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Perfect 2-colorings of the generalized Petersen graph $GP(n, 3)$

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Abstract

In this paper we enumerate the parameter matrices of all perfect 2-colorings of the generalized Petersen graphs $GP(n, 3)$, where $n \geq 7$. We also give some basic results for $GP(n, k)$.

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1. Introduction

The theory of error-correcting codes has always been a popular subject in group theory, combinatorial configuration, covering problems and even diophantine number theory. So, mathematicians always show a lot of interest in this historical research field. The problem of finding all perfect codes was begun by M. Golay in 1949. Perfect code is originally a topic in the theory of error-correcting codes. All perfect codes are known to be completely regular, which were introduces by Delsarte in 1973. A set of vertices, say C, of a simple graph is called completely regular code with covering radius ρ, if the distance partition of the vertex set with respect C is equitable. Therefore, the problem of existence of equitable partitions in graph is of great importance in graph theory. There is another term for this concept in the literature as ”perfect m-coloring”.
As explained above, enumerating parameter matrices in graphs is a key problem to find perfect

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codes in graphs. For example, by the results of this paper, we can easily conclude that the graph $GP(9,3)$ has just two nontrivial completely regular codes with the size of 9, which neither of them is perfect. There has always been a notably interest in enumerating parameter matrices of some popular families of graphs ”johnson graphs”, ”hypercube graphs” and recently ”generalized petersen graphs” (see [1, 2, 3, 4, 5, 6, 7, 8, 9]).

In this article, all parameter matrices of $GP(n,3)$ are enumerated.

2. Definition and Concepts

In this section, some basic definitions and concepts are given.

Definition 2.1. *The generalized petersen graph $GP(n,k)$, also denoted $P(n,k)$, for $n \geq 3$ and $1 \leq k < \frac{n}{2}$, is a connected cubic graph that has vertices, respectively, edges given by*

$$V(GP(n,k)) = \{a_i, b_i : 0 \leq i \leq n-1\},$$
$$E(GP(n,k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} : 0 \leq i \leq n-1\},$$

These graphs were introduced by Coxeter (1950) and named by Watkins (1969). $GP(n,k)$ is isomorphic to $GP(n,n-k)$. It is why we consider $k < \frac{n}{2}$, with no restriction of generality.

Definition 2.2. *For a graph G and an integer m, we call a mapping $T : V(G) \to \{1,\ldots,m\}$ a perfect m-coloring with matrix $A = (a_{ij})_{i,j\in\{1,\ldots,m\}}$, if it is surjective, and for all i, j, for every vertex of color i, the number of its neighbors of color j is equal to a_{ij}. We call the matrix A the parameter matrix of a perfect coloring. In the case $m = 2$, the first color is called white, and the second color black.*

Remark 2.1. In this paper, we consider all perfect 2-colorings, up to renaming the colors; i.e, we identify the perfect 2-coloring with the matrix

$$\begin{bmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{bmatrix},$$

obtained by switching the colors with the original coloring.

3. The Existence of Perfect 2-Colorings of $GP(n,3)$

In this section, we first give some results covering necessary conditions for the existence of perfect 2-colorings of $GP(n,k)$ graphs with a given parameter matrix $A = (a_{ij})_{i,j=1,2}$, and then we enumerate the parameters of all perfect 2-colorings of $GP(n,3)$.

The first and perhaps the simplest necessary condition for the existence of a perfect 2-colorings of $GP(n,k)$ with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$a_{11} + a_{12} = a_{21} + a_{22} = 3.$$

Also, it is clear that neither a_{12} nor a_{21} cannot be equal to zero, otherwise white and black vertices of $GP(n, k)$ would not be adjacent, which is impossible, as the graph is connected.
By the presented conditions, a parameter matrix of a perfect 2-coloring of $GP(n, k)$ must be one of the following matrices:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}.$$

The next proposition provides a formula for calculating the number of white vertices in a perfect 2-coloring (see [4]).

