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Rainbow connection number of comb product of graphs

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Abstract

An edge-colored graph G is called a rainbow connected if any two vertices are connected by a path whose edges have distinct colors. Such a path is called a rainbow path. The smallest number of colors required in order to make G rainbow connected is called the rainbow connection number of G. For two connected graphs G and H with $v \in V(H)$, the comb product between G and H, denoted by $G \triangleright_v H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the i-th copy of H at the vertex v to the i-th vertex of G. In this paper, we give sharp lower and upper bounds for the rainbow connection number of comb product between two connected graphs. We also determine the exact values of rainbow connection number of $G \triangleright_v H$ for some connected graphs G and H.

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1. Introduction

Throughout this paper, all graphs are simple, finite, and undirected. For $h \in \mathbb{N}$, we define a coloring $c : E(G) \rightarrow \{1, 2, \ldots, h\}$ of the edges of G such that the adjacent edges can be

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colored the same. An edge-colored graph G is called a *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. Such a path is called a *rainbow path*. In this case, the edge-coloring c is called a *rainbow h-coloring* of G. The smallest number of colors required in order to make G rainbow connected is called the *rainbow connection number* of G, denoted by $rc(G)$. This concept was introduced by Chartrand et al. in 2008 [8]. It is obvious that $diam(G) \leq rc(G) \leq |E(G)|$, where $diam(G)$ and $|E(G)|$ denote the diameter and the size of G, respectively.

The rainbow connection number has an important application in security systems in a communication network. One of the things that can be done so that any two people in a communication network can communicate securely is by assigning some passwords to a path connecting them (which may have other people as intermediaries) so that there is no repetition of the passwords in it. Of course, the number of passwords that we used are expected to be as minimal as possible. The minimum number of these passwords is represented by the rainbow connection number.

Many previous researchers determined the rainbow connection number of graphs by limiting the study to certain classes of graphs. This is because computing the rainbow connection number of graphs is an NP-Hard problem [7]. Chartrand et al. in [8] determined the rainbow connection number of some classes of graphs, such as complete graphs, trees, cycles, and wheels. These results are given in Theorems 1.1-1.3. Further, Sy et al. determined the rainbow connection number of fans and suns [23], meanwhile Shulhany and Salman determined the rainbow connection number of stellar graphs [20]. Other researchers also interested in studying the color code techniques in rainbow connection like Septyanto and Sugeng did [19].

Theorem 1.1. *[8] Let G be a nontrivial connected graph of size m. Then*

(a) $rc(G) = 1$ *if and only if G is a complete graph,*

(b) $rc(G) = m$ *if and only if G is a tree.*

Theorem 1.2. *[8] For each integer $n \geq 4$, the rainbow connection number of a cycle C_n is* $rc(C_n) = \lceil \frac{n}{2} \rceil$.

Theorem 1.3. *[8] For $n \geq 3$, the rainbow connection number of a wheel W_n is*

$$rc(W_n) = \begin{cases} 1, & \textit{if } n = 3; \\ 2, & \textit{if } n \in \{4, 5, 6\}; \\ 3, & \textit{if } n \geq 7. \end{cases}$$

There are also some results about bounds for rainbow connection number of graphs resulted from graph operations; for instance: Cartesian product graphs [12, 15], composition (lexicographic product) graphs [10, 15], join of graphs [15], direct product and strong product graphs [10], and amalgamation of some graphs [9]. Some other results on rainbow connection number of graphs can be found in [11, 16, 17, 21, 22]. An overview about rainbow connection number can be found in a survey by Li et al. [13] and a book of Li and Sun [14].

Later, Awanis and Salman [2] introduced a new concept called a strong k-rainbow index. A *rainbow tree* in G is a tree whose edges have distinct colors. For an integer $k \in \{2, 3, \ldots, n\}$, the *strong k-rainbow index* of G, denoted by $srx_k(G)$, is the smallest number of colors required in an edge-coloring of G such that every k vertices of G are connected by a rainbow tree with minimum size. If $k = 2$, then the strong 2-rainbow index of G is called the *strong rainbow connection number* of G, denoted by $src(G)$ [8]. Awanis and Salman [2] determined the strong 3-rainbow index of some certain graphs, meanwhile Salman et al. [18] investigated the characterization of graphs whose strong 3-rainbow index equals 2. Other researchers also determined the strong 3-rainbow index of some graph operations which can be found in [2, 3, 4, 5, 6].

In this paper, we study the rainbow connection number of comb product of graphs. The following definition of comb product of two graphs is taken from [1]. Let G and H be two connected graphs. Let v be a vertex of H. The *comb product* between G dan H, denoted by $G \triangleright_v H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the i-th copy of H at the vertex v to the i-th vertex of G. We first determine the lower and upper bounds for the rainbow connection number of $G \triangleright_v H$, then we provide comb product of graphs whose rainbow connection number satisfies the bounds. These results are given in Section 2. We also determine the exact values of rainbow connection number of $G \triangleright_v H$ for some connected graphs G and H which are given in Section 3.

2. Sharp lower and upper bounds for $rc(G \triangleright_v H)$

Let G and H be two connected graphs of order m and n, respectively, with $V(G) = \{g_1, g_2, \ldots, g_m\}$ and $V(H) = \{h_1, h_2, \ldots, h_n\}$. Let v be a vertex of H. According to the definition of comb product, we have $V(G \triangleright_v H) = V(G) \times V(H) = \{(g_i, h_j) : g_i \in V(G), h_j \in V(H)\}$ and two vertices (g_i, h_j) and (g_k, h_l) are adjacent if and only if

(a) $g_i = g_k$ and $h_j h_l \in E(H)$, or

(b) $g_i g_k \in E(G)$ and $h_j = h_l = v$.

