Electronic Journal of Graph Theory and Applications **10** (1) (2022), 113–130

Electronic Journal of
Graph Theory and Applications

Relative g-noncommuting graph of finite groups

Monalisha Sharma, Rajat Kanti Nath

Department of Mathematical Sciences, Tezpur University, Napaam-784028, Sonitpur, Assam, India

monalishasharma2013@gmail.com and rajatkantinath@yahoo.com (corresponding author)

Abstract

Let G be a finite group. For a fixed element g in G and a given subgroup X of G, the relative g-noncommuting graph of G is a simple undirected graph whose vertex set is G and two vertices x and y are adjacent if $x \in X$ or $y \in X$ and $[x, y] \neq g, g^{-1}$. We denote this graph by $\Gamma^g_{X,G}$. In this paper, we obtain computing formulae for degree of any vertex in $\Gamma^g_{X,G}$ and characterize whether $\Gamma^g_{X,G}$ is a tree, star graph, lollipop or a complete graph together with some properties of $\Gamma^g_{X,G}$ involving isomorphism of graphs. We also present certain relations between the number of edges in $\Gamma^g_{X,G}$ and certain generalized commuting probabilities of G which give some computing formulae for the number of edges in $\Gamma^g_{X,G}$. Finally, we conclude this paper by deriving some bounds for the number of edges in $\Gamma^g_{X,G}$.

Keywords: finite group, g-noncommuting graph, commuting probability
Mathematics Subject Classification : 05C25, 20P05
DOI: 10.5614/ejgta.2022.10.1.7

1. Introduction

Throughout the paper, G is a finite non-abelian group and $Z(G) = \{z \in G : zx = xz \,\forall\, x \in G\}$ is the center of G. For any $X \leq G$ (X is a subgroup of G), we write $Z(X, G) = \{x \in X : xy = yx \,\forall\, y \in G\}$ and $Z(G, X) = \{x \in G : xy = yx \,\forall\, y \in X\}$, which implies $Z(G, G) = Z(G)$. For any element $x \in G$, we write $C_X(x) = \{y \in X : xy = yx\}$. Clearly, $Z(X, G) = \bigcap\limits_{x \in G} C_X(x)$. We write $K(X, G) = \{[x, y] : x \in X \text{ and } y \in G\}$, where $[x, y] = x^{-1}y^{-1}xy$, and $[X, G] = \langle K(X, G)\rangle$. Therefore, $[G, G] = G'$, the commutator subgroup of G.

Received: 26 September 2020, Revised: 14 July 2021, Accepted: 23 January 2022

The non-commuting graph of G, denoted by Γ_G, is a simple undirected graph with $G \setminus Z(G)$ as the vertex set and two distinct vertices x and y are adjacent whenever $[x, y] \neq 1$. This graph is originated due to the work of Erdös and Neumann [19] in 1976. After that different mathematicians studied different aspects of Γ_G. For instance, characterization of finite non-abelian groups with isomorphic non-commuting graph is discussed in [1, 3, 13, 16], Laplacian spectrum and energy of Γ_G are computed in [7, 8], expressions for various topological indices of Γ_G are obtained in [14, 15] and a characterization of finite groups through domination number of Γ_G can be found in [27]. Graph theoretic invariants such that clique number, vertex chromatic number, independent number etc. for non-commuting graph of dihedral groups are investigated in [24].

Various interesting generalizations of Γ_G due to Erfanian and his collaborators can be found in [2, 11, 25, 26]. In particular, in the year 2013, Tolue and Erfanian [25] introduced relative non-commuting graph for a given $X \leq G$ which is a simple undirected graph, denoted by $\Gamma_{X,G}$, with vertex set $G \setminus Z(X, G)$ where two vertices x and y ($x \neq y$) are adjacent if $x \in X$ or $y \in X$ and $[x, y] \neq 1$. In the year 2014, Tolue, Erfanian and Jafarzadeh [26] introduced g-noncommuting graph for a given element g of a finite group G which is denoted by Γ_G^g. Recall that g-noncommuting graph of G is a simple undirected graph whose vertex set is G and two vertices x and y ($x \neq y$) are adjacent if $[x, y] \neq g$ and g^{-1}. Fusing the concepts of $\Gamma_{X,G}$ and Γ_G^g, in this paper, we introduce relative g-noncommuting graph of G. For a given $X \leq G$ and $g \in G$, the relative g-noncommuting graph of G, denoted by $\Gamma_{X,G}^g$, is defined as the simple undirected graph whose vertex set is G and two vertices x and y ($x \neq y$) are adjacent if $x \in X$ or $y \in X$ and $[x, y] \neq g$ and g^{-1}. In [23], the induced subgraph of $\Gamma_{X,G}^g$ on $G \setminus Z(X, G)$ is considered and its properties, including connectivity and diameter with special attention to the dihedral groups, are investigated. Note that if $g = 1$ then the induced subgraph of $\Gamma_{X,G}^g$ on $G \setminus Z(X, G)$ is the relative non-commuting graph for a given $X \leq G$, that is $\Gamma_{X,G}$. Also, if $X = G$ then $\Gamma_{G,G}^g = \Gamma_G^g$. The ring theoretic analogues of $\Gamma_{X,G}$, Γ_G^g and $\Gamma_{X,G}^g$ can be found in [6, 17] and [22] respectively.

Let $\mathcal{G}_1 + \mathcal{G}_2$ be the join of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and let $\overline{\mathcal{G}}$ be the complement of $\mathcal{G}$. Then we have the following observations, where K_n is the complete graph on n vertices and $\deg(v)$ denotes the degree of any vertex v in $\Gamma_{X,G}^g$.

Observation 1.1. *Let $X \leq G$ and $g \in G$.*

(a) *If $g \notin K(X, G)$ then $\Gamma_{X,G}^g = \overline{K_{|G|-|X|}} + K_{|X|}$ and so*

$$\deg(x) = \begin{cases} |X|, & \text{if } x \in G \setminus X \\ |G| - 1, & \text{if } x \in X. \end{cases}$$

(b) *If $g = 1$ and $K(X, G) = \{1\}$ then $\Gamma_{X,G}^g = \overline{K_{|G|}}$.*

Observation 1.2. *Let $X \leq G$ and $g \in G \setminus K(X, G)$. Then*

(a) *$\Gamma_{X,G}^g$ is a tree $\iff X = \{1\}$ and $|X| = |G| = 2$.*
(b) *$\Gamma_{X,G}^g$ is a star $\iff X = \{1\}$.*
(c) *$\Gamma_{X,G}^g$ is complete $\iff X = G$.*

Note that if $X = Z(X, G)$ or G is abelian then $K(X, G) = \{1\}$. Therefore, in view of Observation 1.1, we shall consider G to be non-abelian, $X \leq G$ such that $X \neq Z(X, G)$ and $g \in K(X, G)$ throughout this paper.

In Section 2, we obtain computing formulae for degree of any vertex in $\Gamma^g_{X,G}$ and characterize whether $\Gamma^g_{X,G}$ is a tree, star graph, lollipop or a complete graph together with some properties of $\Gamma^g_{X,G}$ involving isomorphism of graphs. In Section 3, we obtain the number of edges in $\Gamma^g_{X,G}$ using $\Pr_g(X, G)$, which is the probability (introduced and studied in [4, 18]) that the commutator of a randomly chosen pair of elements $(x, y) \in X \times G$ equals g. We shall conclude this paper with some bounds for the number of edges in $\Gamma^g_{X,G}$.

2. Vertex degree and other properties

In this section we first obtain computing formula for $\deg(v)$ in terms of $|G|$, $|X|$ and the orders of the centralizers of v. We write $v \sim z$ if v is conjugate to z.