Proposition 3.1. *If W is the set of white vertices in a perfect 2-coloring of a graph G with matrix $A = (a_{ij})_{i,j=1,2}$, then*
$$|W| = |V(G)| \frac{a_{21}}{a_{12} + a_{21}}.$$

Now, we are ready to enumerate the parameter matrices of all perfect 2-colorings of $GP(n, 3)$. In [1] , the parameter matrices of all perfect 2-colorings of $GP(n, k)$ with the matrices of A_1 and A_6 are enumerated. So, we just present theorems in order to enumerate parameter matrices corresponding to perfect 2-colorings of $GP(n, 3)$ with the matrices A_2, A_3, A_4, and A_5.

Perfect 2-colorings of $GP(n, 3)$ with the matrix A_2:

In this part, we show that the graphs $GP(n, 3)$ have no perfect 2-colorings with the matrix A_2.

Theorem 3.1. *The graphs $GP(n, 3)$ have no perfect 2-colorings with the matrix A_2.*

Proof. At first, we claim that for each perfect 2-coloring, say T, of $GP(n, 3)$ with the matrix A_2, there are no consecutive vertices a_i and a_{i+1}, such that $T(a_i) = T(a_{i+1}) = 2$. To prove it, suppose contrary to our claim, without loss of generality, there is a perfect 2-coloring, say T, of $GP(n, 3)$ with the matrix A_2, such that $T(a_1) = T(a_2) = 2$. It imeadiately gives $T(b_1) = T(b_2) = T(a_0) = T(a_3) = 1$ and then $T(b_0) = T(b_3) = T(a_4) = T(b_4) = T(b_5) = 1$. Now, from $T(a_3) = T(a_4) = T(b_4) = 1$, we have $T(a_5) = 2$. Next, from $T(a_4) = T(b_5) = 1$ and $T(a_5) = 2$, we get $T(a_6) = 2$. It gives $T(b_6) = 1$ which is a contradiction with $T(a_3) = T(b_3) = T(b_0) = 1$.
Now, to prove the theorem, suppose the assertion is false. Therefore, there is a perfect 2-coloring, say T, of $GP(n, 3)$ with the matrix A_2. By symmetry, with no loss of generality, we can assume $T(a_0) = T(b_0) = 1$ and $T(a_1) = 2$. By the above claim, we have $T(a_2) = 1$. Now, from $T(a_0) = T(a_2) = 1$ and $T(a_1) = 2$, it follows that $T(b_1) = 2$. This immediately gives $T(b_2) = T(a_3) = T(b_4) = 1$. Next, by $T(b_4) = 1$ and $T(b_1) = 2$, we get $T(a_4) = 1$ and, in consequence, $T(b_3) = T(a_5) = 2$. Again, by using the above claim, we get $T(a_6) = 1$ and then $T(b_6) = 1$, which is a contradiction with $T(a_3) = T(b_0) = 1$ and $T(b_3) = 2$. □

Perfect 2-colorings of $GP(n,3)$ with the matrix A_3:

We will show that the graphs $GP(2m,3)$ have a perfect coloring with the matrix A_3 and the graphs $GP(2m+1,3)$ have no perfect 2-colorings with the matrix A_3.

Theorem 3.2. *All of the graphs $GP(n,3)$, where n is even, have a perfect 2-coloring with the matrix A_3. Also, there are no perfect 2-colorings of $GP(n,3)$, where n is odd, with the matrix A_3.*

Proof. To prove the first part, consider the mapping $T: V(GP(2m,3)) \to \{1,2\}$ by

$$T(a_{2i}) = T(b_{2i}) = 1,$$
$$T(a_{2i+1}) = T(b_{2i+1}) = 2.$$

for $i \geq 0$. It can be easily seen that the given mapping is a perfect 2-coloring of $GP(2m,3)$ with the matrix A_3.
To prove the second part, contrary to our claim, suppose there is a perfect 2-coloring, say T, of $GP(n,3)$, where n is odd, with the matrix A_3. Now, we use lemma in ([1], Lemma 3.4).

