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A generalization of Pappus graph

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Abstract

In this paper, we introduce a new family of cubic graphs $\Gamma(m)$, called Generalized Pappus graphs, where $m \geq 3$. We compute the automorphism group of $\Gamma(m)$ and characterize when it is a Cayley graph.

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1. Introduction

The study of different families of graphs with respect to their group of symmetries is an important aspect of modern algebraic graph theory. Among them the cubic families are one of the most important class of graphs. Various important families of cubic graphs which are extensively studied are Generalized Petersen graphs [2], Double Generalized Petersen graphs [5], Zhou-Ghasemi graphs [8], Zhou-Li graphs [9], Devilliers *et.al.* graphs [1], [4] etc. In this paper, we construct another infinite family of cubic graphs starting from the well-known Pappus graph and study its automorphism group and structural properties.

Pappus graph is a bipartite cubic graph with 18 vertices and 27 edges, formed as the Levi graph of the Pappus configuration. It is named after Pappus of Alexandria who is believed to have discovered the “hexagon theorem” describing the Pappus configuration. Recently, [7] proposed a group theoretic generalization of Pappus configuration from a projective geometric viewpoint. Our

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goal is to generalize it from a graph theoretic viewpoint. To begin with, we define the Genralized Pappus graph next. For other definitons and terminologies, readers are referred to [3].

Definition 1.1. *Let $m \geq 3$ be a positive integer and set $n = 2m$. The generalized Pappus graph $\Gamma(m)$ is defined on the vertex set $V = \{x_i, y_i, z_i : i \in \mathbb{Z}_n\}$, where x_i's, y_i's and z_i's are called the outer vertices, middle vertices and inner vertices respectively. There are four types of edges between these vertices, namely outer edges of the form $x_i \sim x_{i+1}$, spoke edges of the form $x_i \sim y_i$, middle edges of the form $y_i \sim z_{i+1}$ and $y_i \sim z_{i-1}$ and inner edges of the form $z_i \sim z_{i+m}$. (Here $u \sim v$ means u and v are adjacent.)*

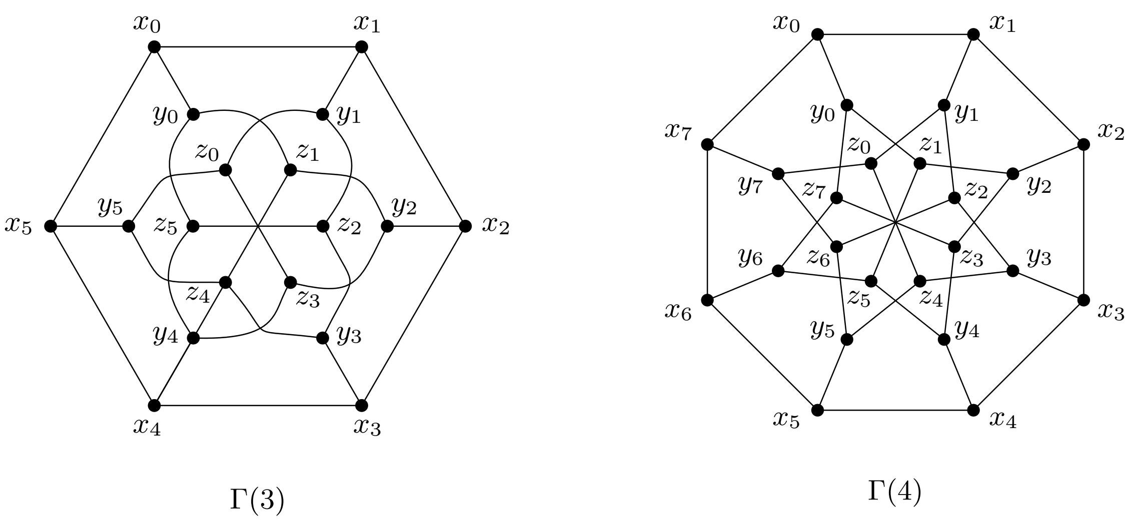

Figure 1. Generalized Pappus Graphs, $\Gamma(3)$ and $\Gamma(4)$.

It is obvious that $\Gamma(m)$ is a cubic graph of order $6m$ and $\Gamma(3)$ is the Pappus graph (See Figure 1). We denote the set of outer, spoke, middle and inner edges by $\Omega, \Sigma, \mathcal{M}$ and $\mathcal{I}$ respectively, and the set of vertices x_i, y_i and z_i by X, Y and Z respectively.

2. Automorphism Group of $\Gamma(m)$

We start by noting some automorphisms of $\Gamma(m)$. It can be easily checked that $\rho : \Gamma(m) \to \Gamma(m)$ and $\tau : \Gamma(m) \to \Gamma(m)$ defined by

$$\begin{aligned} \rho :\quad & x_i \mapsto x_{i+1} \qquad & \tau : x_i \mapsto x_{-i} \\ & y_i \mapsto y_{i+1} & y_i \mapsto y_{-i} \\ & z_i \mapsto z_{i+1} & z_i \mapsto z_{-i} \end{aligned}$$

are automorphisms of $\Gamma(m)$ and $\circ(\rho) = n; \circ(\tau) = 2$ and $\tau\rho\tau = \rho^{-1}$. Thus $H = \langle \rho, \tau \rangle \cong D_n$, the dihedral group of order $2n$. Moreover, if m is odd, it can be shown that $\sigma : \Gamma(m) \to \Gamma(m)$ given by

$$\sigma: \begin{array}{l} x_i \mapsto y_{i+m} \\ y_i \mapsto x_{i+m} \\ z_i \mapsto y_i \end{array} \quad \text{if } i \text{ is even} \qquad \sigma: \begin{array}{l} x_i \mapsto z_{i+m} \\ y_i \mapsto z_i \\ z_i \mapsto x_{i+m} \end{array} \quad \text{if } i \text{ is odd}$$

is an automorphism of $\Gamma(m)$ which does not belong to H. Also we have $\circ(\sigma) = 3$, $\sigma\rho\sigma = \rho$ and $\sigma\tau = \tau\sigma$.

Theorem 2.1. *If m is odd, then $\Gamma(m)$ is a Cayley graph.*

Proof. We prove the theorem by showing that $K = \langle \rho, \sigma \rangle$ acts regularly on $\Gamma(m)$. As $|K| = 3n = |\Gamma(m)|$, it is enough to show that K acts transitively on the vertices of $\Gamma(m)$.

Let us start with x_0. Note that x_0 can be mapped to any x_i by applying suitable powers of ρ. As $\sigma(x_0) = y_m$, we can map x_0 to any y_i by applying suitable powers of ρ on $\sigma(x_0)$. And, as $\sigma\rho(x_0) = z_{m+1}$, we can map x_0 to any z_i by applying suitable powers of ρ on $\sigma\rho(x_0)$. Thus, we can map x_0 to any vertex of $\Gamma(m)$ and vice-versa using the elements of K. Now, if we start with two arbitrary vertices, we can map one to the other via x_0. Hence, K acts transitively on the vertices of $\Gamma(m)$. □

Lemma 2.1. *Let $\varphi \in Aut(\Gamma(m))$, if $\varphi(X) = X$ then $\varphi \in H$.*

Proof. Let $\varphi(x_0) = x_a$, then $\varphi(x_1) = x_{a+1}$ or x_{a-1} (since $\varphi(x_1) \sim \varphi(x_0) = x_a$).

Let $\varphi(x_1) = x_{a+1}$. As $\varphi(X) = X$ we have $\varphi(x_2) = x_{a+2},\ \varphi(x_3) = x_{a+3}, \cdots, \varphi(x_{n-1}) = x_{a+n-1}$. Now as $x_i \sim y_i$, $\varphi(x_i) = x_{a+i} \sim \varphi(y_i)$ so we have $\varphi(y_i) = y_{a+i}$ for all i. Again as $\varphi(y_{i-1}) = y_{a+i-1}$ and $\varphi(y_{i+1}) = y_{a+i+1}$, we have $\varphi(z_i) = z_{a+i}$ for all i. Hence, $\varphi = \rho^a \in H$.