Theorem 2.1. *Let $v \in X$.*

(a) *For $g = 1$, $\deg(v) = |G| - |C_G(v)|$.*

(b) *For $g \neq 1$ and $g^2 \neq 1$,* $\deg(v) = \begin{cases} |G| - |C_G(v)| - 1, & \text{if } v \sim vg \text{ or } vg^{-1}, \\ |G| - 2|C_G(v)| - 1, & \text{if } v \sim vg \text{ and } vg^{-1}. \end{cases}$

(c) *For $g \neq 1$ and $g^2 = 1$, $\deg(v) = |G| - |C_G(v)| - 1$, whenever $v \sim vg$.*

Proof. (a) Let $g = 1$. Then $\deg(v)$ is the number of $z \in G$ such that z does not commute with v. Hence, $\deg(v) = |G| - |C_G(v)|$.

(b) Let $g \neq 1$ and $g^2 \neq 1$. Then $g \neq g^{-1}$. Suppose that $v \sim vg$ or vg^{-1} but not to both. Without any loss we assume that v is conjugate to vg. Then there exits $z \in G$ such that $z^{-1}vz = vg$, that is $[v, z] = v^{-1}z^{-1}vz = g$. Therefore, the set $S_g := \{z \in G : z^{-1}vz = vg\}$ is non-empty. Also, for any $\alpha \in S_g$ we have $[v, \alpha] = g$ which gives that α is not adjacent to z. Thus, $\alpha \in G$ is not adjacent to v if and only if $\alpha = v$ or $\alpha \in S_g$. Therefore, the number of vertices not adjacent to v is equal to $|S_g| + 1$.

Let $z_1 \in S_g$ and $z_2 \in C_G(v)z_1$. Then $z_2 = uz_1$ for some $u \in C_G(v)$. We have

$$z_2^{-1}vz_2 = z_1^{-1}u^{-1}vuz_1 = z_1^{-1}vz_1 = vg.$$

Therefore, $z_2 \in S_g$ and so $C_G(v)z_1 \subseteq S_g$. Suppose that $z_3 \in S_g$. Then $z_1^{-1}vz_1 = z_3^{-1}vz_3$ which implies $z_3z_1^{-1} \in C_G(v)$. Therefore, $z_3 \in C_G(v)z_1$ and so $S_g \subseteq C_G(v)z_1$. Thus $S_g = C_G(v)z_1$ and so $|S_g| = |C_G(v)|$. Hence, the number of vertices not adjacent to v is equal to $|C_G(v)| + 1$ and so $\deg(v) = |G| - |C_G(v)| - 1$.

If v is conjugate to vg and vg^{-1} then $S_g \cap S_{g^{-1}} = \emptyset$, where $S_{g^{-1}} := \{z \in G : z^{-1}vz = vg^{-1}\}$ and $|S_{g^{-1}}| = |C_G(v)|$. In this case, $\alpha \in G$ is not adjacent to v if and only if $\alpha = v$ or $\alpha \in S_g \cup S_{g^{-1}}$. Therefore, the number of vertices not adjacent to v is equal to $|S_g| + |S_{g^{-1}}| + 1 = 2|C_G(v)| + 1$. Hence, $\deg(v) = |G| - 2|C_G(v)| - 1$.

(c) Let $g \neq 1$ and $g^2 = 1$. Then $g = g^{-1}$ and so $vg = vg^{-1}$. Now, if v is conjugate to vg then, as shown in the proof of part (b), we have $\deg(v) = |G| - |C_G(v)| - 1$. □

Theorem 2.2. *Let $v \in G \setminus X$.*

(a) *For $g = 1$, $\deg(v) = |X| - |C_X(v)|$.*
(b) *For $g \neq 1$ and $g^2 \neq 1$,*
$$\deg(v) = \begin{cases} |X| - |C_X(v)|, & \text{if } v \sim vg \text{ or } vg^{-1} \text{ for some element in } X, \\ |X| - 2|C_X(v)|, & \text{if } v \sim vg \text{ and } vg^{-1} \text{ for some element in } X. \end{cases}$$
(c) *For $g \neq 1$ and $g^2 = 1$, $\deg(v) = |X| - |C_X(v)|$, whenever $v \sim vg$, for some element in X.*

Proof. The proof is analogous to the proof of Theorem 2.1. □

It is noteworthy that $g \notin K(X, G)$ if v is not conjugate to vg and vg^{-1}. Therefore, this case does not arise in Theorem 2.1 and Theorem 2.2. The degree of a vertex, in such case, is given by Observation 1.1.

Now, we present some properties of $\Gamma^g_{X,G}$. The following lemmas are useful in this regard.

Lemma 2.1. *If $g \neq 1$ and X has an element of order 3 then $\Gamma^g_{X,G}$ is not triangle free.*

Proof. Let $v \in X$ having order 3. Then the vertices $1, v$ and v^{-1} form a triangle in $\Gamma^g_{X,G}$. Hence, the lemma follows. □

Lemma 2.2. *If $v \in Z(X, G)$ then* $\deg(v) = \begin{cases} 0, & \text{if } g = 1, \\ |G| - 1, & \text{if } g \neq 1. \end{cases}$

Proof. By definition of $Z(X, G)$, it follows that $v \in X$ and $[v, z] = 1$ for all $z \in G$ and so $C_G(v) = G$. Therefore, if $g = 1$ then by Theorem 2.1(a) we have $\deg(v) = 0$. If $g \neq 1$ then all the elements of G except v are adjacent to v. Therefore, $\deg(v) = |G| - 1$. □

As a consequence of Lemma 2.2, we have $\gamma(\Gamma^g_{X,G}) = 1$ if $g \neq 1$ since $\{v\}$ is a dominating set for all $v \in Z(X, G)$, where $\gamma(\Gamma^g_{X,G})$ is the domination number of $\Gamma^g_{X,G}$. If $g \in X$ having even order then it can be seen that $\{g\}$ is also a dominating set in $\Gamma^g_{X,G}$. If $g = 1$ then $\gamma(\Gamma^g_{X,G}) \geq |Z(X, G)| + 1$. This lower bound is sharp because $\gamma(\Gamma^{(1)}_{X,S_3})$ is $2 = |Z(X, S_3)| + 1$, where X is any subgroup of S_3 of order 2. If $g = 1$ then, by Lemma 2.2, we also have that $\Gamma^g_{X,G}$ is disconnected. Hence, $\Gamma^1_{X,G}$ is not a tree and complete graph. Now we determine whether $\Gamma^g_{X,G}$ is a tree, star graph or complete graph if $g \neq 1$.

Theorem 2.3. *Let $X \leq G$ and $|X| \neq 2$. Then $\Gamma^{g \neq 1}_{X,G}$ is not a tree.*

Proof. Suppose for any $X \leq G$, $\Gamma^g_{X,G}$ is a tree, where $g \neq 1$. There exits a vertex v in $\Gamma^g_{X,G}$ of degree one.

Case 1. $v \in X$

By Theorem 2.1, $\deg(v) = |G| - |C_G(v)| - 1 = 1$ or $\deg(v) = |G| - 2|C_G(v)| - 1 = 1$. That is,
$$|G| - |C_G(v)| = 2 \text{ or } |G| - 2|C_G(v)| = 2.$$
Therefore, $|C_G(v)| = 2$ and $|G| = 4, 6$. Since G is non-abelian and $|X| \neq 1, 2$, we must have $G \cong S_3$ and $X = A_3$ or S_3. Therefore, by Lemma 2.1, $\Gamma^g_{X,G}$ has a triangle which is a contradiction.

Case 2. $v \in G \setminus X$

By Theorem 2.2, $\deg(v) = |X| - |C_X(v)| = 1$ or $\deg(v) = |X| - 2|C_X(v)| = 1$. Therefore, $|C_X(v)| = 1$ and $|X| = 2, 3$. However, $|X| \neq 2$ (by assumption). If $|X| = 3$ then, by Lemma 2.1, $\Gamma^g_{X,G}$ has a triangle which is a contradiction. Hence, the result follows. $\square$

The proof of Theorem 2.3 also gives the following result.

