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# Chromatic number of super vertex local antimagic total labelings of graphs

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**Abstract**

Let $G(V, E)$ be a simple graph and $f$ be a bijection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ where $f(V) = \{1, 2, \ldots, |V|\}$. For a vertex $x \in V$, define its weight $w(x)$ as the sum of labels of all edges incident with $x$ and the vertex label itself. Then $f$ is called a super vertex local antimagic total (SLAT) labeling if for every two adjacent vertices their weights are different. The super vertex local antimagic total chromatic number $\chi_{slat}(G)$ is the minimum number of colors taken over all colorings induced by super vertex local antimagic total labelings of $G$. We classify all trees $T$ that have $\chi_{slat}(T) = 2$, present a class of trees that have $\chi_{slat}(T) = 3$, and show that for any positive integer $n \geq 2$ there is a tree $T$ with $\chi_{slat}(T) = n$.

*Keywords:* super vertex local antimagic total labeling, super vertex local antimagic total chromatic number, tree, chromatic number
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## 1. Introduction

All graphs defined in this paper are simple and connected. Introduced by Arumugam et al. [1], a vertex local antimagic labeling is a bijective function $f : E(G) \to \{1, 2, \dots, |E(G)|\}$ such that $w(u) \neq w(v)$ for any adjacent vertices $u$ and $v$, where the *weight* $w(x)$ of a vertex $x \in V$ is the sum of labels of all edges incident with $x$. The minimum number of distinct weights needed for a graph $G$ to have a vertex local antimagic labeling is denoted by $\chi_{la}(G)$. They conjectured that every connected graph other than $K_2$ is a vertex local antimagic graph, which was confirmed by Haslegrave using probabilistic method [4].

Putri et al. [7] introduced a new variant of vertex local antimagic labeling, called vertex local antimagic total labeling. A vertex local antimagic total labeling is a bijective map $f : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that $w(u) \neq w(v)$ for any two adjacent vertices $u$ and $v$. Here $w(x)$ is the sum of labels of all edges incident with $x$ and the label of $x$ itself. The minimum of distinct weights so that a graph $G$ has vertex local antimagic total labeling is denoted by $\chi_{lat}(G)$. The minimum number of distinct weights needed for a graph $G$ to have a vertex local antimagic labeling is denoted by $\chi_{la}(G)$. Lau [5] adopts a result from Haslegrave [4] to show that every connected graph is a vertex local antimagic total graph. For more information on local antimagic or antimagic labelings, we refer the reader to Gallian's survey [3].

Furthermore, Slamin et al. [8] introduced a new variant of the labeling. *A super* vertex local antimagic total labeling is a bijective map $f : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$ where $f(V(G)) = \{1, 2, \dots, |V(G)|\}$ such that $w(u) \neq w(v)$ for any two adjacent vertices $u$ and $v$, where $w(x) = f(x) + \sum_{xy \in E(G)} f(xy)$.The minimum number of distinct weights needed for a graph $G$ to have a super vertex local antimagic labeling is denoted by $\chi_{slat}(G)$. From the definition, we can perceive the super vertex local antimagic labeling as a vertex coloring of a graph with some additional conditions. An easy observation then follows.

**Observation 1.1.** *For any graph $G$, $\chi_{slat}(G) \geq \chi(G)$.*

We limit our current research to some classes of trees; in particular, stars $S_n$, paths $P_n$, caterpillars $S_{n_1,n_2,\dots,n_k}$ and shrubs $\check{S}(n_1, n_2, \dots, n_k)$. *A shrub* $\check{S}(n_1, n_2, \dots, n_k)$ is defined as a tree constructed from a star $S_m$, every leaf of which is adjacent to some number of isolated vertices (see [6]).

Slamin *et al.* [8] proved the following. If $T$ is a tree on $n \geq 2$ vertices with $k$ leaves, then $\chi_{slat}(T) \leq n - k + 1$. For a star $S_n$ and a double star $S_{k,n-k}$, we have $\chi_{slat}(S_{n+1}) = 2$ and $\chi_{slat}(S_{k,n-k}) = 3$. In addition, if $P_n$ is a path, $\chi_{slat}(P_n) = 3$ if $n$ is odd and $n \geq 5$, or $3 \leq \chi_{slat}(P_n) \leq 4$ if $n$ is even and $n \geq 6$.

In this paper, we characterize trees $T$ with $\chi_{slat}(T) = 2$, show existence of trees with $\chi_{slat}(T) = 3$, and construct trees $T$ that have $\chi_{slat}(T) = n$ for any positive integer $n \geq 2$.

## 2. Characterization of Trees with $\chi_{slat}(T) = 2$

We start by determining the lower bound of $\chi_{slat}(T)$. The following Lemma 2.1 shows sufficient condition for vertices having different weights based on their degrees.