Let $\varphi(x_1) = x_{a-1}$. As $\varphi(X) = X$ then $\varphi(x_2) = x_{a-2},\ \varphi(x_3) = x_{a-3}, \cdots, \varphi(x_{n-1}) = x_{a-(n-1)}$. Now as $x_i \sim y_i$, $\varphi(x_i) = x_{a-i} \sim \varphi(y_i)$ so we have $\varphi(y_i) = y_{a-i}$ for all i. Again, as $\varphi(y_{i-1}) = y_{a-(i-1)}$ and $\varphi(y_{i+1}) = y_{a-(i+1)}$, we have $\varphi(z_i) = z_{a-i}$ for all i. Hence we have, $\varphi = \rho^a\tau \in H$. □

Lemma 2.2. *Let $\varphi \in Aut(\Gamma(m))$ and $m \neq 3$. If $\varphi(x_i) = x_j$ and $\varphi(y_i) = y_j$ for some i, j, then $\varphi \in H$.*

Proof. As $\varphi(x_i) = x_j$ and $\varphi(y_i) = y_j$ then $\varphi(x_{i+1}), \varphi(x_{i-1}) \in \{x_{j+1}, x_{j-1}\}$.

Let $\varphi(x_{i+1}) = x_{j+1}$ and $\varphi(x_{i-1}) = x_{j-1}$, then $\varphi(x_{i+2}) = x_{j+2}$ or y_{j+1}. If $\varphi(x_{i+2}) = y_{j+1}$, consider the cycle $C : y_i \sim x_i \sim x_{i+1} \sim x_{i+2} \sim y_{i+2} \sim z_{i+1} \sim y_i$, then $\varphi(C) : y_j \sim x_j \sim x_{j+1} \sim y_{j+1} \sim \varphi(y_{i+2}) \sim \varphi(z_{i+1}) \sim y_j$. For $m \neq 3$, there exists a unique path P of length 3, namely $y_j \sim x_j \sim x_{j+1} \sim y_{j+1}$ between y_j and y_{j+1}. Thus $\varphi(C)$ is not a cycle, which is a contradiction. Hence $\varphi(x_{i+2}) = x_{j+2}$. Similarly, it can be shown that $\varphi(x_{i+k}) = x_{j+k}$ for all k, i.e., $\varphi(X) = X$. Then by Lemma 2.1, we have $\varphi \in H$. Similarly if $\varphi(x_{i+1}) = x_{j-1}$ we can show $\varphi \in H$. □

It is to be noted that if $m = 3$, there exists another path $P' : y_j \sim z_{j-1} \sim z_{j+2} \sim y_{j+1}$ joining y_j and y_{j+1}.

Lemma 2.3. *Let $\varphi \in Aut(\Gamma(m)) \setminus H$. Then φ can not map consecutive spokes into Σ.*

Proof. Let $\varphi([x_i, y_i]) \in \Sigma$. We will show $\varphi([x_{i+1}, y_{i+1}]) \notin \Sigma$. Let $m \neq 3$. As $\varphi \notin H$, then by Lemma 2.2, we have $(\varphi(x_i), \varphi(y_i)) \neq (x_j, y_j)$ for all j. However, as $\varphi([x_i, y_i]) \in \Sigma$, the orientation of the spoke $[x_i, y_i]$ is changed by φ. Thus, we assume that $\varphi(x_i) = y_k$ and $\varphi(y_i) = x_k$ for some k. Now as $\varphi(x_i) \sim \varphi(x_{i+1})$, we must have $\varphi(x_{i+1}) = z_{k-1}$ or z_{k+1}, hence $\varphi([x_{i+1}, y_{i+1}]) \notin \Sigma$.

Now let $m = 3$ and $\varphi([x_i, y_i]) \in \Sigma$. If $\varphi(x_i) = y_k$ and $\varphi(y_i) = x_k$ for some k, then as $\varphi(x_i) \sim \varphi(x_{i+1})$ we have $\varphi(x_{i+1}) = z_{k-1}$ or z_{k+1} and hence $\varphi([x_{i+1}, y_{i+1}]) \notin \Sigma$. So, we assume that $\varphi(x_i) = x_k$ and $\varphi(y_i) = y_k$. Then $\varphi(x_{i+1}), \varphi(x_{i-1}) \in \{x_{k+1}, x_{k-1}\}$.

Let $\varphi(x_{i+1}) = x_{k+1}$, then $\varphi(x_{i+2}), \varphi(y_{i+1}) \in \{x_{k+2}, y_{k+1}\}$. If $\varphi(y_{i+1}) = x_{k+2}$ and $\varphi(x_{i+2}) = y_{k+1}$ then $\varphi([x_{i+1}, y_{i+1}]) = [x_{k+1}, x_{k+2}] \notin \Sigma$. If $\varphi(y_{i+1}) = y_{k+1}$ and $\varphi(x_{i+2}) = x_{k+2}$, consider the cycle $C_1 : x_{i+1} \sim y_{i+1} \sim z_{i+2} \sim z_{i+5} \sim y_i \sim x_i \sim x_{i+1}$. Then $\varphi(C_1) : x_{k+1} \sim y_{k+1} \sim \varphi(z_{i+2}) \sim \varphi(z_{i+5}) \sim y_k \sim x_k \sim x_{k+1}$. As $\varphi(C_1)$ is a cycle then we have $\varphi(z_{i+2}) = z_{k+2}$ and $\varphi(z_{i+5}) = z_{k+5}$ (See Figure 1(Left)). Now consider the cycle $C_2 : y_{i+1} \sim x_{i+1} \sim x_{i+2} \sim x_{i+3} \sim y_{i+3} \sim z_{i+2} \sim y_{i+1}$, then $\varphi(C_2) : y_{k+1} \sim x_{k+1} \sim x_{k+2} \sim \varphi(x_{i+3}) \sim \varphi(y_{i+3}) \sim z_{k+2} \sim y_{k+1}$. As $\varphi(C_2)$ is a cycle then we have $\varphi(x_{i+3}) = x_{k+3}$ and $\varphi(y_{i+3}) = y_{k+3}$. Proceeding this way, we get $\varphi(X) = X$ and $\varphi(Y) = Y$. Then by Lemma 2.1, we have $\varphi \in H$, which is a contradiction, hence $\varphi(y_{i+1}) \neq y_{k+1}$, i.e., $\varphi([x_{i+1}, y_{i+1}]) \notin \Sigma$. Similarly we can proof this if $\varphi(x_{i+1}) = x_{k-1}$. This completes the proof. □

Corollary 2.1. $\mathsf{Stab}(\Sigma) = H$.

Proof. It is clear that H stabilize Σ setwise, i.e., $H \subseteq \mathsf{Stab}(\Sigma)$. Let $\varphi \in \mathsf{Stab}(\Sigma)$. Then $\varphi([x_i, y_i]), \varphi([x_{i+1}, y_{i+1}]) \in \Sigma$ for all i, and thus by Lemma 2.3, we have $\varphi \in H$, i.e., $\mathsf{Stab}(\Sigma) \subseteq H$. □

Lemma 2.4. *Let $\varphi \in Aut(\Gamma(m)) \setminus H$ and $m \neq 3$. If $\varphi([x_i, y_i]) \in \Sigma$, then $\varphi([x_{i+2}, y_{i+2}]) \in \Sigma$.*

Proof. Let $\varphi(x_i) = x_j$ and $\varphi(y_i) = y_j$, then by Lemma 2.2, we have $\varphi \in H$, which is a contradiction. So we assume that $\varphi(x_i) = y_j$ and $\varphi(y_i) = x_j$. Then $\varphi(x_{i+1}), \varphi(x_{i-1}) \in \{z_{j+1}, z_{j-1}\}$.