Lemma 3.1. *[1] For each perfect 2-coloring T of $GP(n,k)$, where k is a positive even integer or $4 \nmid n$, with the matrix A_3, there are two vertices a_i and b_i, for some $0 \leq i \leq n-1$, such that $T(a_i) = T(b_i)$.*

By above Lemma, with no loss of generality, we can assume $T(a_0) = T(b_0) = 1$. By knowing that the given mapping in the first part is not a perfect 2-coloring with the matrix A_3, where n is odd, we should have two cases below.

Case 1. For some positive integer i, $T(a_i) = T(b_i) = T(b_{i+1}) = 1$ and $T(a_{i+1}) = 2$. It immediately gives $T(a_{i+2}) = 2$ and $T(a_{i+3}) = T(b_{i+2}) = 1$. From $T(a_{i+2}) = T(b_i) = 1$, we deduce that $T(b_{i+3}) = 2$ and then $T(a_{i+4}) = 1$. Next, from $T(b_{i+1}) = T(a_{i+4}) = 1$, we have $T(b_{i+4}) = T(a_{i+5}) = 2$. Now, from $T(b_i) = T(a_{i+3}) = 1$, and $T(b_{i+3}) = 2$, it follows that $T(b_{i+6}) = 2$, and then from $T(a_{i+5}) = 2$, we get $T(a_{i+6}) = 1$ and $T(b_{i+5}) = 2$. Using this argument, for $j \geq 0$, we have

$$T(a_{10j+i}) = T(b_{10j+i}) = T(b_{10j+i+1}) = T(b_{10j+i+2}) = T(a_{10j+i+3}) = T(a_{10j+i+4}) = T(a_{10j+i+6}) = T(a_{10j+i+7}) = T(b_{10j+i+8}) = T(b_{10j+i+9}) = 1.$$

and

$$T(a_{10j+i+1}) = T(a_{10j+i+2}) = T(b_{10j+i+3}) = T(b_{10j+i+4}) = T(a_{10j+i+5}) = T(b_{10j+i+5}) = T(b_{10j+i+6}) = T(b_{10j+i+7}) = T(a_{10j+i+8}) = T(a_{10j+i+9}) = 2.$$

It gives $n = 10m$ which contradicts n is odd.

Case 2. For some positive integer i, $T(a_i) = T(b_i) = T(a_{i+2}) = 1$ and $T(a_{i+1}) = T(b_{i+1}) = T(b_{i+2}) = 2$. It immediately gives $T(a_{i+3}) = 1$ and then $T(a_{i+4}) = T(b_{i+3}) = 2$. From $T(a_{i+4}) = T(b_{i+1}) = 2$, we have $T(b_{i+4}) = 1$ and then we deduce that $T(a_{i+5}) = 2$ and $T(b_{i+5}) = T(a_{i+6}) = 1$. From $T(a_{i+3}) = T(b_i) = 1$ and $T(b_{i+3}) = 2$, we get $T(b_{i+6}) = 2$. Now, from $T(a_{i+5}) = T(b_{i+6}) = 2$ and $T(a_{i+6}) = 1$, we have $T(a_{i+7}) = 1$ and then $T(b_{i+7}) = 2$ which is a contradiction of $T(b_{i+4}) = 1$ and $T(a_{i+4}) = T(b_{i+1}) = T(b_{i+7}) = 2$. $\square$

Perfect 2-colorings of $GP(n,3)$ with the matrix A_4:

We show that just the graphs $GP(4m,3)$ among the graphs $GP(n,3)$ have a perfect 2-coloring with the matrix A_4.