**Lemma 2.1.** *Let $T$ be a tree graph which has SLAT-labeling $f$ and $v_1, v_2 \in V(T)$. If $2\deg(v_1) + 1 \leq \deg(v_2)$, then $w(v_1) < w(v_2)$.*

*Proof.* Let $\deg(v_1) = d$ and $|V| = n$, so that $\deg(v_2) \geq 2d+1$ and $|E| = n-1$. By assigning $v_1$ and edges incident with $v_1$ labels such that the weight of $v_1$ is as large as possible, we have

$$\begin{aligned} w(v_1) &\leq (d+1)|V| + d|E| - \sum_{i=1}^{d}(i-1) \\ w(v_1) &\leq (d+1)n + d(n-1) - \frac{(d-1)d}{2} \\ w(v_1) &\leq 2dn + n - \frac{d^2+d}{2}\,. \end{aligned}$$

Then, by assigning $v_2$ and edges incident with $v_2$ labels such that the weight of $v_2$ is as small as possible, we have

$$\begin{aligned} w(v_2) &\geq (2d+1)|V| + \sum_{i=1}^{2d+1}(i+1) \\ w(v_2) &\geq (2d+1)n + \frac{(2d+1)(2d+2)}{2} + 1 \\ w(v_2) &\geq 2dn + n + 2d^2 + 3d + 1. \end{aligned}$$

It can be seen that $w(v_1) < w(v_2)$. □

The following special case where $v_1$ is a leaf will be useful.

**Corollary 2.1.** *For an arbitrary tree, if $v_1$ is a leaf vertex and $v_2$ is a vertex with $\deg(v_2) \geq 3$, then $w(v_1) \neq w(v_2)$.*

Based on [8], $\chi_{slat}(S_n) = 2$. We will show that stars are the only trees with $\chi_{slat}(T) = 2$. In our proof, we provide a labeling different from the one in [8].

**Theorem 2.1.** *Suppose $T$ is a tree graph, then $\chi_{slat}(T) = 2$ if and only if $T \cong S_n$ for $n \in \mathbb{N}$.*

*Proof.* Let $T \cong S_n$ for $n \in \mathbb{N}$, we will show that $\chi_{slat}(T) = 2$. By the fact that $\chi(T) = 2$ and Observation 1.1 we conclude that $\chi_{slat}(T) \geq 2$. To show $\chi_{slat}(T) \leq 2$, define $f : V(T) \cup E(T) \to \{1, 2, \ldots, |V(T)| + |E(T)|\}$ as follows:

$$\begin{aligned} f(c) &= n+1, \\ f(v_i) &= i, 1 \leq i \leq n, \\ f(cv_i) &= 2n+2-i, i \leq i \leq n. \end{aligned}$$

From here, we get

$$\begin{aligned} w(v_i) &= 2n+2, 1 \leq i \leq n, \\ w(c) &= \frac{3}{2}n^2 + \frac{5}{2}n + 1. \end{aligned}$$

Therefore, $\chi_{slat}(T) \leq 2$. We conclude that if $T \cong S_n$, then $\chi_{slat}(T) = 2$.

Now let $\chi_{slat}(T) = 2$, we will show that $T \cong S_n$.

Let the partition of $V(T)$ be $V_1, V_2$. Without loss of generality, let $x_0 \in V_1$ and $P = x_0, y_1, x_1, \dots$ be a diametrical path. Then $x_0$ is of degree one. By Corollary 2.1, all vertices in $V_1$ are of degree at most two and therefore all vertices of $V_2$ belong to $P$. Denote by $p$ the number of leaves in $V_1$ and by $q$ the number of vertices of degree two. We want to show that $q = 0$.

Using this notation, we can see that $P = x_0, y_1, x_1, \dots, x_q, y_{q+1}$ or $P = x_0, y_1, x_1, \dots, x_q, y_{q+1}, x_{q+1}$ and $|V_2| = q + 1$. Therefore, we have

$$|V| = |V_1| + |V_2| = (p + q) + (q + 1) = p + 2q + 1,$$

which yields

$$p = |V| - 2q - 1.$$

Denote by $V_1^i$ the set of vertices of degree $i$ in $V_1$. Then we have $|V_1^1| = p$ and $|V_1^2| = q$. Denote $|V| = m$.