Let $\varphi(x_{i+1}) = z_{j+1}$ and $\varphi(x_{i-1}) = z_{j-1}$. As $\varphi(x_{i+2}) \sim \varphi(x_{i+1})$ and $\varphi(x_i) = y_j$ then $\varphi(x_{i+2}) = y_{j+2}$ or z_{j+1+m}. We claim that $\varphi(x_{i+2}) = y_{j+2}$. If possible, let $\varphi(x_{i+2}) = z_{j+1+m}$. As $\varphi(y_{i+1}) \sim \varphi(x_{i+1}) = z_{j+1}$, $\varphi(x_i) = y_j$ and $\varphi(x_{i+2}) = z_{j+1+m}$, we have $\varphi(y_{i+1}) = y_{j+2}$. Now consider the cycle $C : y_i \sim x_i \sim x_{i+1} \sim x_{i+2} \sim y_{i+2} \sim z_{i+1} \sim y_i$. Then $\varphi(C) : x_j \sim y_j \sim z_{j+1} \sim z_{j+1+m} \sim \varphi(y_{i+2}) \sim \varphi(z_{i+1}) \sim x_j$. As there exists unique path $x_j \sim y_j \sim z_{j+1} \sim z_{j+1+m}$ of length 3 between x_j and z_{j+1+m} for $m \neq 3$, we get $\varphi(C)$ is not a cycle, which is a contradiction. Therefore $\varphi(x_{i+2}) = y_{j+2}$.

Now as $\varphi(y_{i+2}) \sim \varphi(x_{i+2}) = y_{j+2}$ and $\varphi(x_{i+1}) = z_{j+1}$, we have $\varphi(y_{i+2}) = z_{j+3}$ or x_{j+2}. We claim that $\varphi(y_{i+2}) = x_{j+2}$. If possible let $\varphi(y_{i+2}) = z_{j+3}$. Consider the cycle $C : y_i \sim x_i \sim x_{i+1} \sim x_{i+2} \sim y_{i+2} \sim z_{i+1} \sim y_i$, then $\varphi(C) : x_j \sim y_j \sim z_{j+1} \sim y_{j+2} \sim z_{j+3} \sim \varphi(z_{i+1}) \sim x_j$. As there does not exist any path of length 2 between x_j and z_{j+3}, $\varphi(C)$ is not a cycle, which is a contradiction. Therefore $\varphi(y_{i+2}) = x_{j+2}$. Therefore $\varphi([x_i, y_i]) = [y_j, x_j]$ implies $\varphi([x_{i+2}, y_{i+2}]) = [y_{j+2}, x_{j+2}] \in \Sigma$.

Similarly we can proof this if $\varphi(x_{i+1}) = z_{j-1}$ and $\varphi(x_{i-1}) = z_{j+1}$. This completes the proof. □

From Theorem 2.1, we have if $\varphi \notin H$, then either $\emptyset \neq \varphi(\Sigma) \cap \Sigma \subsetneq \Sigma$ or $\varphi(\Sigma) \cap \Sigma = \emptyset$. We will show in Theorem 2.2 that $\emptyset \neq \varphi(\Sigma) \cap \Sigma \subsetneq \Sigma$ is possible only if m is odd. If $\varphi \notin H$ and $m \neq 3$, then from Lemma 2.4, we have $\varphi([x_i, y_i]) = [y_j, x_j] \Rightarrow \varphi([x_{i+2k}, y_{i+2k}]) = [y_{j+2k}, x_{j+2k}]$ for all k. At first for $m \neq 3$ we consider the case when any one even spoke is mapped to an odd spoke, then by Lemma 2.4 we have all even spokes are mapped to all odd spokes and we will show that case appears only if m is odd.

Lemma 2.5. *Let $\varphi \in Aut(\Gamma(m)) \setminus H$ and $m \neq 3$. If set of all even spokes are mapped to set of all odd spokes via φ, then m is odd.*

Proof. Let $[x_e, y_e]$ be an even spoke and $[x_{od}, y_{od}]$ be an odd spoke such that $\varphi([x_e, y_e]) = [x_{od}, y_{od}]$. If $\varphi(x_e) = x_{od}$ and $\varphi(y_e) = y_{od}$ then by Lemma 2.2 we have $\varphi \in H$, which is a contradiction. So $\varphi(x_e) = y_{od}$ and $\varphi(y_e) = x_{od}$. Then $\varphi(x_{e+1}), \varphi(x_{e-1}) \in \{z_{od+1}, z_{od-1}\}$.

Case 1. Let $\varphi(x_{e+1}) = z_{od+1}$ and $\varphi(x_{e-1}) = z_{od-1}$. As $[x_{e+2}, y_{e+2}]$ is an even spoke, let $\varphi(x_{e+2}) = y_{od2}$, where $od2$ is an odd index. As $x_e \sim x_{e+1} \sim x_{e+2}$, applying φ we have $y_{od} \sim z_{od+1} \sim y_{od2}$, and hence $y_{od2} = y_{od+2}$. Again, as $[x_{e+4}, y_{e+4}]$ is an even spoke, let $\varphi(x_{e+2}) = y_{od3}$. Since the distance between x_{e+2} and x_{e+4} is 2, the distance between y_{od+2} and y_{od3} is also 2, and hence $\varphi(x_{e+4}) = y_{od+4}$ and so $\varphi(y_{e+4}) = x_{od+4}$. Proceeding this way, we have $\varphi(x_{e+2k}) = y_{od+2k}$ and $\varphi(y_{e+2k}) = x_{od+2k}$ for $k = 0, \ldots, m-1$. Now as $x_{e+2k} \sim x_{e+2k+1} \sim x_{e+2k+2}$, applying φ, we have $y_{od+2k} \sim \varphi(x_{e+2k+1}) \sim y_{od+2k+2}$. Hence $\varphi(x_{e+2k+1}) = z_{od+2k+1}$ for $k = 0, \cdots, m-1$. Again as $\varphi(y_{e+2k+1}) \sim \varphi(x_{e+2k+1}) = z_{od+2k+1}$ and $\varphi(x_{e+2k}) = y_{od+2k}$, we have $\varphi(y_{e+2k+1}) = z_{od+(2k+1)+m}$. Now as $y_{e+2k-1} \sim z_{e+2k} \sim y_{e+2k+1}$, applying φ we have $z_{od+(2k-1)+m} \sim \varphi(z_{e+2k}) \sim z_{od+(2k+1)+m}$. Thus $\varphi(z_{e+2k}) = y_{od+2k+m}$ for $k = 0, \cdots, m-1$. Note that if m is even then $od + 2k + m$ is an odd integer and so $\varphi(z_{e+2k}) \neq y_{od+2k+m}$ as all even spokes are mapped to all odd spokes. Hence φ is not a graph automorphism when m is even and so m must be odd.

Case 2. Let $\varphi(x_{e+1}) = z_{od-1}$ and $\varphi(x_{e-1}) = z_{od+1}$. Proceeding as in the previous case, it can be shown that $\varphi(x_{e+2k}) = y_{od-2k}$, $\varphi(y_{e+2k}) = x_{od-2k}$ and $\varphi(x_{e-2k}) = y_{od+2k}$, $\varphi(y_{e-2k}) = x_{od+2k}$ for $k = 0, \cdots, m-1$. Now as $x_{e+2k} \sim x_{e+2k+1} \sim x_{e+2k+2}$, applying φ we have, $y_{od-2k} \sim \varphi(x_{e+2k+1}) \sim y_{od-2k-2}$, hence $\varphi(x_{e+2k+1}) = z_{od-(2k+1)}$ and similarly $\varphi(x_{e-(2k+1)}) = z_{od+(2k+1)}$ for $k = 0, \cdots, m-1$. As $\varphi(y_{e+2k+1}) \sim \varphi(x_{e+2k+1}) = z_{od-(2k+1)}$ and $\varphi(x_{e+2k}) = y_{od-2k}$, we get $\varphi(y_{e+2k+1}) = z_{od-(2k+1)+m}$. Similarly $\varphi(y_{e-(2k+1)}) = z_{od+(2k+1)+m}$. Again, as $y_{e+2k-1} \sim z_{e+2k} \sim y_{e+2k+1}$, applying φ we have $z_{od-(2k-1)+m} \sim \varphi(z_{e+2k}) \sim z_{od-(2k+1)+m}$, and hence $\varphi(z_{e+2k}) = y_{od-2k+m}$ for $k = 0, \cdots, m-1$. Note that if m is even then $od - 2k + m$ is an odd integer, and so $\varphi(z_{e+2k}) \neq y_{od-2k+m}$ as all even spokes are mapped to all odd spokes. Hence φ is not a graph automorphism when m is even and so m must be odd. □