Without loss of generality, let $v = h_1$. For each $i \in \{1, 2, \ldots, m\}$, let $H(i)$ denote a subgraph of $G \triangleright_v H$ induced by $\{(g_i, h_j) : j \in \{1, 2, \ldots, n\}\}$, and $G(h_1)$ denote a subgraph of $G \triangleright_v H$ induced by $\{(g_i, h_1) : i \in \{1, 2, \ldots, m\}\}$. For further discussion, we denote $c(X)$ as a set of colors assigned to the edges in $X \subseteq E(G \triangleright_v H)$.

The following theorem provides the sharp lower and upper bounds for the rainbow connection number of comb product of two arbitrary graphs.

Theorem 2.1. *Let G and H be two connected graphs of order m and n, respectively, and let $v \in V(H)$. Then*

$$diam(G \triangleright_v H) \leq rc(G \triangleright_v H) \leq rc(G) + m(rc(H)).$$

Proof. Without loss of generality, let $v = h_1$. It is obvious that $diam(G \triangleright_v H) \leq rc(G \triangleright_v H)$. Let c^1 be a rainbow $rc(G)$-coloring of G and c^2 be a rainbow $rc(H)$-coloring of H. We define an edge-coloring $c : E(G \triangleright_v H) \to \{1, 2, \ldots, rc(G) + m(rc(H))\}$ as follows.

$$c(e) = \begin{cases} c^1(e), & e \in E(G(h_1)); \\ rc(G) + c^2(e) + (p-1)rc(H), & e \in E(H(p)) \text{ for each } p \in \{1, 2, \ldots, m\}. \end{cases}$$

Now, we show that an edge-coloring c above is a rainbow coloring of $G \triangleright_v H$. For $i, k \in \{1, 2, \ldots, m\}$ and $j, l \in \{1, 2, \ldots, n\}$, let $x = (g_i, h_j)$ and $y = (g_k, h_l)$ be two vertices of $G \triangleright_v H$. If $i = k$, then there exists a rainbow $x - y$ path by edge-coloring c corresponding to edge-coloring c^2. If $i \neq k$, there exist a rainbow $(g_i, h_j) - (g_i, h_1)$ path P_1 in $H(i)$, a rainbow $(g_i, h_1) - (g_k, h_1)$ path P_2 in $G(h_1)$, and a rainbow $(g_k, h_1) - (g_k, h_l)$ path P_3 in $H(k)$, so that $c\,(E(P_a)) \cap c\,(E(P_b)) = \emptyset$ for distinct $a, b \in \{1, 2, 3\}$. Then $P = P_1 \cup P_2 \cup P_3$ is a rainbow $x - y$ path. □

Now, we prove the existence of comb product of graphs whose rainbow connection number satisfies either the lower or upper bound in Theorem 2.1. These results are given in the next two theorems.

Theorem 2.2. *Let G be a connected graph of order $m \geq 2$ with $rc(G) = diam(G)$, C_n be a cycle of order $n \geq 3$, and $v \in V(C_n)$. For $m \geq 2$ and even $n \geq 4$, or $m = 2$ and odd $n \geq 3$, $rc\,(G \triangleright_v C_n) = diam\,(G \triangleright_v C_n)$.*

Proof. Let $V(C_n) = \{h_1, h_2, \ldots, h_n\}$ such that $E(C_n) = \{h_j h_{j+1} : j \in \{1, 2, \ldots, n\}$ and $h_{n+1} = h_1\}$. Without loss of generality, let $v = h_1$. By Theorem 2.1, we only need to show that $rc\,(G \triangleright_v C_n) \leq diam\,(G \triangleright_v C_n) = diam(G) + 2\,diam(C_n)$.

For $m \geq 2$ and even $n \geq 4$, $diam\,(G \triangleright_v C_n) = rc(G) + n$. Let c' be a rainbow $rc(G)$-coloring of G. We define an edge-coloring $c : E\,(G \triangleright_v C_n) \to \{1, 2, \ldots, rc(G) + n\}$ as follows.

(i) For each $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, define $c\,((g_i, h_j)\,(g_i, h_{j+1})) = j$.

(ii) For distinct $i, k \in \{1, 2, \ldots, m\}$, define $c\,((g_i, h_1)\,(g_k, h_1)) = c'(g_i g_k) + n$.

Meanwhile for $m = 2$ and odd $n \geq 3$, $diam\,(G \triangleright_v C_n) = n$. We define an edge-coloring $c : E\,(G \triangleright_v C_n) \to \{1, 2, \ldots, n\}$ as follows.

(i) Define $c\,((g_1, h_1)\,(g_2, h_1)) = 1$.

(ii) Define $c\,((g_1, h_j)\,(g_1, h_{j+1})) = j+1$ for each $j \in \left\{1, 2, \ldots, \frac{n+1}{2}\right\}$ and $c\,((g_1, h_j)\,(g_1, h_{j+1})) = j - \frac{n-1}{2}$ for each $j \in \left\{\frac{n+3}{2}, \ldots, n\right\}$.

(iii) Define $c\,((g_2, h_j)\,(g_2, h_{j+1})) = j + \frac{n+1}{2}$ for each $j \in \left\{1, 2, \ldots, \frac{n-1}{2}\right\}$ and $c\,((g_2, h_j)\,(g_2, h_{j+1})) = j$ for each $j \in \left\{\frac{n+1}{2}, \ldots, n\right\}$.