We know that all vertices in $V_1$ have the same weight, call it $w^*$. We first look at the $p$ vertices of degree one, observing that

$$\sum_{x_i \in V_1^1} w(x_i) = pw^*. \tag{1}$$

We also know that

$$\begin{aligned}
\sum_{x_i \in V_1^1} w(x_i) &= \sum_{x_i \in V_1^1} f(x_i) && + \sum_{x_i \in V_1^1, x_i y_j \in E} f(x_i y_j) \\
&\leq \sum_{s=m-p+1}^{m} s && + \sum_{t=2m-p}^{2m-1} t \\
&= \frac{(2m-p+1)p}{2} && + \frac{(4m-p-1)p}{2}
\end{aligned} \tag{2}$$

Combining (1) and (2), we obtain

$$pw^* = \sum_{x_i \in V_1^1} w(x_i) \leq \frac{(2m-p+1)p}{2} + \frac{(4m-p-1)p}{2}, \tag{3}$$

which yields

$$w^* \leq \frac{(2m-p+1)}{2} + \frac{(4m-p-1)}{2} = 3m - p, \tag{4}$$

Now we look at the $q$ vertices of degree two, observing that

$$\sum_{x_i \in V_2^1} w(x_i) = qw^*. \tag{5}$$

We also know that

$$\begin{aligned}\sum_{x_i \in V_1^2} w(x_i) &= \sum_{x_i \in V_1^2} f(x_i) &&+ \sum_{x_i \in V_1^2, x_i y_j \in E} f(x_i y_j) \\ &\geq \sum_{s=1}^{q} s &&+ \sum_{t=m+1}^{m+2q} t \\ &= \frac{(q+1)q}{2} &&+ \frac{(2m+2q+1)(2q)}{2} \end{aligned} \tag{6}$$

Combining (5) and (6), we obtain

$$qw^* = \sum_{x_i \in V_1^2} w(x_i) \geq \frac{(q+1)q}{2} + \frac{(2m+2q+1)(2q)}{2}, \tag{7}$$

which for $q > 0$ yields

$$w^* \geq \frac{q+1}{2} + (2m+2q+1) = 2m + \frac{5q+3}{2}. \tag{8}$$

We noted above that

$$p = |V| - 2q - 1 = m - 2q - 1. \tag{9}$$

Substituting (9) into (4), we have

$$w^* \leq 3m - p = 3m - (m - 2q - 1) = 2m + 2q + 1. \tag{10}$$

Now comparing (8) and (10), we get

$$2m + \frac{5q+3}{2} \leq w^* \leq 2m + 2q + 1, \tag{11}$$

which is impossible for $q > 0$. Hence, $q = 0$. We already noticed that $|V_2| = q + 1 = 1$, which implies that $T$ must be the star $S_p$.

□

In Figure 1, we give an example of SLAT labeling on $S_8$.

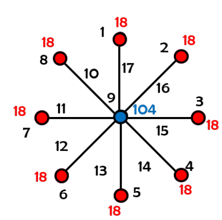



Figure 1: SLAT labeling on $S_8$, $\chi_{slat}(S_8) = 2$.

**Corollary 2.2.** *Suppose $T$ is a non-trivial tree graph and $S_n$ is a star graph. If $T$ is not isomorphic to $S_n$, then $\chi_{slat}(T) \geq 3$.*

## 3. Existence of Trees with $\chi_{slat}(T) = 3$

Slamin et al. in [8] investigated paths $P_n$ and proved that $\chi slat(T_n) = 3$ when $n$ is odd, and $3 \leq \chi_{slat}(T_n) \leq 4$ when $n$ is even. In Theorem 3.1, we present a more straightforward proof.

**Theorem 3.1.** *Let $P_n$ be a path on $n$ vertices, $n \geq 4$. Then $\chi_{slat}(P_n) = 3$ when $n$ is odd or $n \in \{4, 6, 8, 10\}$ and $3 \leq \chi_{slat}(P_n) \leq 4$ when $n$ is even and $n \geq 12$.*

*Proof.* Let $V(P_n) = \{v_i | 1 \leq i \leq n\}$ and $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$ with $n \in \mathbb{N}$. According to Corollary 2.2, graphs that are not isomorphic to a star have $\chi_{slat}(P_n) \geq 3$. To show the upper bound, the problem is divided into two cases, according to the parity of $n$.

**Case 1.** $n$ is odd

Define $f : V(P_n) \cup E(P_n) \to \{1, 2, 3, \ldots, |V| + |E|\}$ as follows

$$f(v_i) = \begin{cases} 2i - 1, & \text{if } \in \{1, 2\}, \\ 2, & \text{if } i = n, \\ n - i + 2, & \text{if } 3 \leq i \leq n - 2, i \text{ is odd}, \\ n - i + 4, & \text{if } 4 \leq i \leq n - 1, i \text{ is even}. \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} 2n - 1, & \text{if } i = 1, \\ n + \frac{i-1}{2}, & \text{if } 3 \leq i \leq n - 2, i \text{ is odd}, \\ \frac{3}{2}(n - 1) + \frac{i}{2}, & \text{if } 2 \leq i \leq n - 1, i \text{ is even}. \end{cases}$$

Then we have the weights as follows.

$$w(v_i) = \begin{cases} 2n, & \text{if } \in \{1, n\}, \\ \frac{7}{2}n - \frac{1}{2}, & \text{if } 3 \leq i \leq n - 2, i \text{ is odd}, \\ \frac{7}{2}n + \frac{3}{2}, & \text{if } 4 \leq i \leq n - 1, i \text{ is even}. \end{cases}$$

Therefore, $\chi_{slat}(P_n) \leq 3$.