Similarly, it can be proved that:

Lemma 2.6. *Let $\varphi \in Aut(\Gamma(m)) \setminus H$ and $m \neq 3$.*

- *If set of all even spokes are mapped to set of all even spokes via φ, then m is odd.*
- *If set of all odd spokes are mapped to set of all odd spokes via φ, then m is odd.*
- *If set of all odd spokes are mapped to set of all even spokes via φ, then m is odd.*

Theorem 2.2. *Let $\varphi \in Aut(\Gamma(m)) \setminus H$. If $\emptyset \neq \varphi(\Sigma) \cap \Sigma \subsetneq \Sigma$, then m is odd.*

Proof. If $m = 3$, then there is nothing to prove. So, we assume that $m \neq 3$. As $\varphi(\Sigma) \cap \Sigma \neq \emptyset$, there exists some i such that $\varphi([x_i, y_i]) \in \Sigma$. Then, by Lemma 2.4, we have $\varphi([x_{i+2k}, y_{i+2k}]) \in \Sigma$ for all k. Now, depending on whether i is odd or even and depending on whether $\varphi([x_i, y_i])$ is an even spoke or an odd spoke, we can apply Lemma 2.5 or Lemma 2.6, to prove that m is odd. □

Corollary 2.2. *If m is even and $\varphi \in Aut(\Gamma(m)) \setminus H$, then $\varphi(\Sigma) \cap \Sigma = \emptyset$.*

Proof. This is the contrapositive form of Theorem 2.2.

Lemma 2.7. *If $\varphi \in Aut(\Gamma(m)) \setminus H$, then $\varphi(\Sigma) \nsubseteq \mathcal{M}$.*

Proof. If possible, let $\varphi(\Sigma) \subseteq \mathcal{M}$. Then $\varphi([x_0, y_0]) \in \mathcal{M}$. Now, two cases may arise. Either $\varphi([x_0, y_0]) = [y_i, z_{i+1}]$ or $\varphi([x_0, y_0]) = [y_i, z_{i-1}]$ for some i.

Case 1. Let $\varphi([x_0, y_0]) = [y_i, z_{i+1}]$. If $\varphi(x_0) = y_i$, $\varphi(y_0) = z_{i+1}$, then as $\varphi(x_1)$, $\varphi(x_{-1})$ are adjacent to $\varphi(x_0)$, we have $\varphi(x_1), \varphi(x_{-1}) \in \{x_i, z_{i-1}\}$. Let $\varphi(x_1) = x_i$ and $\varphi(x_{-1}) = z_{i-1}$. Then $\varphi(y_1) = x_{i+1}$ or x_{i-1}, hence $\varphi(x_1, y_1) \in \Omega$, which is a contradiction. Hence $\varphi(x_1) \neq x_i$ or $\varphi(x_{-1}) \neq z_{i-1}$. Similarly we get a contradiction for $\varphi(x_{-1}) = x_i$ and $\varphi(x_1) = z_{i-1}$. Thus we have $\varphi(x_0) \neq y_i$ or $\varphi(y_0) \neq z_{i+1}$.

If $\varphi(x_0) = z_{i+1}$ and $\varphi(y_0) = y_i$, then as $\varphi(x_1)$, $\varphi(x_{-1})$ are adjacent to $\varphi(x_0)$, we have $\varphi(x_1)$, $\varphi(x_{-1}) \in \{y_{i+2}, z_{i+1+m}\}$. Let $\varphi(x_1) = y_{i+2}$ and $\varphi(x_{-1}) = z_{i+1+m}$. Then $\varphi(x_2)$, $\varphi(y_1) \in \{z_{i+3}, x_{i+2}\}$. As $\varphi(\Sigma) \subset \mathcal{M}$, we have $\varphi(y_1) = z_{i+3}$ and $\varphi(x_2) = x_{i+2}$. Then $\varphi(y_2) \in \{x_{i+1}, x_{i+3}\}$ and hence $\varphi([x_2, y_2]) \in \Omega$, which is a contradiction. Hence $\varphi(x_1) \neq y_{i+2}$ or $\varphi(x_{-1}) \neq z_{i+1+m}$. Similarly we arrive at a contradiction for $\varphi(x_{-1}) = y_{i+2}$ and $\varphi(x_1) = z_{i+1+m}$ and then we have $\varphi(x_0) \neq z_{i+1}$ or $\varphi(y_0) \neq y_i$. Therefore $\varphi([x_0, y_0]) \neq [y_i, z_{i+1}]$.

Case 2. Let $\varphi([x_0, y_0]) = [y_i, z_{i-1}]$. The proof is similar to that in the previous case.

Combining two cases, we get the lemma. □

Lemma 2.8. *If m is even and $\varphi \in Aut(\Gamma(m)) \setminus H$, then $\varphi(\Sigma) \cap \Omega = \emptyset$.*

Proof. If possible let $\varphi(\Sigma) \cap \Omega \neq \emptyset$ and let $\varphi([x_a, y_a]) \in \Omega$. Without loss of generality, we can assume that $\varphi([x_a, y_a]) = [x_0, x_1]$ such that $\varphi(x_a) = x_0$ and $\varphi(y_a) = x_1$.

Let us first explain the rationale behind such an assumption. As $\varphi([x_a, y_a]) \in \Omega$, we have $\varphi([x_a, y_a]) = [x_j, x_{j+1}]$ for some j. Then $\rho^{-j}\varphi([x_a, y_a]) = [x_0, x_1]$. Now, $\rho^{-j}\varphi(\Sigma) \cap \Omega = \emptyset$ if and only if $\varphi(\Sigma) \cap \rho^j(\Omega) = \varphi(\Sigma) \cap \Omega = \emptyset$. Thus, without loss of generality, we can assume that $\varphi([x_a, y_a]) = [x_0, x_1]$. Now, if $\varphi(x_a) = x_1$ and $\varphi(y_a) = x_0$, we can work with $\rho\tau$ in the same manner as $\rho\tau(\Omega) = \Omega$. Thus, the assumption is justified and we start with $\varphi(x_a) = x_0$ and $\varphi(y_a) = x_1$. Then $\varphi(x_{a+1})$, $\varphi(x_{a-1}) \in \{x_{-1}, y_0\}$.

Case 1. Let $\varphi(x_{a+1}) = y_0$ and $\varphi(x_{a-1}) = x_{-1}$. As $x_{a-1} \sim y_{a-1}$, we have $\varphi(y_{a-1}) = x_{-2}$ or y_{-1}. If $\varphi(y_{a-1}) = y_{-1}$, then we have $\varphi([x_{a-1}, y_{a-1}]) = [x_{-1}, y_{-1}] \in \Sigma$. But this contradicts Corollary 2.2. Thus $\varphi(y_{a-1}) = x_{-2}$.

Consider the cycle $C_0 : y_{a-1} \sim x_{a-1} \sim x_a \sim x_{a+1} \sim y_{a+1} \sim z_a \sim y_{a-1}$. Then $\varphi(C_0) : x_{-2} \sim x_{-1} \sim x_0 \sim y_0 \sim \varphi(y_{a+1}) \sim \varphi(z_a) \sim x_{-2}$. As $\varphi(C_0)$ is a cycle, we have $\varphi(y_{a+1}) = z_{-1}$ and $\varphi(z_a) = y_{-2}$.