Now, we show that there exists a rainbow $x - y$ path for any two vertices $x, y \in V\,(G \triangleright_v C_n)$. For $i, k \in \{1, 2, \ldots, m\}$ and $j, l \in \{1, 2, \ldots, n\}$, let $x = (g_i, h_j)$ and $y = (g_k, h_l)$. We consider two cases.

Case 1. $i = k$

Observe that the edge-colorings c above assign n distinct colors to the edges of $C_n(i)$ for $m \geq 2$ and even $n \geq 4$, and $\frac{n+1}{2}$ distinct colors to the edges of $C_n(i)$ for $m = 2$ and odd $n \geq 3$. Hence, it is easy to find a rainbow $x - y$ path in $G \triangleright_v C_n$.

Case 2. $i \neq k$

If $j = l = 1$, there exists a rainbow $x - y$ path in $G \triangleright_v C_n$ by edge-coloring c corresponding to edge-coloring c'. Otherwise, there exist a shortest rainbow $(g_i, h_j) - (g_i, h_1)$ path P_1 in $C_n(i)$, a shortest rainbow $(g_i, h_1) - (g_k, h_1)$ path P_2 in $G(h_1)$, and a shortest rainbow $(g_k, h_1) - (g_k, h_l)$ path P_3 in $C_n(k)$, so that $c(E(P_a)) \cap c(E(P_b)) = \emptyset$ for distinct $a, b \in \{1, 2, 3\}$. Then $P = P_1 \cup P_2 \cup P_3$ is a rainbow $x - y$ path. $\square$

Theorem 2.3. *Let G and H be two arbitrary trees of order m and n, respectively, and let $v \in V(H)$. Then $rc(G \triangleright_v H) = rc(G) + m(rc(H))$.*

Proof. Note that $G \triangleright_v H$ is also a tree with $|E(G \triangleright_v H)| = |E(G)| + m(|E(H)|)$. According to Theorem 1.1(b), $rc(G) = |E(G)|$ if and only if G is a tree. Thus, $rc(G \triangleright_v H) = |E(G \triangleright_v H)| = |E(G)| + m(|E(H)|) = rc(G) + m(rc(H))$. $\square$

For illustration of Theorems 2.2 and 2.3, please see Figures 1 and 2, respectively.

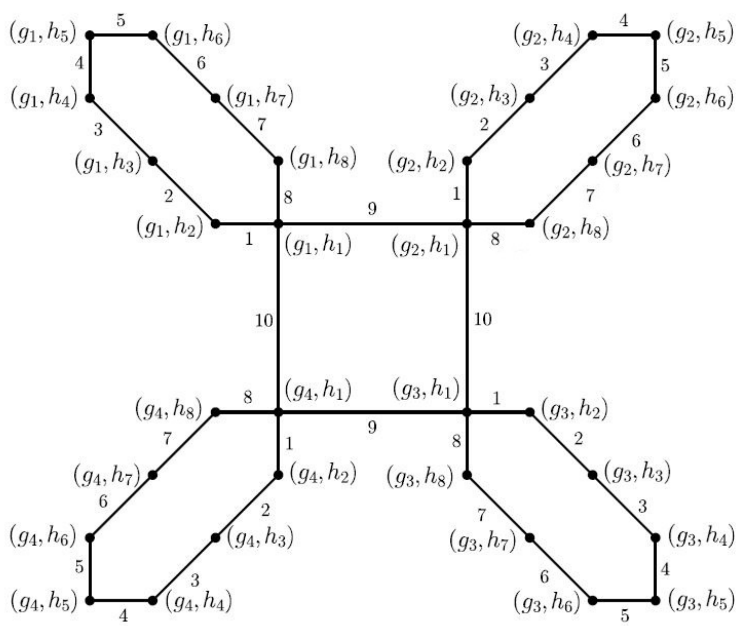

Figure 1. A rainbow 10-coloring of $C_4 \triangleright_v C_8$

3. Rainbow connection number of comb product of some graphs

In Section 2, we have proven the sharpness of the lower and upper bounds in Theorem 2.1. In this section, we provide comb product of graphs $G \triangleright_v H$ for some connected graphs G and H whose rainbow connection number lies between these lower and upper bounds.

Our first result is the rainbow connection number of $P_m \triangleright_v C_n$ for certain values of n, which is given in the following theorem.

Theorem 3.1. *Let P_m be a path of order $m \geq 3$, C_n be a cycle of order $n \geq 3$ where n is odd, and $v \in V(C_n)$. Then $rc(P_m \triangleright_v C_n) = diam(P_m \triangleright_v C_n) + 1 = rc(P_m) + n$.*