**Case 2.** $n$ is even

Define $f : V(P_n) \cup E(P_n) \to \{1, 2, 3, \ldots, |V| + |E|\}$ as follows.

$$f(v_i) = \begin{cases} n, & \text{if } i = 1, \\ n - 1, & \text{if } i = n, \\ n - i - 1, & \text{if } 2 \leq i \leq n - 2, i \text{ is even}, \\ n - i + 1, & \text{if } 3 \leq i \leq n - 1, i \text{ is odd}. \end{cases}$$

$$f(v_i v_{i+1}) = \begin{cases} n + \frac{i}{2}, & \text{if } 2 \leq i \leq n - 2, i \text{ is even}, \\ \frac{3}{2}n + \frac{i-1}{2}, & \text{if } 1 \leq i \leq n - 1, i \text{ is odd}. \end{cases}$$

Then we have the weights as follows.

$$w(v_i) = \begin{cases} \frac{5}{2}n, & \text{if } i = 1, \\ 3n - 2, & \text{if } i = n, \\ \frac{7}{2}n, & \text{if } 3 \leq i \leq n - 2, i \text{ is odd}, \\ \frac{7}{2}n - 2, & \text{if } 2 \leq i \leq n - 1, i \text{ is even}. \end{cases}$$

We conclude that $\chi_{slat}(P_n) = 3$ when $n$ is odd or $n\{4, 6, 8, 10\}$, and $3 \leq \chi_{slat}(P_n) \leq 4$ when $n$ is even and $n \geq 12$. □

In Figure 2 we present SLAT labelings of $P_4, P_6, P_8, P_{10}$ and also $P_7$ as an example for $n$ odd. The labeling is not unique. Here are some labelings for $P_6, P_8, P_{10}$. First bracket is vertex labels, second edges, third weights. The labelings for $P_6$, $P_8$, and $P_{10}$ were found by Branson [2].

$P_6$
$[5, 3, 4, 1, 2, 6][11, 9, 7, 8, 10][16, 23, 20, 16, 20, 16]$

$P_8$
$[8, 4, 1, 6, 5, 3, 2, 7][14, 12, 13, 11, 10, 9, 15][22, 30, 26, 30, 26, 22, 26, 22]$
$[8, 3, 4, 2, 5, 7, 1, 6][13, 14, 9, 10, 12, 11, 15][21, 30, 27, 21, 27, 30, 27, 21]$
$[7, 1, 2, 4, 6, 3, 5, 8][15, 12, 13, 11, 10, 9, 14][22, 28, 27, 28, 27, 22, 28, 22]$
$[5, 2, 7, 6, 8, 3, 4, 1][10, 15, 9, 12, 11, 13, 14][15, 27, 31, 27, 31, 27, 31, 15]$
$[3, 1, 5, 4, 7, 8, 6, 2][12, 15, 10, 14, 9, 11, 13][15, 28, 30, 28, 30, 28, 30, 15]$
$[4, 1, 7, 6, 5, 3, 8, 2][11, 15, 9, 12, 14, 10, 13][15, 27, 31, 27, 31, 27, 31, 15]$
$[7, 2, 3, 6, 1, 5, 4, 8][15, 11, 13, 9, 12, 10, 14][22, 28, 27, 28, 22, 27, 28, 22]$
$[6, 1, 3, 7, 2, 5, 4, 8][15, 12, 11, 10, 14, 9, 13][21, 28, 26, 28, 26, 28, 26, 21]$

$P_{10}$
$[8, 1, 7, 3, 2, 5, 4, 9, 6, 10][19, 14, 15, 16, 18, 11, 12, 13, 17][27, 34, 36, 34, 36, 34, 27, 34, 36, 27]$
$[9, 6, 4, 5, 3, 7, 2, 1, 8, 10][19, 15, 12, 11, 17, 16, 13, 14, 18][28, 40, 31, 28, 31, 40, 31, 28, 40, 28]$

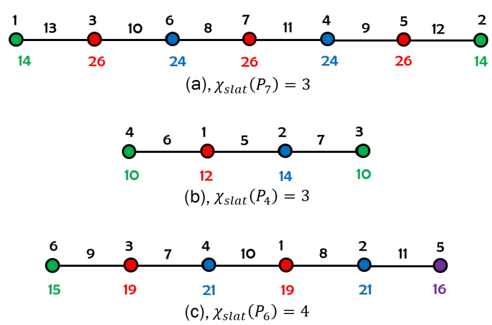



Figure 2: SLAT labeling on $P_7$, $P_4$ and $P_6$.