Again, consider the cycle $C_1 : y_a \sim x_a \sim x_{a+1} \sim x_{a+2} \sim y_{a+2} \sim z_{a+1} \sim y_a$. Then $\varphi(C_1) : x_1 \sim x_0 \sim y_0 \sim \varphi(x_{a+2}) \sim \varphi(y_{a+2}) \sim \varphi(z_{a+1}) \sim x_1$. As $\varphi(C_1)$ is a cycle, we have $\varphi(x_{a+2}) = z_1, \varphi(y_{a+2}) = y_2$ and $\varphi(z_{a+1}) = x_2$.

Proceeding in this way and considering the cycles $C_i : y_{a+i-1} \sim x_{a+i-1} \sim x_{a+i} \sim x_{a+i+1} \sim y_{a+i+1} \sim z_{a+i} \sim y_{a+i-1}$, for $i = 2, \cdots, n-1$, we get $\varphi(C_i) : \varphi(y_{a+i-1}) \sim \varphi(x_{a+i-1}) \sim \varphi(x_{a+i}) \sim \varphi(x_{a+i+1}) \sim \varphi(y_{a+i+1}) \sim \varphi(z_{a+i}) \sim \varphi(y_{a+i-1})$. Hence, we have

$\varphi(x_{a+12k}) = x_{6k}$	$\varphi(y_{a+12k}) = x_{6k+1}$	$\varphi(z_{a+12k}) = y_{6k-2}$
$\varphi(x_{a+12k+1}) = y_{6k}$	$\varphi(y_{a+12k+1}) = z_{6k-1}$	$\varphi(z_{a+12k+1}) = x_{6k+2}$
$\varphi(x_{a+12k+2}) = z_{6k+1}$	$\varphi(y_{a+12k+2}) = y_{6k+2}$	$\varphi(z_{a+12k+2}) = z_{6k-1+m}$
$\varphi(x_{a+12k+3}) = z_{6k+1+m}$	$\varphi(y_{a+12k+3}) = y_{6k+m}$	$\varphi(z_{a+12k+3}) = z_{6k+3}$
$\varphi(x_{a+12k+4}) = y_{6k+2+m}$	$\varphi(y_{a+12k+4}) = z_{6k+3+m}$	$\varphi(z_{a+12k+4}) = x_{6k+m}$
$\varphi(x_{a+12k+5}) = x_{6k+2+m}$	$\varphi(y_{a+12k+5}) = x_{6k+1+m}$	$\varphi(z_{a+12k+5}) = y_{6k+4+m}$
$\varphi(x_{a+12k+6}) = x_{6k+3+m}$	$\varphi(y_{a+12k+6}) = x_{6k+4+m}$	$\varphi(z_{a+12k+6}) = y_{6k+1+m}$
$\varphi(x_{a+12k+7}) = y_{6k+3+m}$	$\varphi(y_{a+12k+7}) = z_{6k+2+m}$	$\varphi(z_{a+12k+7}) = x_{6k+5+m}$
$\varphi(x_{a+12k+8}) = z_{6k+4+m}$	$\varphi(y_{a+12k+8}) = y_{6k+5+m}$	$\varphi(z_{a+12k+8}) = z_{6k+2}$
$\varphi(x_{a+12k+9}) = z_{6k+4}$	$\varphi(y_{a+12k+9}) = y_{6k+3}$	$\varphi(z_{a+12k+9}) = z_{6k+6+m}$
$\varphi(x_{a+12k+10}) = y_{6k+5}$	$\varphi(y_{a+12k+10}) = z_{6k+6}$	$\varphi(z_{a+12k+10}) = x_{6k+3}$
$\varphi(x_{a+12k+11}) = x_{6k+5}$	$\varphi(y_{a+12k+11}) = x_{6k+4}$	$\varphi(z_{a+12k+11}) = y_{6k+7}$

As m is even, m is either $6k$ or $6k+2$ or $6k+4$. Hence $n = 12k$ or $12k+4$ or $12k+8$.

Let $m = 6k$ and $k = 2i$. Then $\varphi(z_{a+1+m}) = \varphi(z_{a+1+12i}) = x_{6i+2}$. As $\varphi(z_{a+1}) \sim \varphi(z_{a+1+m})$, we have $x_2 \sim x_{6i+2}$, i.e., $6i+2 = 3$, i.e., $3k = 1$, i.e., $m = 2$. However, we considered $m \geq 3$. Hence, a contradiction. Again, let $m = 6k$ and $k = 2i+1$, then $\varphi(z_{a+1+m}) = \varphi(z_{a+7+12i}) = x_{6i+5+m}$. As $\varphi(z_{a+1}) \sim \varphi(z_{a+1+m})$, we have $x_2 \sim x_{6i+5+m}$, i.e., hence $6i+5+m = 3$, i.e., $3m = 2$, i.e., $m = 2$. a contradiction. So $m \neq 6k$.

Similarly, it can be shown that $m \neq 6k+2,\ 6k+4$. Then $\varphi(x_{a+1}) \neq y_0$ or $\varphi(x_{a-1}) \neq x_{-1}$.

Case 2. Let $\varphi(x_{a+1}) = x_{-1}$ and $\varphi(x_{a-1}) = y_0$. A similar technique as that in the previous case leads to a contradiction and hence we have $\varphi([x_a, y_a]) \neq [x_0, x_1]$.

Therefore, the lemma holds. □

Lemma 2.9. *If m is even and $\varphi \in Aut(\Gamma(m)) \setminus H$, then $\varphi(\Sigma) \cap \mathcal{I} = \emptyset$.*

Proof. As m is even, by Theorem 2.2, we have $\varphi(\Sigma) \cap \Sigma = \emptyset$. If possible, let $\varphi(\Sigma) \cap \mathcal{I} \neq \emptyset$ and let $[x_i, y_i]$ be a spoke edge which is mapped into $\mathcal{I}$ by φ. Without loss of generality, we can assume that $\varphi([x_i, y_i]) = [z_0, z_m]$ with $\varphi(x_i) = z_0$ and $\varphi(y_i) = z_m$. (*We can do so, because if $\varphi([x_i, y_i]) = [z_j, z_{j+m}]$, then $\rho^{-j}\varphi([x_i, y_i]) = [z_0, z_m]$. Now, if $\rho^{-j}\varphi(\Sigma) \cap \mathcal{I} = \emptyset$, then we also have $\varphi(\Sigma) \cap \rho^j(\mathcal{I}) = \varphi(\Sigma) \cap \mathcal{I} = \emptyset$. Similarly, if $\varphi([x_i, y_i]) = [z_{j+m}, z_j]$, then we need to work with $\tau\rho^{-j}$ instead of ρ^{-j}.*) Therefore $\varphi(x_{i+1}),\ \varphi(x_{i-1}) \in \{y_1, y_{-1}\}$.

Let $\varphi(x_{i+1}) = y_1$ and $\varphi(x_{i-1}) = y_{-1}$. Then $\varphi(y_{i+1}) \in \{z_2, x_1\}$ and $\varphi(y_{i-1}) \in \{z_{-2}, x_{-1}\}$. As $\varphi(\Sigma) \cap \Sigma = \emptyset$ then $\varphi(y_{i+1}) = z_2$ and $\varphi(y_{i-1}) = z_{-2}$. Now consider the cycle $C : y_{i-1} \sim x_{i-1} \sim x_i \sim x_{i+1} \sim y_{i+1} \sim z_i \sim y_{i-1}$. Then $\varphi(C) : z_{-2} \sim y_{-1} \sim z_0 \sim y_1 \sim z_2 \sim \varphi(z_i) \sim z_{-2}$. As $\varphi(z_i)$ is the common neighbour of z_2 and z_{-2}, then $\varphi(z_0) = y_j$, for some j. This implies

$j \equiv 2+1 \equiv -2-1 \ (mod\ 2m)$, i.e, $m = 3$, which is a contradiction as m is even. Hence, we have $\varphi(x_{i+1}) \neq y_1$ or $\varphi(x_{i-1}) \neq y_{-1}$.