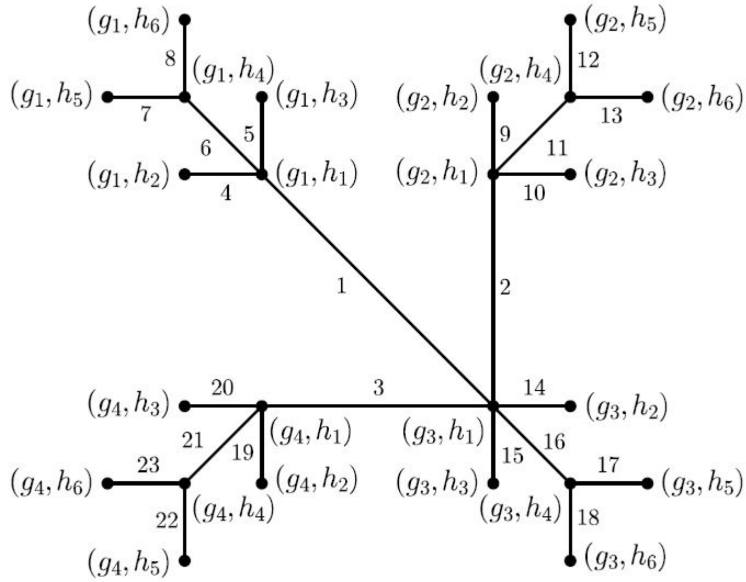

Figure 2. A rainbow 23-coloring of comb product of two trees

Proof. Let $V(P_m) = \{g_1, g_2, \ldots, g_m\}$ such that $E(P_m) = \{g_i g_{i+1} : i \in \{1, 2, \ldots, m-1\}\}$. We can check that $diam(P_m \rhd_v C_n) = rc(P_m) + n - 1$. Without loss of generality, let $v = h_1$. By assigning colors $1, 2, \ldots, rc(P_m)$ to the edges of P_m and colors $rc(P_m)+1, rc(P_m)+2, \ldots, rc(P_m)+n$ to the edges of $C_n(i)$ for each $i \in \{1, 2, \ldots, m\}$, we can find a rainbow $x-y$ path for any two vertices $x, y \in V(P_m \rhd_v C_n)$, where the proof is similar to that used in Theorem 2.2 for case $m \geq 2$ and even $n \geq 4$.

Next,we prove the lower bound. Suppose to the contrary that $rc(P_m \rhd_v C_n) \leq rc(P_m)+n-1$. Let c be a rainbow $(rc(P_m)+n-1)$-coloring of $P_m \rhd_v C_n$ and let $A = \left\{1, 2, \ldots, \frac{n-1}{2}\right\}$, $B = \left\{\frac{n-1}{2}+1, \frac{n-1}{2}+2, \ldots, n-1\right\}$, and $C = \{n, n+1, \ldots, rc(P_m)+n-1\}$ be the sets of colors. Consider two vertices $(g_i, h_j), (g_k, h_l) \in V(P_m \rhd_v C_n)$ so that $d((g_i, h_j), (g_k, h_l)) = rc(P_m) + n - 1$. This condition is satisfied when $i, k \in \{1, m\}$, $i \neq k$, and $j, l \in \{\frac{n+1}{2}, \frac{n+3}{2}\}$. Without loss of generality, let $i = 1$ and $k = m$. Note that there exists only one $(g_1, h_j) - (g_m, h_l)$ path of length $rc(P_m) + n - 1$, which can be obtained by identifying vertices (g_1, h_1) and (g_m, h_1) in a $(g_1, h_j) - (g_1, h_1)$ path of length $\frac{n-1}{2}$, a $(g_1, h_1) - (g_m, h_1)$ path of length $rc(P_m)$, and a $(g_m, h_1)-(g_m, h_l)$ path of length $\frac{n-1}{2}$. Thus, we need at least $rc(P_m)+n-1$ distinct colors to color all edges in $(g_1, h_j) - (g_m, h_l)$ path. First, consider vertices $(g_1, h_{\frac{n+1}{2}})$ and $(g_m, h_{\frac{n+1}{2}})$. Without loss of generality, assign colors from A to all edges in $(g_1, h_{\frac{n+1}{2}}) - (g_1, h_1)$ path, colors from C to all edges in $(g_1, h_1) - (g_m, h_1)$ path, and colors from B to all edges in $(g_m, h_1) - (g_m, h_{\frac{n+1}{2}})$ path. Next, by considering vertices $(g_1, h_{\frac{n+3}{2}})$ and $(g_m, h_{\frac{n+1}{2}})$ and vertices $(g_1, h_{\frac{n+1}{2}})$ and $(g_m, h_{\frac{n+3}{2}})$, we obtain that all edges in $(g_1, h_{\frac{n+3}{2}}) - (g_1, h_1)$ path and all edges in $(g_m, h_1) - (g_m, h_{\frac{n+3}{2}})$ path should be colored with colors from A and B, respectively. Next, consider vertices $(g_1, h_{\frac{n+1}{2}})$

and $(g_{m-1}, h_{\frac{n+1}{2}})$. Note that any $(g_1, h_{\frac{n+1}{2}}) - (g_{m-1}, h_{\frac{n+1}{2}})$ path has length either $rc(P_m) + n - 2$ or $rc(P_m) + n - 1$ and must contains a $(g_1, h_{\frac{n+1}{2}}) - (g_{m-1}, h_1)$ path as a subgraph. Since some edges in any $(g_1, h_{\frac{n+1}{2}}) - (g_{m-1}, h_1)$ path have been colored with colors from $A \cup C \setminus \{c((g_{m-1}, h_1), (g_m, h_1))\}$, this forces all edges in $C_n(m-1)$ should be colored with colors from $B \cup \{c((g_{m-1}, h_1), (g_m, h_1))\}$. However, there is no rainbow $(g_{m-1}, h_{\frac{n+1}{2}}) - (g_m, h_{\frac{n+1}{2}})$ path, a contradiction. $\square$

Our next results are the rainbow connection number of $K_m \triangleright_v H$ where H is either a complete graph, a wheel, or a fan.

Theorem 3.2. *For $m \geq 2$ and $n \geq 3$, Let K_m and K_n be two complete graphs of order m and n, respectively, and let $v \in V(K_n)$. Then*

$$rc(K_m \triangleright_v K_n) = \begin{cases} 3, & \textit{for } m \in \{2,3\}; \\ 4, & \textit{for } m \geq 4. \end{cases}$$

Proof. Without loss of generality, let $v = h_1$. We consider two cases.

Case 1. $m \in \{2,3\}$

We first show that $rc(K_m \triangleright_v K_n) \leq 3$ by defining a rainbow 3-coloring of $K_m \triangleright_v K_n$ as follows.