Based on the labelings of the short even paths above, we state the following.

**Conjecture.** *For any even $n \geq 4$, $\chi_{slat}(P_n) = 3$.*

As a common generalization of caterpillars and shrubs, we introduce a new class of trees called shrubs. A *shrub* $\hat{S}(m, n, p)$ is defined by its vertex and edge set as follows.

$$V(\hat{S}(m, n, p)) = \{c, v_i, v_i^j, u_k | 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\}$$
$$E(\hat{S}(m, n, p)) = \{cv_i, v_iv_i^j, cu_k | 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\}$$

When $p = 0$, then $\hat{S}(m, n, p)$ is a regular shrub (all $u_k$ vertices and $cu_k$ edges are omitted). Else, if $m \leq 2$, then $\hat{S}(m, n, p)$ is a caterpillar. However, when $m = 0$, $n = 0$, or $m + p = 1$, then $\hat{S}(m, n, p)$ is a star. Since we already know that $\chi_{slat}(T) = 2$ for $T \cong S_n$, the case of graph which is isomorphic to a star is omitted.

**Theorem 3.2.** *Suppose $\hat{S}(m, n, p)$ is a modified shrub. For positive $m, n$, non-negative $p$ and $m + p \neq 1$, $\chi_{slat}(\hat{S}(m, n, p)) = 3$.*

*Proof.* Let $\hat{S}(m, n, p) = \{c, v_i, v_i^j, u_k | 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\}$ and $E(\hat{S}(m, n, p)) = \{cv_i, v_i v_i^j, cu_k | 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p\}$. By Corollary 2.2, graphs other than stars have $\chi_{slat}(\hat{S}(m, n, p)) \geq 3$. To show the upper bound, the proof is divided into two cases.

**Case 1.** $p + m \geq n + 1$
The case is divided into three subcases, according to the parity of $n$ and $m$.

**Subcase 1.1.** $n$ is even
Define $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ as follows

$$f(u_k) = k, 1 \leq k \leq p,$$

$$f(v_i^j) = \begin{cases} m(j-1) + p + i, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}, \\ mj - i + p + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}. \end{cases}$$

$$f(v_i) = mn + p + i, 1 \leq i \leq m,$$

$$f(c) = m(n+1) + p + 1,$$

$$f(v_i v_i^j) = \begin{cases} m(2n - j + 1) + p + i + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}, \\ m(2n - j + 2) + p - i + 2, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}. \end{cases}$$

$$f(cv_i) = m(2n+2) + 2p - i + 2, 1 \leq i \leq m,$$

$$f(cu_k) = m(2n+1) + 2p - k + 2, 1 \leq k \leq p.$$

When $p = 0$, then vertices $v_k$ and edges $cv_k$ are omitted.

We have

$$w(u_k) = w(v_i^j) = m(2n+1) + 2p + 2, 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n,$$

$$w(v_i) = m(n(2n + \frac{9}{2}) - \frac{n(n+1)}{2} + 2) + p(n+3) + \frac{3n}{2} + 2, 1 \leq i \leq m,$$

$$w(c) = m((2m+1)(n+1) + p(2n+3) + 2) + \frac{p(3p+5)}{2} - \frac{m(m+1)}{2} + 1.$$

It can be seen that these three weights are different.

**Subcase 1.2.** Both $n$ and $m$ are odd
Define $f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$ as follows

$$f(u_k) = k, 1 \leq k \leq p,$$
$$f(v_i^j) = \begin{cases} m(j-1) + p + i, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}, \\ mj - i + p + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}. \end{cases}$$
$$f(v_i) = \begin{cases} mn + p + \frac{m+1}{2} - i + 1, & \text{if} \quad 1 \leq i \leq \frac{m+1}{2}, \\ m(n+1) + p + \frac{m+1}{2} - i + 1, & \text{if} \quad \frac{m+3}{2} \leq i \leq m. \end{cases}$$
$$f(c) = m(n+1) + p + 1,$$
$$f(v_i v_i^j) = \begin{cases} m(2n - j + 1) + p + i + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}, \\ m(2n - j + 2) + p - i + 2, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}. \end{cases}$$
$$f(cv_i) = \begin{cases} m(2n+1) + 2p + 2i, & \text{if} \quad 1 \leq i \leq \frac{m+1}{2}, \\ 2mn + 2p + 2i, & \text{if} \quad \frac{m+3}{2} \leq i \leq m. \end{cases}$$
$$f(cu_k) = m(2n+1) + 2p - k + 2, 1 \leq k \leq p.$$

When $p = 0$, then vertices $v_k$ and edges $cv_k$ are omitted.