Similarly we can also prove that $\varphi(x_{i+1}) = y_{-1}$ and $\varphi(x_{i-1}) = y_1$ does not hold. Hence, the lemma holds. □

Theorem 2.3. *If m is even, then $Aut(\Gamma(m)) = H = \langle \rho, \tau \rangle$.*

Proof. It is known that H is a subgroup of $Aut(\Gamma(m))$. If possible, let $Aut(\Gamma(m)) \setminus H \neq \emptyset$ and $\varphi \in Aut(\Gamma(m)) \setminus H$. Then by Corollary 2.2, Lemma 2.8 and Lemma 2.9, we have $\varphi(\Sigma) \cap (\Sigma \cup \Omega \cup \mathcal{I}) = \emptyset$. However, as the edge set of $\Gamma(m)$ is the union of $\Sigma, \Omega, \mathcal{M}$ and $\mathcal{I}$, it follows that $\varphi(\Sigma) \subseteq \mathcal{M}$. But this contradicts Lemma 2.7. Thus $Aut(\Gamma(m)) = H = \langle \rho, \tau \rangle$. □

Lemma 2.10. *If m is odd and $m \neq 3, 9$, then $|\mathsf{Stab}(x_0) \cap \mathsf{Stab}(x_1)| = 1$.*

Proof. Let $\varphi \in \mathsf{Stab}(x_0) \cap \mathsf{Stab}(x_1)$. Then $\varphi(x_0) = x_0$ and $\varphi(x_1) = x_1$. As $\varphi(y_0), \varphi(x_{-1})$ are adjacent to $\varphi(x_0)$; and $\varphi(y_1),\ \varphi(x_2)$ are adjacent to $\varphi(x_1)$, we have $\varphi(y_0),\ \varphi(x_{-1}) \in \{y_0, x_{-1}\}$ and $\varphi(y_1), \varphi(x_2) \in \{y_1, x_2\}$.

Case 1. Let $\varphi(y_0) = x_{-1}$ and $\varphi(y_1) = x_2$, and hence $\varphi(x_{-1}) = y_0$ and $\varphi(x_2) = y_1$.

Consider the cycle $C_0 : y_{-1} \sim x_{-1} \sim x_0 \sim x_1 \sim y_1 \sim z_0 \sim y_{-1}$. Then $\varphi(C_0) : \varphi(y_{-1}) \sim \varphi(x_{-1}) \sim x_0 \sim x_1 \sim x_2 \sim \varphi(z_0) \sim \varphi(y_{-1})$. As $\varphi(C_0)$ is a cycle, we have $\varphi(x_{-1}) = y_0$, $\varphi(y_{-1}) = z_1$ and $\varphi(z_0) = y_2$.

Now consider the cycle $C_1 : y_0 \sim x_0 \sim x_1 \sim x_2 \sim y_2 \sim z_1 \sim y_0$. Then $\varphi(C_1) : x_{-1} \sim x_0 \sim x_1 \sim \varphi(x_2) \sim \varphi(y_2) \sim \varphi(z_1) \sim x_{-1}$. As $\varphi(C_1)$ is a cycle, we have $\varphi(x_2) = y_1$, $\varphi(y_2) = z_0$ and $\varphi(z_1) = y_{-1}$.

Proceeding in this way and considering the cycles $C_i : y_{i-1} \sim x_{i-1} \sim x_i \sim x_{i+1} \sim y_{i+1} \sim z_i \sim y_{i-1}$, for $i = 2, \cdots, n-1$, we get $\varphi(C_i) : \varphi(y_{i-1}) \sim \varphi(x_{i-1}) \sim \varphi(x_i) \sim \varphi(x_{i+1}) \sim \varphi(y_{i+1}) \sim \varphi(z_i) \sim \varphi(y_{i-1})$. Thus, we have

$\varphi(x_{12k}) = x_{6k}$	$\varphi(y_{12k}) = x_{6k-1}$	$\varphi(z_{12k}) = y_{6k+2}$
$\varphi(x_{12k+1}) = x_{6k+1}$	$\varphi(y_{12k+1}) = x_{6k+2}$	$\varphi(z_{12k+1}) = y_{6k-1}$
$\varphi(x_{12k+2}) = y_{6k+1}$	$\varphi(y_{12k+2}) = z_{6k}$	$\varphi(z_{12k+2}) = x_{6k+3}$
$\varphi(x_{12k+3}) = z_{6k+2}$	$\varphi(y_{12k+3}) = y_{6k+3}$	$\varphi(z_{12k+3}) = z_{6k+m}$
$\varphi(x_{12k+4}) = z_{6k+2+m}$	$\varphi(y_{12k+4}) = y_{6k+1+m}$	$\varphi(z_{12k+4}) = z_{6k+4}$
$\varphi(x_{12k+5}) = y_{6k+3+m}$	$\varphi(y_{12k+5}) = z_{6k+4+m}$	$\varphi(z_{12k+5}) = x_{6k+1+m}$
$\varphi(x_{12k+6}) = x_{6k+3+m}$	$\varphi(y_{12k+6}) = x_{6k+2+m}$	$\varphi(z_{12k+6}) = y_{6k+5+m}$
$\varphi(x_{12k+7}) = x_{6k+4+m}$	$\varphi(y_{12k+7}) = x_{6k+5+m}$	$\varphi(z_{12k+7}) = y_{6k+2+m}$
$\varphi(x_{12k+8}) = y_{6k+4+m}$	$\varphi(y_{12k+8}) = z_{6k+3+m}$	$\varphi(z_{12k+8}) = x_{6k+6+m}$
$\varphi(x_{12k+9}) = z_{6k+5+m}$	$\varphi(y_{12k+9}) = y_{6k+6+m}$	$\varphi(z_{12k+9}) = z_{6k+3}$
$\varphi(x_{12k+10}) = z_{6k+5}$	$\varphi(y_{12k+10}) = y_{6k+4}$	$\varphi(z_{12k+10}) = z_{6k+7+m}$
$\varphi(x_{12k+11}) = y_{6k+6}$	$\varphi(y_{12k+11}) = z_{6k+7}$	$\varphi(z_{12k+11}) = x_{6k+4}$

As m is odd, m is either $6k+1$ or $6k+3$ or $6k+5$. Hence $n = 12k+2$ or $12k+6$ or $12k+10$.

If $m = 6k+1$ then $\varphi(x_{12k+2}) = \varphi(x_0)$, i.e., $y_{6k+1} = x_0$, which is a contradiction. Thus $\varphi(y_0) \neq x_{-1}$ or $\varphi(y_1) \neq x_2$.

If $m = 6k + 5$ then $\varphi(x_{12k+10}) = \varphi(x_0)$, i.e., $z_{6k+5} = x_0$, which is a contradiction. Thus $\varphi(y_0) \neq x_{-1}$ or $\varphi(y_1) \neq x_2$.

Now let $m = 6k + 3$, consider $y_2 \sim z_1 \sim z_{1+m}$ then $\varphi(y_2) = z_0 \sim \varphi(z_1) = y_{-1} \sim \varphi(z_{1+m})$. If $k = 2i$ is even, then $z_{1+m} = z_{6k+4} = z_{12i+4}$ and hence $\varphi(z_{1+m}) = z_{6i+4}$. Therefore when k is even, we get $6i + 4 \equiv -2\ (mod\, n)$, i.e., $6i + 6 \equiv 0\ (mod\, n)$, i.e, $3k + 6 \equiv 0\ (mod\ 12k + 6)$. This happens only when $k = 0$, i.e., $m = 3$.

On the other hand, if $k = 2i + 1$ is odd, then $z_{1+m} = z_{12i+10} = z_{6i+7+m}$, hence $\varphi(z_{1+m}) = z_{6i+7+m}$. Thus when k is odd, we have $6i+7+m \equiv -2\ (mod\, n)$, i.e, $9(k+1) \equiv 0\ (mod\ 12k+6)$. This happens only when $k = 1$, i.e., $m = 9$.