(i) For each $i \in \{1, \ldots, m\}$, assign colors i to all edges of $K_n(i)$.

(ii) If $m = 2$, assign color 3 to the edge $(g_1, h_1)(g_2, h_1)$. If $m = 3$, assign color 3 to the edge $(g_1, h_1)(g_2, h_1)$, color 1 to the edge $(g_2, h_1)(g_3, h_1)$, and color 2 to the edge $(g_1, h_1)(g_3, h_1)$.

By the edge-coloring above, it is easy to find a rainbow $x - y$ path for any two vertices $x, y \in V(K_m \triangleright_v K_n)$. Meanwhile for the lower bound, note that $diam(K_m \triangleright_v K_n) = 3$. Thus, we get $rc(K_m \triangleright_v K_n) \geq 3$ by Theorem 2.1.

Case 2. $m \geq 4$

We show that $rc(K_m \triangleright_v K_n) \leq 4$ by defining a rainbow 4-coloring of $K_m \triangleright_v K_n$ as follows.

(i) For each $i \in \{1, 2, \ldots, m\}$ and $j \in \{2, 3, \ldots, n\}$, assign color 1 to the edges $(g_i, h_1)(g_i, h_j)$ for even j, color 2 to the edges $(g_i, h_1)(g_i, h_j)$ for odd j, and color 3 to the remaining edges of $K_n(i)$.

(ii) Assign color 4 to the edges of $K_m(h_1)$.

For $i, k \in \{1, 2, \ldots, m\}$ and $j, l \in \{1, 2, \ldots, n\}$, let $x = (g_i, h_j)$ and $y = (g_k, h_l)$ be two vertices of $K_m \triangleright_v K_n$. If $i = k$ and $j \neq l$, or if $i \neq k$ and $j = l = 1$, then the edge xy is a rainbow $x - y$ path. If $i \neq k$, $j = 1$, and $l \in \{2, 3, \ldots, n\}$, then $P = (g_i, h_1), (g_k, h_1), (g_k, h_l)$ is a rainbow $x - y$ path. Next, we may further consider cases when $i \neq k$ and $j, l \in \{2, 3, \ldots, n\}$ as follows.

- j and l have same parity. If $l \neq n$, then $P = (g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_{l+1}), (g_k, h_l)$ is a rainbow $x - y$ path. Otherwise, $P = (g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_{l-1}), (g_k, h_l)$ is a rainbow $x - y$ path.

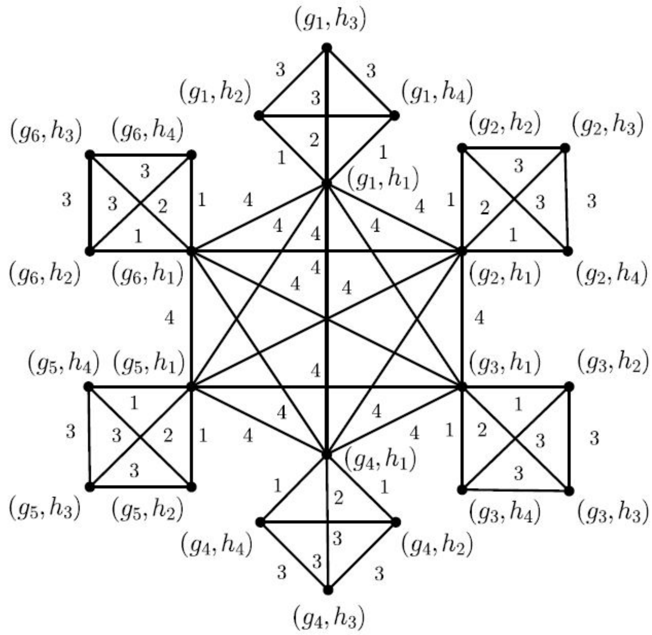

Figure 3. A rainbow 4-coloring of $K_6 \triangleright_v K_4$

- j and l have distinct parity. Then $P = (g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_l)$ is a rainbow $x - y$ path.

Figure 3 gives an illustration of a rainbow 4-coloring of $K_6 \triangleright_v K_4$.

For the lower bound, suppose to the contrary that $rc\,(K_m \triangleright_v K_n) \leq 3$. Let c be a rainbow 3 coloring of $K_m \triangleright_v K_n$. Observe that for distinct $i, k \in \{1, 2, \ldots, m\}$ and $j, l \in \{2, 3, \ldots, n\}$, any $(g_i, h_j) - (g_k, h_l)$ path has length at least 3. This forces $(g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_l)$ is the only possible $(g_i, h_j) - (g_k, h_l)$ path, where $c((g_i, h_1)(g_i, h_j)) \neq c((g_k, h_1)(g_k, h_l))$. First, consider vertices (g_1, h_2) and (g_2, h_2). Without loss of generality, let $c((g_1, h_1)(g_1, h_2)) = 1$, $c((g_1, h_1)(g_2, h_1)) = 2$, and $c((g_2, h_1)(g_2, h_2)) = 3$. Next, consider vertices (g_1, h_2) and (g_i, h_2) and vertices (g_2, h_2) and (g_i, h_2) for all $i \in \{3, 4, \ldots, m\}$, successively. Thus, $c((g_i, h_1)(g_i, h_2)) = 2$ for all $i \in \{3, 4, \ldots, m\}$. However, there is no rainbow $(g_i, h_2) - (g_k, h_2)$ path for distinct $i, k \in \{3, 4, \ldots, m\}$, a contradiction. □

A *wheel* of order $n + 1$, denoted by W_n, is a graph formed by joining a new vertex to all vertices of a cycle C_n. Let $V(W_n) = \{h_1, h_2, \ldots, h_{n+1}\}$ such that $E(W_n) = \{h_1 h_i, h_i h_{i+1} : i \in \{2, 3, \ldots, n + 1\}$ and $h_{n+2} = h_2\}$. The vertex h_1 is called the *center vertex* of W_n, and the edge $h_1 h_i$ for each $i \in \{2, 3, \ldots, n + 1\}$ is called the *spoke* of W_n.