We have

$$w(u_k) = w(v_i^j) = m(2n+1) + 2p + 2, 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n,$$
$$w(v_i) = m(2n(n+2) + (1-n)\frac{n+1}{2} + 1) + p(n+3) + \frac{m+3n}{2} + 2, 1 \leq i \leq m,$$
$$w(c) = m((2m+1)(n+1) + p(2n+3) + 2) + \frac{p(3p+5)}{2} - \frac{m(m+1)}{2} + 1.$$

It can be seen that these three weights are different.

**Subcase 1.3.** $n$ is odd and $m$ is even
Define $f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$ as follows

$$f(u_k) = k, 1 \leq k \leq p,$$
$$f(v_i^j) = \begin{cases} m(j-1) + p + i, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}, \\ mj - i + p + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}. \end{cases}$$
$$f(v_i) = \begin{cases} mn + p + \frac{i+1}{2} + 1, & \text{if} \quad 1 \leq i \leq m, i \text{ is odd}, \\ mn + p + \frac{m+i}{2} + 1, & \text{if} \quad 1 \leq i \leq m, i \text{ is even}. \end{cases}$$
$$f(c) = mn + p + \frac{m}{2} + 1,$$
$$f(v_i v_i^j) = \begin{cases} m(2n - j + 1) + p + i + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}, \\ m(2n - j + 2) + p - i + 2, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}. \end{cases}$$

$$f(cv_i) = \begin{cases} m(2n+1)+2p+\frac{i}{2}+1, & \text{if } 1 \le i \le m, i \text{ is even}, \\ m(2n+1)+2p+\frac{m+i+1}{2}+1, & \text{if } 1 \le i \le m, i \text{ is odd}. \end{cases}$$

$$f(cu_k) = m(2n+1)+2p-k+2, 1 \le k \le p.$$

When $p = 0$, then vertices $v_k$ and edges $cv_k$ are omitted.

We have

$$w(u_k) = w(v_i^j) = m(2n+1)+2p+2, 1 \le k \le p, 1 \le i \le m, 1 \le j \le n,$$

$$w(v_i) = m(2n(n+2)+(1-n)\frac{n+1}{2}+1)+p(n+3)+\frac{m+3n+1}{2}+2, 1 \le i \le m,$$

$$w(c) = m((2m+1)(n+1)+p(2n+3)+\frac{3}{2})+\frac{p(3p+5)}{2}-\frac{m(m+1)}{2}+1.$$

It can be seen that these three weights are different.

**Case 2.** $p+m < n+1$
The case is divided into three subcases according to the parity of $n$ and $m$.

**Subcase 2.1.** $n$ is even
Define $f : V \cup E \to \{1, 2, 3, \ldots, |V|+|E|\}$ as follows.

$$f(u_k) = mn+k, 1 \le k \le p,$$

$$f(v_i^j) = \begin{cases} m(j-1)+i, & \text{if } 1 \le i \le m, 1 \le j \le n, j \text{ is odd}, \\ mj-i+1, & \text{if } 1 \le i \le m, 1 \le j \le n, j \text{ is even}. \end{cases}$$

$$f(v_i) = mn+p+i+1, 1 \le i \le m,$$

$$f(c) = mn+p+1,$$

$$f(v_iv_i^j) = \begin{cases} m(2n-j+2)+2p+i+1, & \text{if } 1 \le i \le m, 1 \le j \le n, j \text{ is even}, \\ m(2n-j+3)+2p-i+2, & \text{if } 1 \le i \le m, 1 \le j \le n, j \text{ is odd}. \end{cases}$$

$$f(cv_i) = m(n+2)+p-i+2, 1 \le i \le m,$$

$$f(cu_k) = m(n+2)+2p-k+2, 1 \le k \le p.$$

When $p = 0$, then vertices $v_k$ and edges $cv_k$ are omitted.

We have

$$w(u_k) = w(v_i^j) = m(2n+2)+2p+2, 1 \le k \le p, 1 \le i \le m, 1 \le j \le n,$$

$$w(v_i) = m(n(2n+\frac{9}{2})-\frac{n(n+1)}{2}+2)+2p(n+1)+\frac{3n}{2}+3, 1 \le i \le m,$$

$$w(c) = m((m+2)(n+2)+p)+\frac{p(3p+5)}{2}-\frac{m(m+1)}{2}+1.$$

It can be seen that these three weights are different.