Theorem 3.3. *All the graphs $GP(n,3)$, where $4 \mid n$, have a perfect 2-coloring with the matrix A_4. Also, there are no perfect 2-coloring of $GP(n,3)$, where $4 \nmid n$, with this matrix.*

Proof. For the first part, consider the mapping $T : V(GP(4m,3)) \to \{1,2\}$ by

$$T(a_{4i}) = T(b_{4i+2}) = 1,$$
$$T(b_{4i}) = T(a_{4i+1}) = T(b_{4i+1}) = T(a_{4i+2}) = T(a_{4i+3}) = T(b_{4i+3}) = 2.$$

for $i \geq 0$. It can be easily checked that the given mapping is a perfect 2-coloring with the matrix A_4.
To prove the second part, contrary to our claim, suppose that there is a perfect 2-coloring of $GP(n,3)$ with the matrix A_4, say T. with no restriction of generality, let $T(a_0) = 1$. It follows that $T(a_1) = T(b_0) = T(a_{n-1}) = T(b_{n-1}) = 2$. From $T(a_1) = 2$ and $T(a_0) = 1$ we get $T(b_1) = T(a_2) = 2$. Now, we should have two cases below.

Case 1. $T(b_2) = 1$. It immediately gives

$$T(a_{4i}) = T(b_{4i+2}) = 1,$$
$$T(b_{4i}) = T(a_{4i+1}) = T(b_{4i+1}) = T(a_{4i+2}) = T(a_{4i+3}) = T(b_{4i+3}) = 2.$$

for $i \geq 0$. It clearly gives $n = 4m$ which is a contradiction of $4 \nmid n$.

Case 2. $T(b_2) = 2$. It immediately gives $T(a_3) = 1$ and $T(b_3) = T(a_4) = 2$. From $T(a_4) = 2$ and $T(a_3) = 1$, we get $T(b_4) = T(a_5) = 2$. Then, from $T(a_2) = T(b_2) = T(b_{n-1}) = 2$, we have $T(b_5) = 1$. So, we immidiately conclude that $T(a_6) = 2$. Now, from $T(b_0) = T(b_3) = 2$ and $T(a_3) = 1$, we have $T(b_6) = 2$. It gives $T(a_7) = 1$ and then $T(b_7) = 2$ which is a contradiction of $T(b_0) = T(b_4) = T(a_4) = 2$. □

Perfect 2-colorings of $GP(n,3)$ with the matrix A_5

Here, we show that just the graphs $GP(5m,3)$, where $m \in \mathbb{N}$, among the graphs $GP(n,3)$ have a perfect 2-coloring with the matrix A_5.

Theorem 3.4. *The graphs $GP(5m,5t+2)$ and $GP(5m,5t+3)$, where $t \geq 0$, have a perfect 2-coloring with the matrix A_5. $GP(n,k)$ graphs for n such that $5 \nmid n$, have no perfect colorings with the matrix A_5.*

Proof. For the first part, consider the mapping $T : V(GP(5m,5t+2)) \to \{1,2\}$ by

$$T(a_{5i}) = T(a_{5i+2}) = T(a_{5i+3}) = T(b_{5i}) = T(b_{5i+1}) = T(b_{5i+4}) = 2,$$
$$T(a_{5i+1}) = T(a_{5i+4}) = T(b_{5i+2}) = T(b_{5i+3}) = 1,$$

for $i \geq 0$. It can be easily checked that the given mapping gives a perfect 2-coloring with the matrix A_5. The mapping $T : V(GP(5m,5t+3)) \to \{1,2\}$ by the exactly above definition is also a perfect 2-coloring with the matrix A_5. Moreover, the second part can be proved by Proposition 3.1. □

Remark 3.1. There is no information for the cases $GP(5m, 5t)$, $GP(5m, 5t + 1)$ and $GP(5m, 5t + 4)$ in Theorem 3.4. So, we leave these cases as an open problem.

Finally, we summerize the obtained results from enumerating the parameter matrices of $GP(n, k)$ in the following table.

Table 1. Eumerating the parameter matrices of $GP(n, k)$

	$GP(n, 2)$	$GP(n, 3)$	$GP(n, k)$
A_1	all graphs	all graphs	all graphs
A_2	just $GP(3m, 2)$	no graphs	?
A_3	no graphs	just $GP(2m, 3)$	?
A_4	no graphs	just $GP(4m, 3)$	?
A_5	just $GP(5m, 2)$	just $GP(5m, 3)$	?
A_6	no graphs	just $GP(2m, 3)$	just $GP(2m, 2t + 1)$

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