Theorem 2.4. *Let $X \leq G$ and $|X| \neq 2, 3$. Then $\Gamma^{g\neq 1}_{X,G}$ is not a lollipop. Further, if $|X| \neq 2, 3, 6$ then $\Gamma^{g\neq 1}_{X,G}$ has no vertex of degree 1.*

As a consequence of Theorem 2.3 we have the following results.

Corollary 2.1. *Let $X \leq G$ and $|X| \neq 2$. Then $\Gamma^{g\neq 1}_{X,G}$ is not a star graph.*

Corollary 2.2. *If $g \neq 1$ and G is a group of odd order then $\Gamma^g_{X,G}$ is not a tree and hence not a star.*

Theorem 2.5. *If $g \neq 1$ then $\Gamma^g_{X,G}$ is a star $\iff G \cong S_3$ and $|X| = 2$.*

Proof. By Lemma 2.2, $\deg(1) = |G| - 1$. Suppose that $\Gamma^g_{X,G}$ is a star graph. Then $\deg(v) = 1 \;\; \forall$ $1 \neq v \in G$. Since $g \in K(X, G)$ and $g \neq 1$ we have $X \neq \{1\}$. Suppose that $1 \neq y \in X$. If $g^2 = 1$, then by Theorem 2.1, we get $1 = \deg(y) = |G| - |C_G(y)| - 1$ which gives $|G| = 4$, a contradiction since G is non-abelian. If $g^2 \neq 1$, then by Theorem 2.1, we get $1 = \deg(y) = |G| - |C_G(y)| - 1$ or $|G| - 2|C_G(y)| - 1$ which gives $|G| = 6$. Therefore, $G \cong S_3$, $g = (123), (132)$ and $X = \{(1), (12)\}, \{(1), (13)\}, \{(1), (23)\}$ or $X = \{(1), (123), (132)\}$. If $|X| = 3$ then, by Lemma 2.1, Γ^g_{X,S_3} is not a star. If $|X| = 2$ then it is easy to see that Γ^g_{X,S_3} is a star. This completes the proof. $\square$

Theorem 2.6. *If $g \neq 1$ then $\Gamma^g_{X,G}$ is not complete.*

Proof. Let $\Gamma^g_{X,G}$ be complete graph where $g \neq 1$. Then $\deg(v) = |G| - 1 \;\; \forall \;\; v \in G$. Since $g \in K(X, G)$ and $g \neq 1$ we have $X \neq \{1\}$. Suppose that $1 \neq y \in X$. Then by Theorem 2.1, we get $|G| - 1 = \deg(y) = |G| - |C_G(y)| - 1$ or $|G| - 1 = \deg(y) = |G| - 2|C_G(y)| - 1$. Therefore, $|C_G(y)| = 0$, a contradiction. Hence, $\Gamma^g_{X,G}$ is not complete. $\square$

For X is a normal subgroup of G, we write $X \trianglelefteq G$.

Theorem 2.7. *Let $X \trianglelefteq G$ and $g \sim h$. Then $\Gamma^g_{X,G} \cong \Gamma^h_{X,G}$.*

Proof. Let $h = g^x := x^{-1}gx$ for some $x \in G$. Then for any two elements $a_1, a_2 \in G$, we have

$$[a_1^x, a_2^x] = h \text{ or } h^{-1} \iff [a_1, a_2] = g \text{ or } g^{-1}. \tag{2.1}$$

Consider the bijection $\phi : V(\Gamma^g_{X,G}) \to V(\Gamma^h_{X,G})$ given by $\phi(a) = a^x$ for all $a \in G$. We shall show that ϕ preserves adjacency.

Suppose that $a_1, a_2 \in V(\Gamma^h_{X,G})$. If a_1 and a_2 are not adjacent in $\Gamma^g_{X,G}$ then $[a_1, a_2] = g$ or g^{-1}. Therefore, by (2.1), it follows that $\phi(a_1)$ and $\phi(a_2)$ are not adjacent in $\Gamma^h_{X,G}$. If a_1 and a_2 are adjacent then atleast one of a_1 and a_2 must belong to X and $[a_1, a_2] \neq g, g^{-1}$. Without any loss assume that $a_1 \in X$. Since $X \trianglelefteq G$ we have $\phi(a_1) \in X$. By (2.1), we have $[\phi(a_1), \phi(a_2)] \neq h, h^{-1}$. Thus $\phi(a_1)$ and $\phi(a_2)$ are adjacent in $\Gamma^h_{X,G}$. Hence, the result follows. $\square$

A pair of isomorphisms (ϕ, ψ) is called a relative isoclinism between the pairs of groups (X_1, G_1) and (X_2, G_2), where $X_i \le G_i$ for $i = 1, 2$, $\phi : \frac{G_1}{Z(X_1,G_1)} \to \frac{G_2}{Z(X_2,G_2)}$ and $\psi : [X_1, G_1] \to [X_2, G_2]$, if

$$\phi\left(\frac{X_1}{Z(X_1, G_1)}\right) = \frac{X_2}{Z(X_2, G_2)} \text{ and } \psi \circ a_{(X_1,G_1)} = a_{(X_2,G_2)} \circ (\phi \times \phi),$$

where $a_{(X_i,G_i)} : \frac{X_i}{Z(X_i,G_i)} \times \frac{G_i}{Z(X_i,G_i)} \to [X_i, G_i]$ is given by

$$a_{(X_i,G_i)}((h_i Z(X_i, G_i), g_i Z(X_i, G_i))) = [h_i, g_i]$$

and $\phi \times \phi : \frac{X_1}{Z(X_1,G_1)} \times \frac{G_1}{Z(X_1,G_1)} \to \frac{X_2}{Z(X_2,G_2)} \times \frac{G_2}{Z(X_2,G_2)}$ is given by

$$(\phi \times \phi)((h_1 Z(X_1, G_1), g_1 Z(X_1, G_1))) = (\phi(h_1 Z(X_1, G_1)), \phi(g_1 Z(X_1, G_1))).$$

Thus for all $h_1 \in X_1$ and $g_1 \in G_1$ we must have $\psi([h_1, g_1]) = [h_2, g_2]$, where $g_2 \in \phi(g_1 Z(X_1, G_1))$ and $h_2 \in \phi(h_1 Z(X_1, G_1))$.

The pairs (X_1, G_1) and (X_2, G_2) are called relative isoclinic if there is a relative isoclinism between them. The concept of relative isoclinism between two pairs of groups was introduced in [18, 21, 25]. This coincides with one of the fascinating concepts of Hall [12] known as isoclinism between two groups if $X_i = G_i$ for $i = 1, 2$. In [25, Theorem 4.5], it was shown that Γ_{X_1,G_1} is isomorphic to Γ_{X_2,G_2} if (X_1, G_1) and (X_2, G_2) are relative isoclinic satisfying certain conditions. Tolue et al. [26, Theorem 2.16], also proved that $\Gamma^g_{G_1}$ is isomorphic to $\Gamma^{\psi(g)}_{G_2}$ if G_1 and G_2 are isoclinic such that $|Z(G_1)| = |Z(G_2)|$. We conclude Section 2 with Theorem 2.8 which generalizes [26, Theorem 2.16].