Theorem 3.3. *For $m \geq 2$ and $n \geq 4$, let K_m be a complete graph of order m, W_n be a wheel of order $n + 1$, and v be the center vertex of W_n. Then*

$$rc(K_m \triangleright_v W_n) = \begin{cases} 3, & \text{for } m \in \{2, 3\} \text{ and } n \in \{4, 5, 6\}; \\ 4, & \text{for } m \geq 4 \text{ and } n \in \{4, 5, 6\}, \text{ or } m \geq 2 \text{ and } n \geq 7. \end{cases}$$

Proof. We consider two cases.

Case 1. $n \in \{4, 5, 6\}$

We consider two subcases.

Subcase 1.1. $m \in \{2, 3\}$

Let $i \in \{1, \ldots, m\}$. We show that $rc\,(K_m \triangleright_v W_n) \leq 3$ by defining a rainbow 3-coloring of $K_m \triangleright_v W_n$ as follows.

(i) If $m = 2$, assign color 3 to the edge $(g_1, h_1)(g_2, h_1)$. Otherwise, assign color 3 to the edge $(g_1, h_1)(g_2, h_1)$, color 1 to the edge $(g_2, h_1)(g_3, h_1)$, and color 2 to the edge $(g_1, h_1)(g_3, h_1)$.

(ii) Assign colors i to the edges $(g_i, h_1)(g_i, h_j)$ for all $j \in \{2, 3, \ldots, n + 1\}$.

(iii) For $n = 4$, assign color 1 to the edges $(g_i, h_2)(g_i, h_3)$ and $(g_i, h_4)(g_i, h_5)$ and color 2 to the edges $(g_i, h_3)(g_i, h_4)$ and $(g_i, h_2)(g_i, h_5)$.

(iv) For $n = 5$, assign color 1 to the edges $(g_i, h_2)(g_i, h_3)$ and $(g_i, h_5)(g_i, h_6)$, color 2 to the edges $(g_i, h_3)(g_i, h_4)$ and $(g_i, h_2)(g_i, h_6)$, and color 3 to the edge $(g_i, h_4)(g_i, h_5)$.

(v) For $n = 6$, assign color 1 to the edges $(g_i, h_2)(g_i, h_3)$ and $(g_i, h_5)(g_i, h_6)$, color 2 to the edges $(g_i, h_3)(g_i, h_4)$ and $(g_i, h_6)(g_i, h_7)$, and color 3 to the edges $(g_i, h_4)(g_i, h_5)$ and $(g_i, h_2)(g_i, h_7)$.

For $i, k \in \{1, \ldots, m\}$ and $j, l \in \{1, 2, \ldots, n + 1\}$, let $x = (g_i, h_j)$ and $y = (g_k, h_l)$ be two vertices of $K_m \triangleright_v W_n$. We show that there exists a rainbow $x - y$ path by considering the following two subcases.

- $i = k$. Without loss of generality, let $j < l$. If $d(x, y) = 1$, then it is clearly that edge xy is a rainbow $x - y$ path. If $d(x, y) = 2$, then a shortest $x - y$ path which contained in the cycle C_n is a rainbow $x - y$ path.
- $i \neq k$. If $j = l = 1$, then edge xy is a rainbow $x - y$ path. If $j = 1$ and $l \in \{2, 3, \ldots, n + 1\}$, then a path $P = (g_i, h_1), (g_k, h_1), (g_k, h_l)$ is a rainbow $x - y$ path. Otherwise, a path $P = (g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_l)$ is a rainbow $x - y$ path.

For the lower bound, note that $diam(K_m \triangleright_v W_n) = 3$. Thus, $rc(K_m \triangleright_v K_n) \geq 3$ by Theorem 2.1.

Subcase 1.2. $m \geq 4$

We show that $rc\,(K_m \triangleright_v W_n) \leq 4$ by defining a rainbow 4-coloring of $K_m \triangleright_v W_n$. Let $i \in \{1, 2, \ldots, m\}$ and $j \in \{2, 3, \ldots, n + 1\}$. We assign color 1 to the edges $(g_i, h_1)(g_i, h_j)$ for even j, color 2 to the edges $(g_i, h_1)(g_i, h_j)$ for odd j, color 3 to the remaining edges of $W_n(i)$, and color 4 to all edges of $K_m(h_1)$. For $i, k \in \{1, 2, \ldots, m\}$ and $j, l \in \{1, 2, \ldots, n + 1\}$, let $x = (g_i, h_j)$ and $y = (g_k, h_l)$ be two vertices of $K_m \triangleright_v W_n$. We show that there exists a rainbow $x - y$ path by considering the following two subcases.

- $i = k$. Without loss of generality, let $j < l$. It is clearly that edge xy is a rainbow $x - y$ path if $d(x, y) = 1$. Hence, we may further consider cases when $d(x, y) = 2$. If j and l have same parity, then $P = (g_i, h_j), (g_i, h_1), (g_i, h_{l-1}), (g_i, h_l)$ is a rainbow $x - y$ path. Otherwise, $P = (g_i, h_j), (g_i, h_1), (g_i, h_l)$ is a rainbow $x - y$ path.

