac{m}{2}+1} v_{i-2}^{\frac{m}{2}+2})$.

Let S be a 3-subset of $V(P_n^o \triangleright_{\vec{e}} C_m)$. Similar to the proof of Subcase 1.2, we can find a rainbow Steiner S-tree in $P_n^o \triangleright_{\vec{e}} C_m$. Figure 7 illustrates a strong 3-rainbow coloring of $P_5^o \triangleright_{\vec{e}} C_{10}$. □

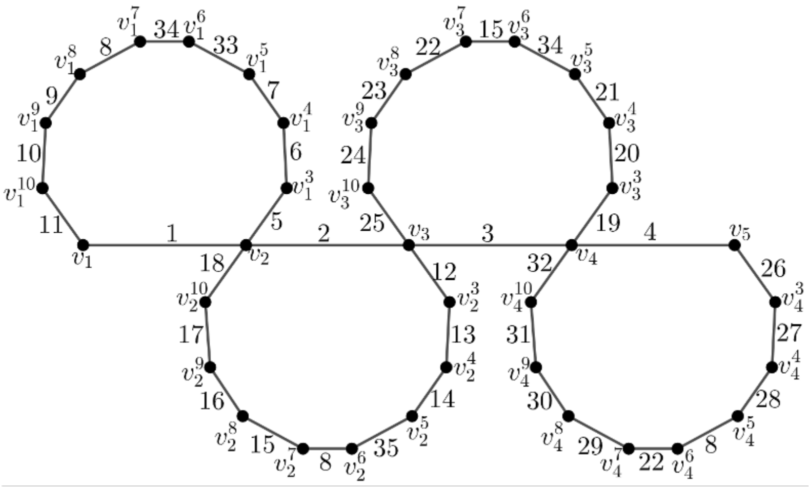


Figure 7. A strong 3-rainbow coloring of $P_5^o \triangleright_{\vec{e}} C_{10}$.

Following Theorem 3.4, we obtain that $srx_3(P_n^o \triangleright_{\vec{e}} C_4)$ is equal to the upper bound in Theorem 2.3, meanwhile for other values of m, the $srx_3(P_n^o \triangleright_{\vec{e}} C_m)$ is not equal to the upper bound.

4. Conclusion

We have shown that H is a tree if and only if $srx_3(P_n^o \rhd_{\vec{e}} H) = |E(P_n^o \rhd_{\vec{e}} H)|$. Further, we have also provided a sharper upper bound for $srx_3(P_n^o \rhd_{\vec{e}} H)$, that is $srx_3(P_n^o \rhd_{\vec{e}} H) \leq srx_3(H)(n-1)$, and have determined the exact values of $srx_3(P_n^o \rhd_{\vec{e}} H)$ for some connected graphs H.

There are many classes of connected graphs H for which the $srx_3(P_n^o \rhd_{\vec{e}} H)$ is not known. Hence, it is interesting to continue the study by determining the exact value of $srx_3(P_n^o \rhd_{\vec{e}} H)$ for other connected graphs H. These results are expected to help characterize the connected graphs H with $srx_3(P_n^o \rhd_{\vec{e}} H) = srx_3(H)(n-1)$. Since a path is one of classes of trees, it is also interesting to study the srx_3 of edge-comb product of a tree and a connected graph.

Acknowledgement

This research has been supported by “Program Penelitian dan Pengabdian kepada Masyarakat (P3MI)”, Institut Teknologi Bandung.