Theorem 2.8. *Let (ϕ, ψ) be a relative isoclinism between the pairs of groups (X_1, G_1) and (X_2, G_2). If $|Z(X_1, G_1)| = |Z(X_2, G_2)|$ then $\Gamma^g_{X_1,G_1}$ is isomorphic to $\Gamma^{\psi(g)}_{X_2,G_2}$.*

Proof. Since $\phi : \frac{G_1}{Z(X_1,G_1)} \to \frac{G_2}{Z(X_2,G_2)}$ is an isomorphism such that $\phi\left(\frac{X_1}{Z(X_1,G_1)}\right) = \frac{X_2}{Z(X_2,G_2)}$. So we have $|\frac{X_1}{Z(X_1,G_1)}| = |\frac{X_2}{Z(X_2,G_2)}|$ and $|\frac{G_1}{Z(X_1,G_1)}| = |\frac{G_2}{Z(X_2,G_2)}|$. Let $\left|\frac{X_1}{Z(X_1,G_1)}\right| = |\frac{X_2}{Z(X_2,G_2)}| = m$ and $\left|\frac{G_1}{Z(X_1,G_1)}\right| = |\frac{G_2}{Z(X_2,G_2)}| = n$. Given $|Z(X_1, G_1)| = |Z(X_2, G_2)|$, so $\exists$ a bijection $\theta : Z(X_1, G_1) \to Z(X_2, G_2)$. Let $\{h_1, h_2, \dots, h_m, g_{m+1}, \dots, g_n\}$ and $\{h'_1, h'_2, \dots, h'_m, g'_{m+1}, \dots, g'_n\}$ be two transversals of $\frac{G_1}{Z(X_1,G_1)}$ and $\frac{G_2}{Z(X_2,G_2)}$ respectively where $\{h_1, h_2, \dots, h_m\}$ and $\{h'_1, h'_2, \dots, h'_m\}$ are transversals of $\frac{X_1}{Z(X_1,G_1)}$ and $\frac{X_2}{Z(X_2,G_2)}$ respectively. Let us define ϕ as $\phi(h_i Z(X_1, G_1)) = h'_i Z(X_2, G_2)$ and $\phi(g_j Z(X_1, G_1)) = g'_j Z(X_2, G_2)$ for $1 \le i \le m$ and $m + 1 \le j \le n$.

Let $\mu : G_1 \to G_2$ be a map such that $\mu(h_i z) = h'_i \theta(z)$, $\mu(g_j z) = g'_j \theta(z)$ for $z \in Z(X_1, G_1)$, $1 \le i \le m$ and $m + 1 \le j \le n$. Clearly μ is a bijection. Suppose two vertices x and y in $\Gamma^g_{X_1,G_1}$ are adjacent. Then $x \in X_1$ or $y \in X_1$ and $[x, y] \ne g, g^{-1}$. Without any loss of generality, let us assume that $x \in X_1$. Then $x = h_i z_1$ for $1 \le i \le m$ and $y = k z_2$ where $z_1, z_2 \in Z(X_1, G_1)$, $k \in \{h_1, h_2, \dots, h_m, g_{m+1}, \dots, g_n\}$. Therefore, for some $k' \in \{h'_1, \dots, h'_m, g'_{m+1}, \dots, g'_n\}$, we have

$$\begin{aligned}
\psi([h_iz_1, kz_2]) = \psi([h_i, k]) &= \psi \circ a_{(X_1,G_1)}\left((h_iZ(X_1, G_1), kZ(X_1, G_1))\right)\\
&= a_{(X_2,G_2)} \circ (\phi \times \phi)\left((h_iZ(X_1, G_1), kZ(X_1, G_1))\right)\\
&= a_{(X_2,G_2)}((h_i'Z(X_2, G_2), k'Z(X_2, G_2)))\\
&= [h_i', k'] = [h_i'z_1', k'z_2'],
\end{aligned} \tag{2.2}$$

where $z_1', z_2' \in Z(X_2, G_2)$. Also,

$$\begin{aligned}
&[h_iz_1, kz_2] \neq g, g^{-1}\\
\Rightarrow\ &\psi([h_iz_1, kz_2]) \neq \psi(g), \psi(g^{-1})\\
\Rightarrow\ &[h_i'z_1', k'z_2'] \neq \psi(g), \psi^{-1}(g) \text{ (using (2.2))}\\
\Rightarrow\ &[h_i'\theta(z_1), k'\theta(z_2)] \neq \psi(g), \psi^{-1}(g)\\
\Rightarrow\ &[\mu(h_iz_1), \mu(kz_2)] \neq \psi(g), \psi^{-1}(g)\\
\Rightarrow\ &[\mu(x), \mu(y)] \neq \psi(g), \psi^{-1}(g).
\end{aligned}$$

Thus $\mu(x)$ is adjacent to $\mu(y)$ in $\Gamma^{\psi(g)}_{X_2,G_2}$ since $\mu(x) \in X_2$. Hence, the graphs $\Gamma^{g}_{X_1,G_1}$ and $\Gamma^{\psi(g)}_{X_2,G_2}$ are isomorphic under the map μ. □

3. Relation between $\Gamma^g_{X,G}$ and $\mathrm{Pr}_g(X, G)$

The commuting probability of a finite group G is the probability that a randomly chosen pair of elements of it commute with each other. The popularity of this probability have been constantly increasing since its inception which is attributed to the works of Erdös and Turán [9] published in the year 1968. Many mathematicians worked on commuting probability and its generalizations and obtained valuable results towards classification of finite groups. Results related to this notion can be found in [5] and the references listed there. Two most striking generalizations of commuting probability due to Pournaki et. al [20] and Erfanian et. al [10] are given by

$$\mathrm{Pr}_g(G) := \frac{|\{(c, d) \in G^2 : [c, d] = g\}|}{|G|^2}$$

and

$$\mathrm{Pr}_1(X, G) := \frac{|\{(c, d) \in X \times G : [c, d] = 1\}|}{|X||G|}$$

respectively. Blending these notions, Nath together with Das and Yadav [4, 18] considered the following generalization of commuting probability in their study

$$\mathrm{Pr}_g(X, G) := \frac{|\{(c, d) \in X \times G : [c, d] = g\}|}{|X||G|}.$$

In [25], Tolue and Erfanian established some relations between $\mathrm{Pr}_1(X, G)$ and relative non-commuting graphs of finite groups. In [26], Tolue et al. also established relations between Γ^g_G and $\mathrm{Pr}_g(G)$.

Their results stimulate us to obtain relations between $\Gamma^g_{X,G}$ and $\mathrm{Pr}_g(X,G)$. We obtain the number of edges of $\Gamma^g_{X,G}$, denoted by $|E(\Gamma^g_{X,G})|$, in terms of $\mathrm{Pr}_g(X,G)$. Clearly, if $g \notin K(X,G)$ then from Observation 1.1, we get

$$2|E(\Gamma^g_{X,G})| = 2|X||G| - |X|^2 - |X|.$$

The following theorem gives expressions for $|E(\Gamma^g_{X,G})|$, in terms of $\mathrm{Pr}_g(X,G)$ where $g \in K(X,G)$.

Theorem 3.1. *Let* $\mathrm{Pr}_{x\neq g}(X,G) := 1 - \mathrm{Pr}_g(X,G)$*, where* $X \neq \{1\}$.

(a) $2|E(\Gamma^1_{X,G})| = 2|X||G|\mathrm{Pr}_{x\neq 1}(X,G) - |X|^2(1 - \mathrm{Pr}_1(X))$.

(b) *If* $g \neq 1$ *and* $g^2 = 1$ *then*

$$2|E(\Gamma^g_{X,G})| = \begin{cases} 2|X||G|\mathrm{Pr}_{x\neq g}(X,G) - |X|^2(1 - \mathrm{Pr}_g(X)) - |X|, & \text{if } g \in X, \\ 2|X||G|\mathrm{Pr}_{x\neq g}(X,G) - |X|^2 - |X|, & \text{if } g \in G \setminus X. \end{cases}$$

(c) *If* $g \neq 1$ *and* $g^2 \neq 1$ *then*

$$2|E(\Gamma^g_{X,G})| = \begin{cases} 2|X||G|(1 - \sum\limits_{u=g,g^{-1}} \mathrm{Pr}_u(X,G)) & \\ \qquad - |X|^2(1 - \sum\limits_{u=g,g^{-1}} \mathrm{Pr}_u(X)) - |X|, & \text{if } g \in X, \\ 2|X||G|(1 - \sum\limits_{u=g,g^{-1}} \mathrm{Pr}_u(X,G)) & \\ \qquad - |X|^2 - |X|, & \text{if } g \in G \setminus X. \end{cases}$$

Proof. Let $E_1 = \{(c,d) \in X \times G : c \neq d, [c,d] \neq g$ and $[c,d] \neq g^{-1}\}$ and $E_2 = \{(c,d) \in G \times X : c \neq d, [c,d] \neq g$ and $[c,d] \neq g^{-1}\}$. Clearly we have a bijection from E_1 to E_2 defined by $(c,d) \mapsto (d,c)$. So $|E_1| = |E_2|$. It is easy to see that $|E(\Gamma^g_{X,G})|$ is equal to half $|E_1 \cup E_2|$. Therefore,

$$2|E(\Gamma^g_{X,G})| = 2|E_1| - |E_1 \cap E_2|, \tag{3.1}$$

where $E_1 \cap E_2 = \{(c,d) \in X \times X : c \neq d, [c,d] \neq g$ and $[c,d] \neq g^{-1}\}$.