- $i \neq k$. If $j = l = 1$, then edge xy is a rainbow $x - y$ path. If $j = 1$ and $l \in \{2, 3, \ldots, n+1\}$, then $P = (g_i, h_1), (g_k, h_1), (g_k, h_l)$ is a rainbow $x - y$ path. Next, we may further consider cases when $j, l \in \{2, 3, \ldots, n+1\}$. If j and l have same parity with $l \neq 2$, then $P = (g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_{l-1}), (g_i, h_l)$ is a rainbow $x - y$ path. If j and l have same parity with $l = 2$, then $P = (g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_{l+1}), (g_i, h_l)$ is a rainbow $x - y$ path. Otherwise, $P = (g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_l)$ is a rainbow $x - y$ path.

For the lower bound, suppose to the contrary that $rc\,(K_m \triangleright_v W_n) \leq 3$. Let c be a rainbow 3-coloring of $K_m \triangleright_v W_n$. Observe that for distinct $i, k \in \{1, 2, \ldots, m\}$ and $j, l \in \{2, 3, \ldots, n+1\}$, the only possible $(g_i, h_j) - (g_k, h_l)$ path of length 3 is $(g_i, h_j), (g_i, h_1), (g_k, h_1), (g_k, h_l)$. This forces $c((g_i, h_1)(g_i, h_j)) \neq c((g_k, h_1)(g_k, h_l))$ for all distinct $i, k \in \{1, 2, \ldots, m\}$ and $j, l \in \{2, 3, \ldots, n+1\}$. However, $m \geq 4$, implying that we need at least 4 distinct colors to color edges $(g_i, h_1)(g_i, h_j)$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{2, 3, \ldots, n+1\}$, which is impossible.

Case 2. $n \geq 7$

By using the same 4-rainbow coloring as in Subcase 1.2, we have $rc\,(K_m \triangleright_v K_n) \leq 4$. For the lower bound, suppose to the contrary that $rc\,(K_m \triangleright_v W_n) \leq 3$. Let c be a rainbow 3-coloring of $K_m \triangleright_v W_n$. First, consider vertices (g_1, h_j) and (g_2, h_l) for $j, l \in \{2, 3, \ldots, n+1\}$. Since path $(g_1, h_j), (g_1, h_1), (g_2, h_1), (g_2, h_l)$ is the only possible $(g_1, h_j) - (g_2, h_l)$ path of length 3, without loss of generality, let $c((g_1, h_1)(g_1, h_j)) = 1$, $c((g_1, h_1)(g_2, h_1)) = 2$, and $c((g_2, h_1)(g_2, h_l)) = 3$ for all $j, l \in \{2, 3, \ldots, n+1\}$. Next, consider vertices (g_1, h_j) and (g_1, h_l) for distinct $j, l \in \{2, 3, \ldots, n+1\}$. Since all spokes of $W_n(1)$ have the same color, which is 1, a rainbow $(g_1, h_j) - (g_1, h_l)$ path should be a subgraph of C_n. Since $n \geq 7$, it follows by Theorem 1.2 that we need at least 4 distinct colors assigned to the edges of C_n so that there exists a rainbow $(g_1, h_j) - (g_1, h_l)$ as a subgraph of C_n, which is impossible. $\square$

For illustration of Theorem 3.3, please see Figure 4.

A *fan* F_n of order $n+1$ is a graph formed by joining a new vertex to all vertices of a path P_n. Let $V(F_n) = \{h_1, h_2, \ldots, h_{n+1}\}$ such that $E(F_n) = \{h_1 h_i : i \in \{2, 3, \ldots, n+1\}\} \cup \{h_i h_{i+1} : i \in \{2, 3, \ldots, n\}\}$. The vertex h_1 is called the *center vertex* of F_n, and the edge $h_1 h_i$ for each $i \in \{2, 3, \ldots, n+1\}$ is called the *spoke* of F_n.

Theorem 3.4. *For $m \geq 2$ and $n \geq 3$, let K_m be a complete graph of order m, F_n be a fan of order $n+1$, and v be the center vertex of F_n. Then*

$$rc(K_m \triangleright_v F_n) = \begin{cases} 3, & \text{for } m \in \{2,3\} \text{ and } n \in \{3,4\}; \\ 4, & \text{for } m \geq 4 \text{ and } n \in \{3,4\}, \text{ or } m \geq 2 \text{ and } n \geq 5. \end{cases}$$

Proof. We consider two cases.

Case 1. $n \in \{3, 4\}$

We consider two subcases.

Subcase 1.1 $m \in \{2, 3\}$

Let $i \in \{1, \ldots, m\}$. We show that $rc\,(K_m \triangleright_v F_n) \leq 3$ by defining a rainbow 3-coloring of $K_m \triangleright_v F_n$ as follows.

(i) If $m = 2$, assign color 3 to the edge $(g_1, h_1)(g_2, h_1)$. Otherwise, assign color 3 to the edge $(g_1, h_1)(g_2, h_1)$, color 1 to the edge $(g_2, h_1)(g_3, h_1)$, and color 2 to the edge $(g_1, h_1)(g_3, h_1)$.