Therefore we have if m is odd and $m \neq 3, 9$, then there does not exists $\varphi \in \mathsf{Stab}(x_0) \cap \mathsf{Stab}(x_1)$ such that **Case 1** holds.

Case 2. Let $\varphi(y_0) = y_0$ and $\varphi(y_1) = x_2$. As $\varphi([x_0, y_0]) = [x_0, y_0]$, by Lemma 2.2, we have for $m \neq 3$, $\varphi \in H$, i.e., $\varphi = \mathsf{id}$.

Case 3. Let $\varphi(y_0) = x_{-1}$ and $\varphi(y_1) = y_1$. As $\varphi([x_1, y_1]) = [x_1, y_1]$, by Lemma 2.2, we have for $m \neq 3$, $\varphi \in H$, i.e., $\varphi = \mathsf{id}$.

Case 4. Let $\varphi(y_0) = y_0$ and $\varphi(y_1) = y_1$. Then φ maps consecutive spokes $[x_0, y_0], [x_1, y_1]$ into Σ. Then by Lemma 2.3, we have $\varphi \in H$, i.e., $\varphi = \mathsf{id}$.

Therefore combining all the 4 cases, we have the lemma. □

Lemma 2.11. *Let m is odd and $m \neq 3, 9$, then*

$$|\{\varphi \in Aut(\Gamma(m)) : \varphi(x_0) = x_0, \varphi(x_1) = x_{-1}\}| = 1.$$

Proof. The proof is similar to that of Lemma 2.10. □

Lemma 2.12. *Let m is odd and $m \neq 3, 9$, then there does not exist any $\varphi \in Aut(\Gamma(m))$ such that $\varphi(x_0) = x_0$ and $\varphi(x_1) = y_0$.*

Proof. Let $\varphi(x_0) = x_0$ and $\varphi(x_1) = y_0$. As $\varphi(y_0), \varphi(x_{-1})$ are adjacent to $\varphi(x_0)$; and $\varphi(y_1), \varphi(x_2)$ are adjacent to $\varphi(x_1)$, then we have $\varphi(y_0),\ \varphi(x_{-1}) \in \{x_1, x_{-1}\}$ and $\varphi(y_1), \varphi(x_2) \in \{z_1, z_{-1}\}$.

Case 1. Let $\varphi(y_0) = x_1$ and $\varphi(y_1) = z_1$, then $\varphi(x_2) = z_{-1}$. Now consider the cycle $C : y_0 \sim x_0 \sim x_1 \sim x_2 \sim y_2 \sim z_1 \sim y_0$. Then $\varphi(C) : x_1 \sim x_0 \sim y_0 \sim z_{-1} \sim \varphi(y_2) \sim \varphi(z_1) \sim x_1$. As $\varphi(C)$ is a cycle, for $m = 3$ we have $\varphi(y_2) = z_2$ and $\varphi(z_1) = y_1$, but for $m \neq 3$, $\varphi(C)$ is not a cycle, which is a contradiction.

Case 2. Let $\varphi(y_0) = x_{-1}$ and $\varphi(y_1) = z_{-1}$, then $\varphi(x_2) = z_1$. This leads to a contradiction, as in previous case.

Case 3. Let $\varphi(y_0) = x_1$ and $\varphi(y_1) = z_{-1}$. Then $\varphi(x_2) = z_1$. Consider the cycle $C_0 : y_{-1} \sim x_{-1} \sim x_0 \sim x_1 \sim y_1 \sim z_0 \sim y_{-1}$. Then $\varphi(C_0) : \varphi(y_{-1}) \sim \varphi(x_{-1}) \sim x_0 \sim y_0 \sim z_{-1} \sim \varphi(z_0) \sim \varphi(y_{-1})$. As $\varphi(C_0)$ is a cycle, we have $\varphi(x_{-1}) = x_{-1}$, $\varphi(y_{-1}) = x_{-2}$ and $\varphi(z_0) = y_{-2}$.

Again consider the cycle $C_1 : y_0 \sim x_0 \sim x_1 \sim x_2 \sim y_2 \sim z_1 \sim y_0$. Then $\varphi(C_1) : x_1 \sim x_0 \sim y_0 \sim z_1 \sim \varphi(y_2) \sim \varphi(z_1) \sim x_1$. As $\varphi(C_1)$ is a cycle, we have $\varphi(y_2) = y_2$ and $\varphi(z_1) = x_2$.

Proceeding in this way and considering the cycles $C_i : y_{i-1} \sim x_{i-1} \sim x_i \sim x_{i+1} \sim y_{i+1} \sim z_i \sim y_{i-1}$, for $i = 2, \cdots, n-1$, we have $\varphi(C_i) : \varphi(y_{i-1}) \sim \varphi(x_{i-1}) \sim \varphi(x_i) \sim \varphi(x_{i+1}) \sim \varphi(y_{i+1}) \sim \varphi(z_i) \sim \varphi(y_{i-1})$. Hence we get

$\varphi(x_{12k}) = x_{6k}$	$\varphi(y_{12k}) = x_{6k+1}$	$\varphi(z_{12k}) = y_{6k-2}$
$\varphi(x_{12k+1}) = y_{6k}$	$\varphi(y_{12k+1}) = z_{6k-1}$	$\varphi(z_{12k+1}) = x_{6k+2}$
$\varphi(x_{12k+2}) = z_{6k+1}$	$\varphi(y_{12k+2}) = y_{6k+2}$	$\varphi(z_{12k+2}) = z_{6k+m-1}$
$\varphi(x_{12k+3}) = z_{6k+1+m}$	$\varphi(y_{12k+3}) = y_{6k+m}$	$\varphi(z_{12k+3}) = z_{6k+3}$
$\varphi(x_{12k+4}) = y_{6k+2+m}$	$\varphi(y_{12k+4}) = z_{6k+3+m}$	$\varphi(z_{12k+4}) = x_{6k+m}$
$\varphi(x_{12k+5}) = x_{6k+2+m}$	$\varphi(y_{12k+5}) = x_{6k+1+m}$	$\varphi(z_{12k+5}) = y_{6k+4+m}$
$\varphi(x_{12k+6}) = x_{6k+3+m}$	$\varphi(y_{12k+6}) = x_{6k+4+m}$	$\varphi(z_{12k+6}) = y_{6k+1+m}$
$\varphi(x_{12k+7}) = y_{6k+3+m}$	$\varphi(y_{12k+7}) = z_{6k+2+m}$	$\varphi(z_{12k+7}) = x_{6k+5+m}$
$\varphi(x_{12k+8}) = z_{6k+4+m}$	$\varphi(y_{12k+8}) = y_{6k+5+m}$	$\varphi(z_{12k+8}) = z_{6k+2}$
$\varphi(x_{12k+9}) = z_{6k+4}$	$\varphi(y_{12k+9}) = y_{6k+3}$	$\varphi(z_{12k+9}) = z_{6k+6+m}$
$\varphi(x_{12k+10}) = y_{6k+5}$	$\varphi(y_{12k+10}) = z_{6k+6}$	$\varphi(z_{12k+10}) = x_{6k+3}$
$\varphi(x_{12k+11}) = x_{6k+5}$	$\varphi(y_{12k+11}) = x_{6k+4}$	$\varphi(z_{12k+11}) = y_{6k+7}$

As m is odd, m is either $6k+1$ or $6k+3$ or $6k+5$ and hence $n = 12k+2$ or $12k+6$ or $12k+10$. If $m = 6k+1$ then $\varphi(x_{12k+2}) = \varphi(x_0)$, i.e., $z_{6k+1} = x_0$, which is a contradiction. If $m = 6k+5$ then $\varphi(x_{12k+10}) = \varphi(x_0)$, i.e., $y_{6k+5} = x_0$, which is a contradiction.

Now let $m = 6k+3$. Consider $z_1 \sim z_{1+m} \sim y_{2+m}$. Then applying φ, we get $x_2 \sim \varphi(z_{1+m}) \sim \varphi(y_{2+m})$.