References

[1] Z.Y. Awanis, A. Salman, S.W. Saputro, M. Bača, and A. Semaničová-Feňovčíková, The strong 3-rainbow index of edge-amalgamation of some graphs, *Turkish Journal of Mathematics* **44** (2020), 446–462. doi:10.3906/mat-1911-49

[2] Z.Y. Awanis, A.N.M. Salman, and S.W. Saputro, The strong 3-rainbow index of comb product of a tree and a connected graph, *Journal of Information Processing* **28** (2020), 865–875. doi: 10.2197/ipsjjip.28.865

[3] Z.Y. Awanis and A.N.M. Salman, The strong 3-rainbow index of some certain graphs and its amalgamation, *Submitted.*

[4] Z.Y. Awanis and A.N.M. Salman, The 3-rainbow index of amalgamation of some graphs with diameter 2, *IOP Conference Series: Journal of Physics* **1127** (2019), 012058. doi:10.1088/1742-6596/1127/1/012058

[5] M. Bača, A.N.M. Salman, R. Simanjuntak, and B.H. Susanti, Rainbow 2-connectivity of edge-comb product of a cycle and a Hamiltonian graph, *Proceedings Mathematical Sciences* **130** (2020), 1–12. doi:10.1007/s12044-019-0549-x

[6] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster, On rainbow connection, *The Electronic Journal of Combinatorics* **15** (2008), R57. doi:10.37236/781

[7] S. Chakraborty, E. Fischer, A. Matsliah, and R. Yuster, Hardness and algorithms for rainbow connectivity, *Journal of Combinatorial Optimization* **21** (2011), 330–347.

[8] G. Chartrand, G.L. Johns, K.A. McKeon, and P. Zhang, Rainbow connection in graphs, *Mathematica Bohemica* **133**(1) (2008), 85–98. doi:10.21136/MB.2008.133947

[9] G. Chartrand, F. Okamoto, and P. Zhang, Rainbow trees in graphs and generalized connectivity, *Networks* **55** (2010), 360–367. doi:10.1002/net.20339

[10] L. Chen, X. Li, K. Yang, and Y. Zhao, The 3-rainbow index of a graph, *Discussiones Mathematicae Graph Theory* **35**(1) (2015), 81–94. doi:10.7151/dmgt.1780

[11] R. Diestel, *Graph Theory* 5ed. (Springer, 2017).

[12] D. Fitriani and A.N.M. Salman, Rainbow connection number of amalgamation of some graphs, *AKCE International Journal of Graphs and Combinatorics* **13** (2016), 90–99. doi:10.1016/j.akcej.2016.03.004

[13] D. Kartika and A.N.M. Salman, The 3-rainbow index of some graphs that constructed by joining a graph with a trivial graph, *IOP Conference Series: Journal of Physics* **1127** (2019), 012060. doi:10.1088/1742-6596/1127/1/012060

[14] I.S. Kumala and A.N.M. Salman, The rainbow connection number of a flower (C_m, K_n) graph and a flower (C_3, F_n) graph, *Procedia Computer Science* **74** (2015), 168–172. doi:10.1016/j.procs.2015.12.094

[15] X. Li, Y. Shi, and Y. Sun, Rainbow connections of graphs: a survey, *Graphs and Combinatorics* **29**(1) (2013), 1–38.

[16] X. Li and Y. Sun, An updated survey on rainbow connections of graphs - a dynamic survey, *Theory and Applications of Graphs* **0**(1) (2017), Article 3. doi:10.20429/tag.2017.000103

[17] T. Liu and Y. Hu, The 3-rainbow index of graph operations, *WSEAS Transactions on Mathematics* **13** (2014), 161–170.

[18] S. Nabila and A.N.M. Salman, The rainbow connection number of origami graphs and pizza graphs, *Procedia Computer Science* **74** (2015), 162–167. doi:10.1016/j.procs.2015.12.093

[19] D. Resty, and A.N.M. Salman, The rainbow connection number of an n-crossed prism graph and its corona product with a trivial graph, *Procedia Computer Science* **74** (2015), 143–150. doi:10.1016/j.procs.2015.12.090

[20] D.N.S. Simamora and A.N.M. Salman, The rainbow (vertex) connection number of pencil graphs, *Procedia Computer Science* **74** (2015), 138-142. doi:10.1016/j.procs.2015.12.089