(a) If $g = 1$ then we have

$$\begin{aligned} |E_1| &= |\{(c,d) \in X \times G : [c,d] \neq 1\}| \\ &= |X||G| - |\{(c,d) \in X \times G : [c,d] = 1\}| \\ &= |X||G|(1 - \mathrm{Pr}_g(X,G)) \end{aligned}$$

and

$$\begin{aligned} |E_1 \cap E_2| &= |\{(c,d) \in X \times X : [c,d] \neq 1\}| \\ &= |X|^2 - |\{(c,d) \in X \times X : [c,d] = 1\}| \\ &= |X|^2(1 - \mathrm{Pr}_g(X)). \end{aligned}$$

Hence, the result follows from (3.1).

(b) If $g \neq 1$ and $g^2 = 1$ then we have

$$\begin{aligned}|E_1| &= |\{(c,d) \in X \times G : c \neq d, [c,d] \neq g\}|\\ &= |X||G| - |\{(c,d) \in X \times G : [c,d] = g\}| - |\{(c,d) \in X \times G : c = d\}|\\ &= |X||G|(1 - \mathrm{Pr}_g(X,G)) - |X|.\end{aligned}$$

Now, if $g \in X$ then

$$\begin{aligned}|E_1 \cap E_2| &= |\{(c,d) \in X \times X : c \neq d, [c,d] \neq g\}|\\ &= |X|^2 - |\{(c,d) \in X \times X : [c,d] = g\}| - |\{(c,d) \in X \times X : c = d\}|\\ &= |X|^2(1 - \mathrm{Pr}_g(X)) - |X|.\end{aligned}$$

If $g \in G \setminus X$ then

$$|E_1 \cap E_2| = |X|^2 - |X|.$$

Hence, the result follows from (3.1).

(c) If g $\neq 1$ and $g^2 \neq 1$ then we have

$$\begin{aligned}|E_1| &= |\{(c,d) \in X \times G : c \neq d, [c,d] \neq g \text{ and } [c,d] \neq g^{-1}\}|\\ &= |X||G| - |\{(c,d) \in X \times G : [c,d] = g\}|\\ &\quad - |\{(c,d) \in X \times G : [c,d] = g^{-1}\}| - |\{(c,d) \in X \times X : c = d\}|\\ &= |X||G|(1 - \sum_{u=g,g^{-1}} \mathrm{Pr}_u(X,G)) - |X|.\end{aligned}$$

Now, if $g \in X$ then

$$\begin{aligned}|E_1 \cap E_2| &= |\{(c,d) \in X \times X : c \neq d, [c,d] \neq g \text{ and } [c,d] \neq g^{-1}\}|\\ &= |X|^2 - |\{(c,d) \in X \times X : [c,d] = g\}|\\ &\quad - |\{(c,d) \in X \times X : [c,d] = g^{-1}\}| - |\{(c,d) \in X \times X : c = d\}|\\ &= |X|^2(1 - \sum_{u=g,g^{-1}} \mathrm{Pr}_u(X)) - |X|.\end{aligned}$$

If $g \in G \setminus X$ then

$$|E_1 \cap E_2| = |X|^2 - |X|.$$

Hence, the result follows from (3.1). □

For an abelian group X we have

$$\mathrm{Pr}_g(X) = \begin{cases} 1, & \text{if } g = 1,\\ 0, & \text{if } g \neq 1.\end{cases}$$

Using these values in Theorem 3.1 we get Corollary 3.1.

Corollary 3.1. *Let $\{1\} \neq X \leq G$ be abelian.*

(a) $|E(\Gamma^1_{X,G})| = |X||G|(\mathrm{Pr}_{x\neq 1}(X,G))$.

(b) *If $g \neq 1$ and $g^2 = 1$ then*

$$2|E(\Gamma^g_{X,G})| = 2|X||G|(\mathrm{Pr}_{x\neq g}(X,G)) - |X|^2 - |X|.$$

(c) *If $g \neq 1$ and $g^2 \neq 1$ then*

$$2|E(\Gamma^g_{X,G})| = 2|X||G|(1 - \sum_{u=g,g^{-1}} \mathrm{Pr}_u(X,G)) - |X|^2 - |X|.$$

Proposition 3.1. *Let $X \leq G$ and $g \in K(X,G)$. Let $|[X,G]| = p$, the smallest prime dividing $|G|$.*

(a) $2p|E(\Gamma^1_{X,G})| = (p-1)[2|G|(|X| - |Z(X,G)|) - |X|(|X| - |Z(X)|)]$.

(b) *If $g \neq 1$ and $g^2 = 1$ then*

$$2p|E(\Gamma^g_{X,G})| = \begin{cases} 2|G|((p-1)|X| + |Z(X,G)|) \\ \quad - |X|((p-1)|X| + |Z(X)| + p), & \text{if } g \in X, \\ 2|G|((p-1)|X| + |Z(X,G)|) \\ \quad - p|X|(|X|+1), & \text{if } g \in G \setminus X. \end{cases}$$

(c) *If $g \neq 1$ and $g^2 \neq 1$ then*

$$2p|E(\Gamma^g_{X,G})| = \begin{cases} 2|G|((p-2)|X| + 2|Z(X,G)|) \\ \quad - |X|((p-2)|X| + 2|Z(X)| + p), & \text{if } g \in X, \\ 2|G|((p-2)|X| + 2|Z(X,G)|) \\ \quad - p|X|(|X|+1), & \text{if } g \in G \setminus X. \end{cases}$$

Proof. By [18, Lemma 3], we have

$$\mathrm{Pr}_g(X,G) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{|X:Z(X,G)|}\right), & \text{if } g = 1, \\ \frac{1}{p}\left(1 - \frac{1}{|X:Z(X,G)|}\right), & \text{if } g \neq 1. \end{cases}$$

Hence, the result follows from Theorem 3.1. □

It is worth mentioning that, in view of [18, Theorem B], the conclusion of Proposition 3.1 also holds if G is nilpotent such that $|[X,G]| = p$, where p is not necessarily the smallest prime. We also have the following corollary.

Corollary 3.2. *Let $X \leq G$ where X is abelian and G is nilpotent. Let $|[X,G]| = p$ be any prime and $g \in K(X,G)$.*

(a) $p|E(\Gamma^1_{X,G})| = (p-1)|G|(|X| - |Z(X,G)|)$.

(b) *If* $g \neq 1$ *and* $g^2 = 1$ *then*

$$2p|E(\Gamma^g_{X,G})| = 2|G|(|Z(X,G)| + (p-1)|X|) - p|X|(1 + |X|).$$

(c) *If* $g \neq 1$ *and* $g^2 \neq 1$ *then*

$$2p|E(\Gamma^g_{X,G})| = 2|G|(2|Z(X,G)| + (p-2)|X|) - p|X|(1 + |X|).$$

In [26, Proposition 2.14], Toule et al. obtained a relation between $|E(\Gamma^g_G)|$ and $\Pr_g(G)$. It is noteworthy that their result can also be obtained from the next proposition considering $X = G$, where $k(X)$ denotes the number of conjugacy classes in X.

Proposition 3.2. *Let* $\{1\} \neq X \trianglelefteq G$ *and* $g \in K(X, G)$.