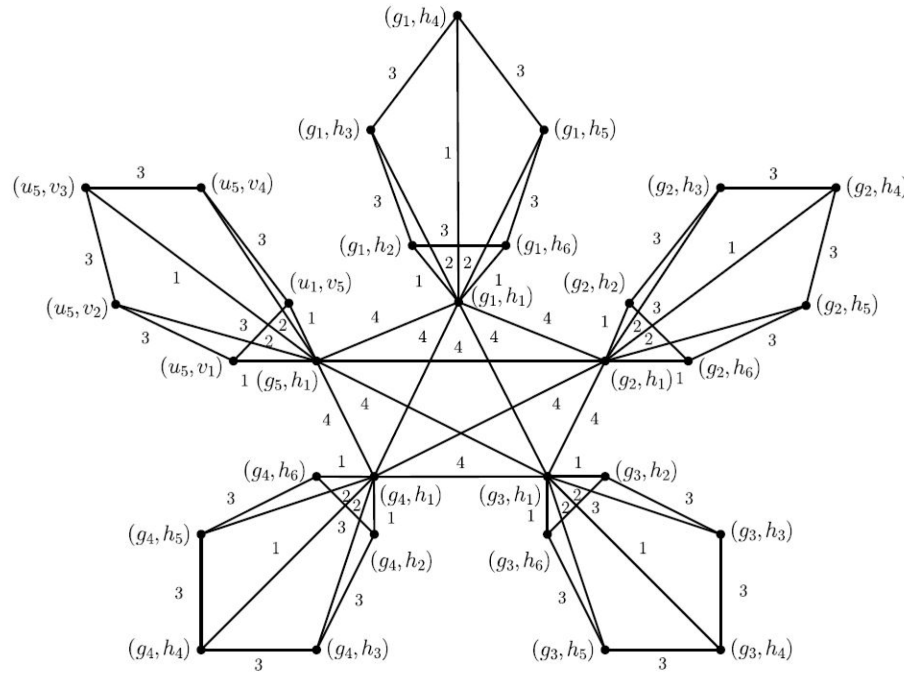

Figure 4. A rainbow 4-coloring of $K_5 \triangleright_v W_5$

(ii) Assign colors i to the edges $(g_i, h_1)(g_i, h_j)$ for all $j \in \{2, 3, \ldots, n+1\}$.

(iii) For $j \in \{2, 3, \ldots, n\}$, assign colors $j-1$ to the edges $(g_i, h_j)(g_i, h_{j+1})$.

By using a similar argument as in the proof of Subcase 1.1 in Theorem 3.3, we can show that there exists a rainbow $x - y$ path for any two distinct vertices x and y of $K_m \triangleright_v F_n$. Meanwhile for the lower bound, since $diam(K_m \triangleright_v F_n) = 3$, it follows by Theorem 2.1 that $rc(K_m \triangleright_v F_n) \geq 3$.

Subcase 1.2. $m \geq 4$

Arguments similar to that used in the proof of Subcase 1.2 in Theorem 3.3 (both for the proof of upper and lower bounds) will verify that $rc\,(K_m \triangleright_v F_n) = 4$.

Case 2. $n \geq 5$

By using the same 4-rainbow coloring as in Subcase 1.2 in Theorem 3.3, we obtain that $rc\,(K_m \triangleright_v F_n) \leq 4$. For the lower bound, suppose to the contrary that $rc\,(K_m \triangleright_v F_n) \leq 3$. By using a similar argument as Case 2 in Theorem 3.3, we will obtain that all spokes of $F_n(i)$ for each $i \in \{1, 2\}$ have the same color. Thus, any rainbow $(g_1, h_j) - (g_1, h_l)$ path for distinct $j, l \in \{2, 3, \ldots, n+1\}$ should be a subgraph of P_n. However, $n \geq 5$. Thus, by Theorem 1.1(b), we need at least 4 distinct colors assigned to the edges of P_n so that there exists a rainbow $(g_1, h_j) - (g_1, h_l)$ as a subgraph of P_n, which is impossible. □

For illustration of Theorem 3.4, please see Figure 5.

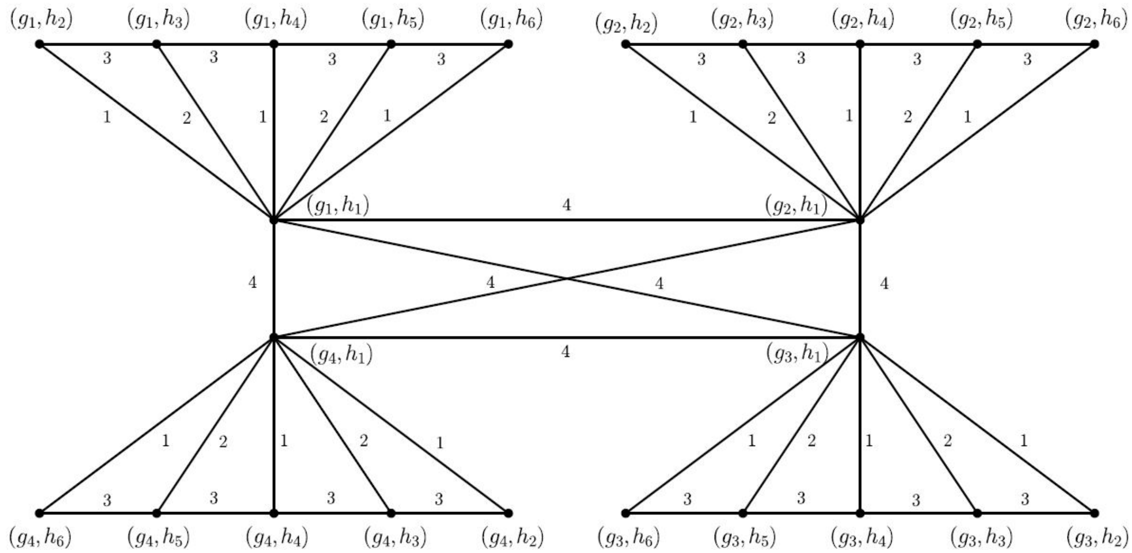

Figure 5. A rainbow 4-coloring of $K_4 \triangleright_v F_5$

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