**Subcase 2.2.** Both $n$ and $m$ are odd
Define $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ as follows.

$$f(u_k) = mn + k, 1 \leq k \leq p,$$

$$f(v_i^j) = \begin{cases} m(j-1) + i, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}, \\ mj - i + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}. \end{cases}$$

$$f(v_i) = \begin{cases} mn + p + \frac{m+1}{2} - i + 2, & \text{if} \quad 1 \leq i \leq \frac{m+1}{2}, \\ m(n+1) + p + \frac{m+1}{2} - i + 2, & \text{if} \quad \frac{m+3}{2} \leq i \leq m. \end{cases}$$

$$f(c) = mn + p + 1,$$

$$f(v_i v_i^j) = \begin{cases} m(2n - j + 2) + 2p + i + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}, \\ m(2n - j + 3) + 2p - i + 2, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}. \end{cases}$$

$$f(cv_i) = \begin{cases} m(n+1) + p + 2i, & \text{if} \quad 1 \leq i \leq \frac{m+1}{2}, \\ mn + p + 2i, & \text{if} \quad \frac{m+3}{2} \leq i \leq m. \end{cases}$$

$$f(cu_k) = m(n+2) + 2p - k + 2, 1 \leq k \leq p.$$

When $p = 0$, then vertices $v_k$ and edges $cv_k$ are omitted.

We have

$$w(u_k) = w(v_i^j) = m(2n+2) + 2p + 2, 1 \leq k \leq p, 1 \leq i \leq m, 1 \leq j \leq n,$$

$$w(v_i) = m(2n(n+2) + (1-n)\frac{n+1}{2} + 1) + 2p(n+1) + \frac{m+3n}{2} + 3, 1 \leq i \leq m,$$

$$w(c) = m((m+1)(n + \frac{3}{2}) + p(n+3)) + \frac{p(3p+5)}{2} + 1.$$

It can be seen that these three weights are different.

**Subcase 2.3.** $n$ is odd and $m$ is even
Define $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ as follows.

$$f(u_k) = mn + k, 1 \leq k \leq p,$$

$$f(v_i^j) = \begin{cases} m(j-1) + i, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is odd}, \\ mj - i + 1, & \text{if} \quad 1 \leq i \leq m, 1 \leq j \leq n, j \text{ is even}. \end{cases}$$

$$f(v_i) = \begin{cases} mn + p + \frac{i+1}{2}, & \text{if} \quad 1 \leq i \leq m, i \text{ is odd}, \\ mn + p + \frac{m+i}{2} + 1, & \text{if} \quad 1 \leq i \leq m, i \text{ is even}. \end{cases}$$

$$f(c) = mn + p + \frac{m}{2} + 1,$$

$$f(v_iv_i^j) = \begin{cases} m(2n-j+2)+2p+i+1, & \text{if } 1 \le i \le m, 1 \le j \le n, j \text{ is even,} \\ m(2n-j+3)+2p-i+2, & \text{if } 1 \le i \le m, 1 \le j \le n, j \text{ is odd.} \end{cases}$$

$$f(cv_i) = \begin{cases} m(n+1)+p+\frac{i}{2}+1, & \text{if } 1 \le i \le m, i \text{ is even,} \\ m(n+1)+p+\frac{m+i+1}{2}+1, & \text{if } 1 \le i \le m, i \text{ is odd.} \end{cases}$$

$$f(cu_k) = m(2n+1)+2p-k+2, 1 \le k \le p.$$

When $p=0$, then vertices $v_k$ and edges $cv_k$ are omitted.

We have

$$w(u_k) = w(v_i^j) = m(2n+2)+2p+2, 1 \le k \le p, 1 \le i \le m, 1 \le j \le n,$$

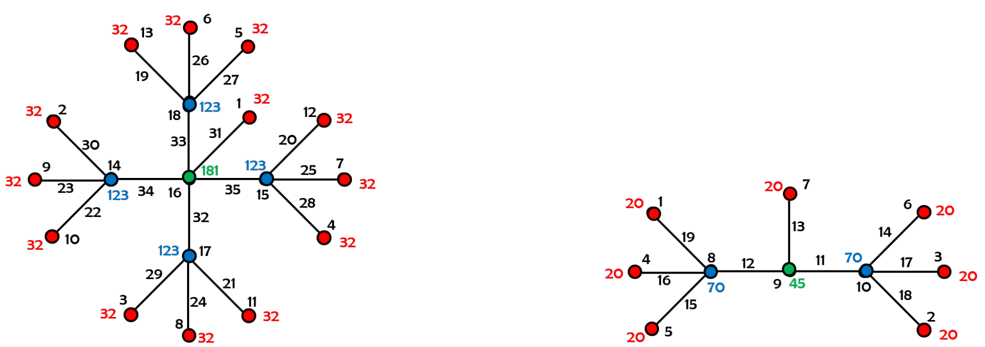
$$w(v_i) = m(2n(n+2)+(1-n)\frac{n+1}{2}+1)+2p(n+1)+\frac{m+3n+1}{2}+2, 1 \le i \le m,$$

$$w(c) = m((m+1)(n+\frac{3}{2})+p(2n+3)+\frac{1}{2})+\frac{p(3p+5)}{2}+1.$$

It can be seen that these three weights are different.