(a) $2|E(\Gamma^1_{X,G})| = (2|G| - |X|)(|X| - k(X))$.

(b) *If* $g \neq 1$ *and* $g^2 = 1$ *then*

$$2|E(\Gamma^g_{X,G})| = 2|X||G|(\Pr_{x \neq g}(X,G)) - |X|^2(1 - \Pr_g(X)) - |X|.$$

(c) *If* $g \neq 1$ *and* $g^2 \neq 1$ *then*

$$2|E(\Gamma^g_{X,G})| = 2|X||G|(1 - 2\Pr_g(X,G)) \; - |X|^2(1 - 2\Pr_g(X)) - |X|.$$

Proof. If $g = 1$ then by [4, Corollary 2.4] we have

$$\Pr_g(X,G) = \Pr_g(X) = \frac{k(X)}{|X|}.$$

Hence, part (a) follows from Theorem 3.1. Parts (b) and (c) also follow from Theorem 3.1 noting that the case $g \in G \setminus X$ does not arise (since $g \in X$ if X is normal) and $\Pr_g(X,G) = \Pr_{g^{-1}}(X,G)$ (as shown in [4, Proposition 2.1]). □

Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of G. If $X \trianglelefteq G$ then by [4, Equation (6)] we have

$$\Pr_g(X,G) = \frac{1}{|G|} \sum_{\vartheta \in \mathrm{Irr}(G)} \langle \vartheta_X, \vartheta_X \rangle \frac{\vartheta(g)}{\vartheta(1)},$$

where ϑ_X is the restriction of $\vartheta \in \mathrm{Irr}(G)$ on X and $\langle , \rangle$ represents inner product of class functions. By the above expression for $\Pr_g(X,G)$ and Proposition 3.2 we get the following character theoretic formula for $|E(\Gamma^g_{X,G})|$.

Corollary 3.3. *Let* $\{1\} \neq X \trianglelefteq G$ *and* $g \in K(X, G)$.

(a) $2|E(\Gamma^1_{X,G})| = (|X| - |\mathrm{Irr}(X)|)(2|G| - |X|)$.

(b) *If* $g \neq 1$ *and* $g^2 = 1$ *then*

$$2|E(\Gamma^g_{X,G})| = 2|X| \left(|G| - \sum_{\vartheta \in \mathrm{Irr}(G)} \langle \vartheta_X, \vartheta_X \rangle \frac{\vartheta(g)}{\vartheta(1)} \right) - |X| \left(|X| - \sum_{\vartheta \in \mathrm{Irr}(X)} \frac{\vartheta(g)}{\vartheta(1)} \right) - |X|.$$

(c) *If* $g \neq 1$ *and* $g^2 \neq 1$ *then*

$$2|E(\Gamma^g_{X,G})| = 2|X| \left(|G| - 2\sum_{\vartheta \in \mathrm{Irr}(G)} \langle \vartheta_X, \vartheta_X \rangle \frac{\vartheta(g)}{\vartheta(1)} \right) - |X| \left(|X| - 2\sum_{\vartheta \in \mathrm{Irr}(X)} \frac{\vartheta(g)}{\vartheta(1)} \right) - |X|.$$

Corollary 3.4. *Let* $g \in K(G)$*.*

(a) $2|E(\Gamma^1_G)| = |G|(|G| - |\operatorname{Irr}(G)|)$.

(b) *If* $g \neq 1$ *then*

$$2|E(\Gamma^g_G)| = \begin{cases} |G| \left(|G| - 1 - \sum\limits_{\vartheta \in \mathrm{Irr}(G)} \frac{\vartheta(g)}{\vartheta(1)} \right), & \text{if } g^2 = 1, \\ |G| \left(|G| - 1 - 2\sum\limits_{\vartheta \in \mathrm{Irr}(G)} \frac{\vartheta(g)}{\vartheta(1)} \right), & \text{if } g^2 \neq 1. \end{cases}$$

4. Bounds for $|E(\Gamma^g_{X,G})|$

In [25, Section 3], Tolue and Erfanian obtained bounds for $|E(\Gamma_{X,G})|$. In this section some bounds for the number of edges in $\Gamma^g_{X,G}$ are obtained. By Theorem 3.1, we have

$$2|E(\Gamma^g_{X,G})| + |X|^2 + |X| = \begin{cases} |X|^2 \mathrm{Pr}_g(X) + 2|X||G|(\mathrm{Pr}_{x \neq g}(X,G)), & \text{if } g \in X, \\ 2|X||G|(\mathrm{Pr}_{x \neq g}(X,G)), & \text{if } g \in G \setminus X, \end{cases} \tag{4.1}$$

if $g \neq 1$ but $g^2 = 1$ and

$$2|E(\Gamma^g_{X,G})| + |X|^2 + |X| = \begin{cases} |X|^2 \sum\limits_{u = g, g^{-1}} \mathrm{Pr}_u(X) + 2|X||G|(1 - \sum\limits_{u=g,g^{-1}} \mathrm{Pr}_u(X,G)), & \text{if } g \in X, \\ 2|X||G|(1 - \sum\limits_{u=g,g^{-1}} \mathrm{Pr}_u(X,G)), & \text{if } g \in G \setminus X, \end{cases} \tag{4.2}$$

if $g \neq 1$ and $g^2 \neq 1$.

Proposition 4.1. *Let $X \leq G$ and $g \neq 1$.*

(a) *If $g^2 = 1$ then*

$$|E(\Gamma^g_{X,G})| \geq \begin{cases} \frac{|G||Z(X,G)|+|X|(|G|-1)+3|Z(X)|^2-|X|^2}{2}, & \text{if } g, \in X, \\ \frac{|G||Z(X,G)|+|X|(|G|-1)-|X|^2}{2}, & \text{if } g \in G \setminus X. \end{cases}$$

(b) *If $g^2 \neq 1$ then*

$$|E(\Gamma^g_{X,G})| \geq \begin{cases} \frac{2|G||Z(X,G)|+6|Z(X)|^2-|X|^2-|X|}{2}, & \text{if } g \in X, \\ \frac{2|G||Z(X,G)|-|X|^2-|X|}{2}, & \text{if } g \in G \setminus X. \end{cases}$$

Proof. By [4, Proposition 3.3], we get

$$1 - \Pr_g(X,G) \geq \frac{|X| + |Z(X,G)|}{2|X|} \text{ and } 1 - \sum_{u=g,g^{-1}} \Pr_u(X,G) \geq \frac{|Z(X,G)|}{|X|}. \tag{4.3}$$

Again, by [4, Proposition 3.1 (iii)], we have

$$\Pr_g(X) \geq \frac{3|Z(X)|^2}{|X|^2}. \tag{4.4}$$

(a) We have $g^2 = 1$. Therefore, if $g \in X$ then, using (4.1), (4.3) and (4.4), we get

$$2|E(\Gamma^g_{X,G})| + |X|^2 + |X| \geq |X|^2 \left(\frac{3|Z(X)|^2}{|X|^2}\right) + 2|X||G| \left(\frac{|X| + |Z(X,G)|}{2|X|}\right). \tag{4.5}$$

If $g \in G \setminus X$ then, using (4.1) and (4.3), we get

$$2|E(\Gamma^g_{X,G})| + |X|^2 + |X| \geq 2|X||G| \left(\frac{|X| + |Z(X,G)|}{2|X|}\right). \tag{4.6}$$

Hence, the result follows from (4.5) and (4.6).