If $k = 2i$ is even, then $y_{2+m} = y_{6k+5} = y_{12i+5}$ and $z_{1+m} = z_{6k+4} = z_{12i+4}$. Hence $\varphi(y_{2+m}) = x_{6i+1+m}$ and $\varphi(z_{1+m}) = x_{6i+m}$. Therefore when k is even, $x_2 \sim x_{6i+m} \sim x_{6i+1+m}$. Hence we have $6i+m \equiv 3 \ (mod\, n)$, i.e, $3k+6k+3 \equiv 3 \ (mod\, n)$, i.e., $9k \equiv 0 \ (mod\, 12k+6)$. This happens only when $k = 0$, i.e., $m = 3$.

On the other hand, if $k = 2i+1$ is odd, then $y_{2+m} = y_{12i+11}$ and $z_{1+m} = z_{12i+10}$. Hence $\varphi(y_{2+m}) = x_{6i+4}$ and $\varphi(z_{1+m}) = x_{6i+3}$. Thus when k is odd, we have $x_2 \sim x_{6i+3} \sim x_{6i+4}$. This implies $6i+3 \equiv 3 \ (mod\, n)$, i.e, $6i = 3(k-1) \equiv 0 \ (mod\, 12k+6)$. This happens only when $k = 1$, i.e., $m = 9$.

Therefore we have if m is odd and $m \neq 3, 9$, then φ as in **Case 3** does not exist.

Case 4. Let $\varphi(y_0) = x_{-1}$ and $\varphi(y_1) = z_1$. In this case also, it can be shown, similarly as in **Case 3**, that such a φ does not exist.

Therefore combining all 4 cases, the lemma follows. □

Theorem 2.4. *If m is odd and $m \neq 3, 9$, then $Aut(\Gamma(m)) = \langle \rho, \tau, \sigma \rangle$.*

Proof. It was already noted that $\langle \rho, \tau, \sigma \rangle$ is a subgroup of $Aut(\Gamma(m))$ of order $6n$. Moreover, as m is odd by the Lemma 2.1, we have $\Gamma(m)$ is Cayley and hence vertex-transitive. Thus, by orbit-stabilizer theorem, we have

$$\frac{|Aut(\Gamma(m))|}{|\mathsf{Stab}(x_0)|} = |\Gamma(m)| = 3n, \text{ i.e., } |Aut(\Gamma(m))| = 3n \cdot |\mathsf{Stab}(x_0)|.$$

Thus, to prove the theorem, it suffices to show that $|\mathsf{Stab}(x_0)| = 2$.

It is clear that $\mathsf{id}, \tau \in \mathsf{Stab}(x_0)$. Let $\varphi \in \mathsf{Stab}(x_0)$. As $x_0 \sim x_1$, therefore $\varphi(x_1) = x_1$ or x_{-1} or y_0. If $\varphi(x_1) = x_1$, by Lemma 2.10, $\varphi = \mathsf{id}$. If $\varphi(x_1) = x_{-1}$, then by Lemma 2.11, $\varphi = \tau$. And finally, Lemma 2.12 shows that no φ exists such that $\varphi(x_1) = y_0$. Hence, $\mathsf{Stab}(x_0) = \{\mathsf{id}, \tau\}$ and the theorem follows. □

We now note a few automorphisms which occurs only when $m = 3$ or 9. Define

$$\eta = (x_4, y_5)(z_1, z_5)(x_2, y_1)(y_3, z_3)(y_4, z_4)(x_3, z_0)(y_2, z_2) \text{ and}$$

$$\varepsilon_i = (x_{6i+3}, z_{6i+4}, z_{6i-4})(x_{6i+2}, y_{6i-1}, z_{6i-1})(x_{6i-2}, y_{6i+1}, z_{6i+1})(x_{6i+1}, x_{6i-1}, y_{6i})(y_{6i+2}, y_{6i-2}, z_{6i}).$$

It can be shown that $\eta, \varepsilon_0 \in Aut(\Gamma(3))$ and $\zeta = \varepsilon_0\varepsilon_1\varepsilon_2 \cdot (z_3, z_9, z_{15}) \cdot (y_3, y_9, y_{15}) \in Aut(\Gamma(9))$. Moreover, it was checked using SageMath [6] that

$$Aut(\Gamma(3)) = \langle \rho, \tau, \sigma, \eta, \varepsilon_0 \rangle \text{ and } Aut(\Gamma(3)) = \langle \rho, \tau, \sigma, \zeta \rangle.$$

Thus, summarizing all the cases, i.e., Theorem 2.3, Theorem 2.4 and the above discussion, we get the following theorem.

Theorem 2.5. *Let $m \geq 3$ and $\Gamma(m)$ be the Generalized Pappus graph. Then*

$$Aut(\Gamma(m)) = \begin{cases} \langle \rho, \tau \rangle, & \text{if } m \text{ is even,} \\ \langle \rho, \tau, \sigma \rangle, & \text{if } m \text{ is odd and } m \neq 3, 9, \\ \langle \rho, \tau, \sigma, \zeta \rangle, & \text{if } m = 9, \\ \langle \rho, \tau, \sigma, \eta, \varepsilon_0 \rangle, & \text{if } m = 3. \end{cases}$$

Corollary 2.3. *The following are true:*

1. $\Gamma(3)$ *and* $\Gamma(9)$ *are arc-transitive.*
2. *For* $m \neq 3, 9$, $\Gamma(m)$ *is not edge-transitive.*
3. $\Gamma(m)$ *is Cayley if and only if* m *is odd.*

Proof. The arc-transitivity of $\Gamma(3)$ and $\Gamma(9)$ can be easily checked in SageMath [6]. As $\Gamma(m)$ has $9m$ edges, in order that $\Gamma(m)$ is edge-transitive, the order of $Aut(\Gamma(m))$ must be a multiple of $9m$. However, if $m \neq 3, 9$, the order of $Aut(\Gamma(m))$ is either $4m$ or $12m$ depending upon whether m is even or odd. Thus the second statement holds.

For the last statement, the sufficiency is proved in Theorem 2.1. For the necessity, the order of a vertex-transitive graph should divide the order of its automorphism group. However, if m is even, then $|\Gamma(m)| = 6m < 4m = |Aut(\Gamma(m))|$. □

3. Open Issues

Another thing which is very much related to graphs with large symmetry groups, is its hamiltonicity, viz, Lovasz' conjecture. Our observation based on first few values of m, made us to strongly believe that $\Gamma(m)$ is Hamiltonian for all value of $m \geq 3$. This can be an interesting topic for further research.

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References

[1] A. Devillers, M. Giudici, C.H. Li, and C.E. Praeger, An infinite family of biquasiprimitive 2-arc transitive cubic graphs, *Journal of Algebraic Combinatorics*, **35** (2021), 173–192.

[2] R. Frucht, J.E. Graver, and M.E. Watkins, The groups of the generalized Petersen graphs, *Proceedings of the Cambridge Philosophical Society*, **70** (2) (1971), 211–218.

[3] C. Godsil, and G.F. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, 207, Springer-Verlag, 2001.

[4] D. Koreas, Some bound of the edge chromatic surplus of certain cubic graphs, *Electron. J. Graph Theory Appl.*, **6** (2) (2018), 310–316.

[5] K. Kutnar and P. Petecki, On automorphisms and structural properties of double generalized Petersen graphs, *Discrete Math.*, **339** (2016), 2861–2870.

[6] W. Stein and others, Sage Mathematics Software (Version 7.3), Release Date: 04.08.2016, http://www.sagemath.org.

[7] M.J. Stroppel, Generalizing the Pappus and Reye configurations, *Australas. J. Combin.*, **72** (2) (2018), 249–272.

[8] J.X. Zhou and M. Ghasemi, Automorphisms of a family of cubic graphs, *Algebra Colloquium*, **20** (3) (2013), 495–506.

[9] J.X. Zhou and Yan-Tao Li, Cubic non-normal Cayley graphs of order $4p^2$, *Discrete Math.* **312** (2012), 1940–1946, doi:10.1016/j.disc.2012.03.012.