From the above cases, we can conclude that $\chi_{slat}(\check{S}'(m,n,p)) \le 3$. Hence, $\chi_{slat}(\check{S}'(m,n,p)) = 3$. □

In Figure 3, we have examples of two cases in the preceding theorem.



(a) $T \cong \hat{S}(4,3,1)$.

(b) $T \cong \hat{S}(2,3,1)$.

Figure 3: SLAT labeling on $T$, $\chi_{slat}(T) = 3$.

**Corollary 3.1.** *If a tree $T$ is isomorphic to a regular shrub $\check{S}(n,n,n,\ldots,n)$ or a caterpillar $S_{n_1,n_2}$ or $S_{n_1,n_2,n_1}$, then $\chi_{slat}(T) = 3$.*

## 4. Construction of Trees $T$ with $\chi_{slat}(T) = n$ for any $n \in \mathbb{N}$

Motivated by the fact that for any tree (an in fact for any bipartite graph) the regular chromatic number $\chi(T) = 2$, it is natural to ask whether there exists $k \in \mathbb{N}$ such that for every tree $T$, $\chi_{slat}(T) \le k$. In the following theorem we show that no such bound exists.

**Theorem 4.1.** *For every $n \geq 2$, there exists a tree $T$ such that $\chi_{slat}(T) = n$.*

*Proof.* The assertion for $n = 2$ follows from Theorem 2.1 and for $n = 3$ from Theorem 3.1. Therefore, we only construct examples fro $n \geq 4$.

We construct a tree $T$ starting with the path $P_{n+1}$. For every $i = 2, 3, \ldots, n$ we define $t_i = \left\lfloor \frac{i}{2} \right\rfloor + 1$ and join vertex $v_i$ to $2^{t_i} - 3$ isolated vertices.
From this construction, we obtain $\deg(v_i) = 2^{t_i} - 1$, for $2 \leq i \leq n$.

First, we need to show that $\chi_{slat}(T) \geq n$. According to the definition of SLAT-labeling, adjacent vertices must have different weights, therefore $w(v_i) \neq w(v_{i+1})$ for $1 \leq i \leq n$.

By the graph construction, for any $1 \leq i, j \leq n$ such that $j \geq i + 2$ the vertices $v_i, v_j$ are non-adjacent and satisfy the condition $2\deg(v_i) + 1 \leq \deg(v_j)$. It then follows from Lemma 2.1 that $w(v_i) \neq w(v_j)$. In addition, it follows from Corollary 2.1 that the weights of vertices of degree at least three are all greater than the weights of all leaves. Thus, the graph needs at least $n$ distinct weights, which means $\chi_{slat}(T) \geq n$.

To show $\chi_{slat}(T) \leq n$, we define a labeling $f$ as follows. For $i = 2, 3, \ldots, n$ and $l = 1, 2, \ldots, t_i$ we denote by $e_{i,l}$ the pendant edges incident with vertex $v_i$ and by $v_{i,l}$ the leaf incident with $e_{i,l}$. First we label edge $v_1v_2$ with label $|V| + 1$. Then we label the remaining pendant edges starting with the lowest available edge label $|V| + 2$ in lexicographic order; that is, $f(e_{i,l}) < f(e_{i,s})$ for any $1 \leq l < s \leq t_i$ and $f(e_{i,l}) < f(i, s)$ for any $2 \leq i < j \leq n$ and any $l$ and $s$. Next, label the leaf incident with an edge $e_{i,l}$ (or $v_1v_2$) so that the sum of the edge and vertex label equals $2|V| - n + 2$.

Then, label the vertices $v_2, v_3, \ldots, v_n$ starting from $f(v_2) = |V| - n + 2$ consecutively in increasing order. Finally, label the remaining edges starting from $f(v_2v_3) = 2|V| - n + 2$consecutively in increasing order. From this labeling, we have $w(v_{i,l}) = 2|V| - n + 2$ for every leaf vertex $v_{i,l}$, while $2|V| - n + 2 < w(v_i) \leq w(v_j)$ for $2 \leq i < j \leq n$. Hence, $\chi_{slat}(T) \leq n$.

We can conclude that $\chi_{slat}(T) = n$. □

## 5. Open Problems

To conclude, we state some obvious open problems.

1. Characterize trees with $\chi_{slat}(T) = 3$.
2. Determine $\chi_{slat}(G)$ for other natural classes of graphs.

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