(b) We have $g^2 \neq 1$. Therefore, if $g \in X$ then, using (4.2), (4.3) and (4.4), we get

$$2|E(\Gamma^g_{X,G})| + |X|^2 + |X| \geq 2|X||G| \left(\frac{|Z(X,G)|}{|X|}\right) + |X|^2 \left(\frac{6|Z(X)|^2}{|X|^2}\right). \tag{4.7}$$

If $g \in G \setminus X$ then, using (4.2) and (4.3), we have

$$2|E(\Gamma^g_{X,G})| + |X|^2 + |X| \geq \left(\frac{2|Z(X,G)||X||G|}{|X|}\right). \tag{4.8}$$

Hence, the result follows from (4.7) and (4.8). □

Proposition 4.2. *Let $X \leq G$ and $g \neq 1$.*

(a) *If $g^2 = 1$ then*

$$|E(\Gamma^g_{X,G})| \leq \begin{cases} \frac{4|X||G|-8|Z(X,G)||Z(G,X)|-|X|^2-|X|(|Z(X)|+2)}{4}, & \text{if } g \in X, \\ \frac{2|X||G|-4|Z(X,G)||Z(G,X)|-|X|^2-|X|}{2}, & \text{if } g \in G \setminus X. \end{cases}$$

(b) *If $g^2 \neq 1$ then*

$$|E(\Gamma^g_{X,G})| \leq \begin{cases} \frac{2|X||G|-8|Z(X,G)||Z(G,X)|-|X|(|Z(X)|+1)}{2}, & \text{if } g \in X \\ \frac{2|X||G|-8|Z(X,G)||Z(G,X)|-|X|^2-|X|}{2}, & \text{if } g \in G \setminus X. \end{cases}$$

Proof. By [4, Proposition 3.1 (ii)], we get

$$1 - \Pr_g(X, G) \leq \frac{|X||G| - 2|Z(X, G)||Z(G, X)|}{|X||G|} \tag{4.9}$$

and

$$1 - \sum_{u=g,g^{-1}} \Pr_u(X, G) \leq \frac{|X||G| - 4|Z(X, G)||Z(G, X)|}{|X||G|}. \tag{4.10}$$

Also, by [4, Proposition 3.3], we get

$$\Pr_g(X) \leq \frac{|X| - |Z(X)|}{2|X|}. \tag{4.11}$$

(a) We have $g^2 = 1$. Therefore, if $g \in X$ then, using (4.1), (4.9) and (4.11), we get

$$\begin{aligned} 2|E(\Gamma^g_{X,G})|&+|X|^2 + |X| \\ &\leq |X|^2 \left(\frac{|X| - |Z(X)|}{2|X|}\right) + 2|X||G| \left(\frac{|X||G| - 2|Z(X, G)||Z(G, X)|}{|X||G|}\right). \end{aligned} \tag{4.12}$$

If $g \in G \setminus X$ then, using (4.1) and (4.9), we get

$$2|E(\Gamma^g_{X,G})| + |X|^2 + |X| \leq 2|X||G| \left(\frac{|X||G| - 2|Z(X, G)||Z(G, X)|}{|X||G|}\right). \tag{4.13}$$

Hence, the result follows from (4.12) and (4.13).
(b) We have $g^2 \neq 1$. Therefore, if $g \in X$ then, using (4.2), (4.10) and (4.11), we get

$$\begin{aligned} 2|E(\Gamma^g_{X,G})| + |X|^2 + |X| \leq\, & 2|X||G| \left(\frac{|X||G| - 4|Z(X, G)||Z(G, X)|}{|X||G|}\right) \\ & + |X|^2 \left(\frac{|X| - |Z(X)|}{|X|}\right). \end{aligned} \tag{4.14}$$

If $g \in G \setminus X$ then, using (4.2) and (4.10), we get

$$2|E(\Gamma^g_{X,G})| + |X|^2 + |X| \leq 2|X||G| \left(\frac{|X||G| - 4|Z(X, G)||Z(G, X)|}{|X||G|}\right). \tag{4.15}$$

Hence, the result follows from (4.14) and (4.15). $\square$

In the remaining results p stands for the smallest prime such that $p \mid |G|$ and $g \neq 1$.

Proposition 4.3. (a) *If $g^2 = 1$ then*

$$|E(\Gamma_{X,G}^g)| \geq \begin{cases} \frac{2(p-1)|X||G|+2|Z(X,G)||G|-p|X|^2+3p|Z(X)|^2-p|X|}{2p}, & \text{if } g \in X, \\ \frac{2(p-1)|X||G|+2|Z(X,G)||G|-p|X|^2-p|X|}{2p}, & \text{if } g \in G \setminus X. \end{cases}$$

(b) *If $g^2 \neq 1$ then*

$$|E(\Gamma_{X,G}^g)| \geq \begin{cases} \frac{2(p-2)|X||G|+4|Z(X,G)||G|-p|X|^2+6p|Z(X)|^2-p|X|}{2p}, & \text{if } g \in X, \\ \frac{2(p-2)|X||G|+4|Z(X,G)||G|-p|X|^2-p|X|}{2p}, & \text{if } g \in G \setminus X. \end{cases}$$

Proof. By [4, Proposition 3.3], we get

$$1 - \Pr_g(X, G) \geq \frac{|Z(X,G)| + (p-1)|X|}{p|X|} \tag{4.16}$$

and

$$1 - \sum_{u=g,g^{-1}} \Pr_u(X, G) \geq \frac{2|Z(X,G)| + (p-2)|X|}{p|X|}. \tag{4.17}$$

(a) We have $g^2 = 1$. Therefore, if $g \in X$ then, using (4.1), (4.16) and (4.4), we get

$$2|E(\Gamma_{X,G}^g)| + |X|^2 + |X| \geq |X|^2 \left(\frac{3|Z(X)|^2}{|X|^2}\right) + 2|X||G| \left(\frac{|Z(X,G)| + (p-1)|X|}{p|X|}\right). \tag{4.18}$$

If $g \in G \setminus X$ then, using (4.1) and (4.16), we get

$$2|E(\Gamma_{X,G}^g)| + |X|^2 + |X| \geq 2|X||G| \left(\frac{|Z(X,G)| + (p-1)|X|}{p|X|}\right). \tag{4.19}$$

Hence, the result follows from (4.18) and (4.19).
(b) We have $g^2 \neq 1$. Therefore, if $g \in X$ then, using (4.2), (4.17) and (4.4), we get

$$2|E(\Gamma_{X,G}^g)| + |X|^2 + |X| \geq 2|X||G| \left(\frac{(p-2)|X| + 2|Z(X,G)|}{p|X|}\right) + |X|^2 \left(\frac{6|Z(X)|^2}{|X|^2}\right). \tag{4.20}$$

If $g \in G \setminus X$ then, using (4.2) and (4.17), we have

$$2|E(\Gamma_{X,G}^g)| + |X|^2 + |X| \geq 2|X||G| \left(\frac{(p-2)|X| + 2|Z(X,G)|}{p|X|}\right). \tag{4.21}$$

Hence, the result follows from (4.20) and (4.21). □

Proposition 4.4. (a) *If $g^2 = 1$ then*

$$|E(\Gamma_{X,G}^g)| \leq \begin{cases} \frac{2p|X||G|-4p|Z(X,G)||Z(G,X)|-(p-1)|X|^2-|X||Z(X)|-p|X|}{2p}, & \text{if } g \in X, \\ \frac{2|X||G|-4|Z(X,G)||Z(G,X)|-|X|^2-|X|}{2}, & \text{if } g \in G \setminus X. \end{cases}$$

(b) *If* $g^2 \neq 1$ *then*

$$|E(\Gamma^g_{X,G})| \leq \begin{cases} \frac{2p|X||G|-8p|Z(X,G)||Z(G,X)|-(p-2)|X|^2-2|X||Z(X)|-p|X|}{2p}, & \text{if } g \in X, \\ \frac{2|X||G|-8|Z(X,G)||Z(G,X)|-|X|^2-|X|}{2}, & \text{if } g \in G \setminus X. \end{cases}$$

Proof. By [4, Proposition 3.3], we get

$$\Pr_g(X) \leq \frac{|X|-|Z(X)|}{p|X|}. \tag{4.22}$$

(a) We have $g^2 = 1$. Therefore, if $g \in X$ then, using (4.1), (4.9) and (4.22), we get

$$\begin{aligned} 2|E(\Gamma^g_{X,G})|+&|X|^2+|X| \\ &\leq |X|^2\left(\frac{|X|-|Z(X)|}{p|X|}\right)+2|X||G|\left(\frac{|X||G|-2|Z(X,G)||Z(G,X)|}{|X||G|}\right). \end{aligned} \tag{4.23}$$

If $g \in G \setminus X$ then, using (4.1) and (4.9), we get

$$2|E(\Gamma^g_{X,G})|+|X|^2+|X| \leq 2|X||G|\left(\frac{|X||G|-2|Z(X,G)||Z(G,X)|}{|X||G|}\right). \tag{4.24}$$

Hence, the result follows from (4.23) and (4.24).
(b) We have $g^2 \neq 1$. Therefore, if $g \in X$ then, using (4.2), (4.10) and (4.22), we get

$$\begin{aligned} 2|E(\Gamma^g_{X,G})|+&|X|^2+|X| \\ &\leq 2|X|^2\left(\frac{|X|-|Z(X)|}{p|X|}\right)+2|X||G|\left(\frac{|X||G|-4|Z(X,G)||Z(G,X)|}{|X||G|}\right). \end{aligned} \tag{4.25}$$

If $g \in G \setminus X$ then, using (4.2) and (4.10), we get

$$2|E(\Gamma^g_{X,G})|+|X|^2+|X| \leq 2|X||G|\left(\frac{|X||G|-4|Z(X,G)||Z(G,X)|}{|X||G|}\right). \tag{4.26}$$

Hence, the result follows from (4.25) and (4.26). □

Note that several other bounds for $|E(\Gamma^g_{X,G})|$ can be obtained using different combinations of the bounds for $\Pr_g(X,G)$ and $\Pr_g(X)$. We conclude this paper with certain bounds for $|E(\Gamma^g_G)|$ which are obtained by putting $X = G$ in the above propositions.

Corollary 4.1. (a) *If* $g^2 = 1$ *then*

$$\frac{3|G|^2-8|Z(G)|^2-|G|(|Z(G)|+2)}{4} \geq |E(\Gamma^g_G)| \geq \frac{|G||Z(G)|+3|Z(G)|^2-|G|}{2}.$$

(b) *If* $g^2 \neq 1$ *then*

$$\frac{2|G|^2-8|Z(G)|^2-|G|(|Z(G)|+1)}{2} \geq |E(\Gamma^g_G)| \geq \frac{2|G||Z(G)|+6|Z(G)|^2-|G|^2-|G|}{2}$$

Corollary 4.2. (a) *If* $g^2 = 1$ *then*

$$\frac{(p+1)|G|^2 - 4p|Z(G)|^2 - |G||Z(G)| - p|G|}{2p} \geq |E(\Gamma_G^g)| \geq \frac{(p-2)|G|^2 + 2|Z(G)||G| + 3p|Z(G)|^2 - p|G|}{2p}.$$

(b) *If* $g^2 \neq 1$ *then*

$$\frac{(p+2)|G|^2 - 8p|Z(G)|^2 - 2|G||Z(G)| - p|G|}{2p} \geq |E(\Gamma_G^g)| \geq \frac{(p-4)|G|^2 + 4|Z(G)||G| + 6p|Z(G)|^2 - p|G|}{2p}.$$

Acknowledgment

The first author would like to thank DST for the INSPIRE Fellowship. The authors would like to thank the referee for his/her valuable comments and suggestions.

References

[1] A. Abdollahi, S. Akbari, and H. R. Maimani, Non-commuting graph of a group, *J. Algebra* **298**(2006), 468–492.

[2] Z. Barati, A. Erfanian, K. Khashyarmanesh, and K. Nafar, A generalization of non-commuting graph via automorphisms of a group, *Comm. Algebra* **42**(2014), 174–185.

[3] M.R. Darafsheh, Groups with the same non-commuting graph, *Discret. Appl. Math.* **157**(2009), 833–837.

[4] A.K. Das and R.K. Nath, On generalized relative commutativity degree of a finite group, *Internat. Electron. J. Algebra* **7**(2010), 140–151.

[5] A.K. Das, R.K. Nath, and M.R. Pournaki, A survey on the estimation of commutativity in finite groups, *Southeast Asian Bull. Math.* **37**(2013), 161–180.

[6] J. Dutta, D.K. Basnet and R.K. Nath, On generalized non-commuting graph of a finite ring, *Algebra Colloq.* **25**(2018), 149–160.

[7] P. Dutta, J. Dutta, and R.K. Nath, On Laplacian spectrum of non-commuting graph of finite groups, *Indian J. Pure Appl. Math.* **49**(2018), 205–216.

[8] P. Dutta and R.K. Nath, On Laplacian energy of non-commuting graphs of finite groups, *J. Linear Top. Algebra* **7**(2018), 121–132.

[9] P. Erdős and P. Turán, On some problems of a statistical group-theory, IV, *Acta. Math. Acad. Sci. Hungar.* **19**(1968), 413–435.

[10] A. Erfanian, R. Rezaei, and P. Lescot, On the relative commutativity degree of a subgroup of a finite group, *Comm. Algebra* **35**(2007), 4183–4197.

[11] S. Ghayekhloo, A. Erfanian, and B. Tolue, The generalised non-commuting graph of a finite group, *Proc. Bulg. Acad. Sci.* **67**(2014), 1037–1044.

[12] P. Hall, The classification of prime-power groups, *J. Reine. Angew. Math.* **182**(1940), 130–141.

[13] M. Jahandideh, M.R. Darafsheh, N.H. Sarmin, and S.M.S. Omer, Conditions on the edges and vertices of non-commuting graph, *J. Tech.* **74**(2015), 73–76.

[14] M. Jahandideh, M.R. Darafsheh, and N. Shirali, Computation of topological indices of non-commuting graphs, *Ital. J. Pure Appl. Math.* **34**(2015), 299–310.

[15] M. Jahandideh, N.H. Sarmin, and S.M.S. Omer, The topological indices of noncommuting graph of a finite group, *Int. J. Pure Appl. Math.* **105**(2015), 27–38.

[16] A.R. Moghaddamfar, W.J. Shi, W. Zhou, and A.R. Zokayi, On the non-commuting graph associated with a finite group, *Sib. Math. J.* **46**(2005), 325–332.

[17] R.K. Nath, M. Sharma, P. Dutta, and Y. Shang, On r-noncommuting graph of finite rings, *Axioms* **10** (2021), 233-1–233-14.

[18] R.K. Nath and M.K. Yadav, Some results on relative commutativity degree, *Rend. Circ. Mat. Palermo* **64** (2015), 229–239.

[19] B.H. Neumann, A problem of Paul Erdös on groups, *J. Aust. Math. Soc.* **21** (1976), 467–472.

[20] M.R. Pournaki and R. Sobhani, Probability that the commutator of two group elements is equal to a given element, *J. Pure Appl. Algebra* **212** (2008), 727–734.

[21] A.R. Salemkar, F. Saeedi, and T. Karimi, The structures of isoclinism of pair of groups, *Southeast Asian Bull. Math.* **31** (2007), 1173–1181.

[22] M. Sharma and R.K. Nath, Relative r-noncommuting graph of finite rings, Preprint.

[23] M. Sharma, R.K. Nath, and Y. Shang, On g-noncommuting graph of a finite group relative to its subgroups, *Mathematics* **9** (2021), 3147-1–3147-13.

[24] A.A. Talebi, On the non-commuting graphs of group D_{2n}, *Int. J. Algebra* **2**(2008), 957–961.

[25] B. Tolue and A. Erfanian, Relative non-commuting graph of a finite group, *J. Algebra Appl.* **12** (2013), 1250157-1–1250157-12.

[26] B. Tolue, A. Erfanian, and A. Jafarzadeh, A kind of non-commuting graph of finite groups, *J. Sci. Islam. Repub. Iran* **25** (2014), 379–384.

[27] E. Vatandoost and M. Khalili, Domination number of the non-commuting graph of finite groups, *Electron. J. Graph Theory Appl.* **6** (2) (2018), 228–237.