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Non-inclusive and inclusive distance irregularity strength for the join product of graphs

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Abstract

A function $\phi : V(G) \to \{1, 2, \dots, k\}$ of a simple graph G is said to be a non-inclusive distance vertex irregular k-labeling of G if the sums of labels of vertices in the open neighborhood of every vertex are distinct and is said to be an inclusive distance vertex irregular k-labeling of G if the sums of labels of vertices in the closed neighborhood of each vertex are different. The minimum k for which G has a non-inclusive (resp. an inclusive) distance vertex irregular k-labeling is called a non-inclusive (resp. an inclusive) distance irregularity strength and is denoted by $\mathrm{dis}(G)$ (resp. by $\widehat{\mathrm{dis}}(G)$). In this paper, the non-inclusive and inclusive distance irregularity strength for the join product graphs are investigated.

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1. Introduction

All graphs considered here are assumed to be simple, finite and undirected. Let G be a graph with vertex-set $V(G) = V$ and edge-set $E(G) = E$. For a vertex $v \in V$, the *degree* of v, denoted by $\deg_G(v)$, is the number of vertices adjacent to v. The *open* and *closed neighborhood* of v is defined as $N_G(v) = \{u : uv \in E\}$ and $N_G[v] = \{v\} \cup N_G(v)$, respectively. The *maximum degree* of vertices in G is denoted by $\Delta(G)$. By *graph labeling* we mean any mapping that carries some sets of graph elements to a set of non-negative integers, called *labels*. There are many types of graph labelings that have been developed. A survey of recent results on graph labelings is provided by Gallian [8].

Let k be a positive integer and let a graph G be given. A function $\phi : V \to \{1, 2, \ldots, k\}$ is said to be a *non-inclusive distance vertex irregular k-labeling* of G if the weights are distinct for every pair of two distinct vertices, where the weight of a vertex v is defined as the sum of labels of vertices in the open neighborhood of v in G. The *non-inclusive distance irregularity strength* of G, denoted by $\mathrm{dis}(G)$, is the minimum integer k for which G has a non-inclusive distance vertex irregular k-labeling. Furthermore, the labeling ϕ is called an *inclusive distance vertex irregular k-labeling* of G if for each two vertices u and v, there is $wt_\phi(u) = \sum_{x \in N_G[u]} \phi(x) \neq \sum_{y \in N_G[v]} \phi(y) = wt_\phi(v)$. The least integer k for which G has an inclusive distance vertex irregular k-labeling is called the *inclusive distance irregularity strength*, $\widehat{\mathrm{dis}}(G)$. We will say that $\mathrm{dis}(G) = \infty$ and $\widehat{\mathrm{dis}}(G) = \infty$ whenever such a non-inclusive and an inclusive distance vertex irregular labeling does not exist, respectively.

The notion of non-inclusive distance vertex irregular labelings was intoduced in 2017 by Slamin [13]. Meanwhile, Bača *et al.* [3] developed inclusive distance vertex irregular labelings one year later as a variation of the non-inclusive irregularity strength of graphs. These graph invariants are then generalized by Bong *et al.* [5] to non-inclusive and inclusive d-distance irregularity strength of graphs where d is an integer arbitrarily taken from 1 up to diameter of the graph. Thus, a non-inclusive 1-distance vertex irregular labeling is called a non-inclusive distance vertex irregular labeling. Similarly, we call an inclusive 1-distance vertex irregular labeling as an inclusive distance vertex irregular labeling.

A number of research results on non-inclusive and inclusive d-distance irregularity strengths have been found as seen in [3, 4, 11, 13, 14, 15, 16, 17] when $d = 1$ and in [5, 18] when $d > 1$. In the literature, it was investigated the total version of this concept, see [19, 20]. Furthermore, related topics on the subjects can also be found in, for example, [1, 6, 9], and for some new results, see [2, 10, 12].

The following lemmas give the necessary and sufficient condition for a graph G to have finite $\mathrm{dis}(G)$ and $\widehat{\mathrm{dis}}(G)$.

Lemma 1.1. *[7] Let G be a graph. Then $\mathrm{dis}(G) < \infty$ if and only if $N_G(u) \neq N_G(v)$ for every two vertices $u, v \in V$.*

Lemma 1.2. *[3] Let G be a graph. Then $\widehat{\mathrm{dis}}(G) < \infty$ if and only if $N_G[u] \neq N_G[v]$ for every two vertices $u, v \in V$.*

In the present paper, we deal with a so-called product of graphs namely a join product. The *join product* of two graphs G and H, denoted by $G \oplus H$, is a graph obtained from G and H by joining an edge from each vertex of G to each vertex of H. We represent the vertex-set of $G \oplus H$ with $V(G \oplus H) = V(G) \cup V(H)$ and the edge-set with $E(G \oplus H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. We here consider the following problems.

Problem 1. *Given two graphs G and H with $\mathrm{dis}(G)$ and $\mathrm{dis}(H)$, respectively, what is the value of $\mathrm{dis}(G \oplus H)$ going to be?*

Problem 2. *Similarly, if two graphs G and H with $\widehat{\mathrm{dis}}(G)$ and $\widehat{\mathrm{dis}}(H)$, respectively, are given, what is the value of $\widehat{\mathrm{dis}}(G \oplus H)$ going to be?*

Using Lemma 1.1, it is easy to show that $\mathrm{dis}(G \oplus H) = \infty$ if and only if either $\mathrm{dis}(G)$ or $\mathrm{dis}(H)$ is infinite. Also, it is not hard to show, by Lemma 1.2, that $\widehat{\mathrm{dis}}(G \oplus H) = \infty$ if and only if one of the following statements holds:

(i) either $\widehat{\mathrm{dis}}(G)$ or $\widehat{\mathrm{dis}}(H)$ is infinite; or
(ii) both $\Delta(G) = |V(G)| - 1$ and $\Delta(H) = |V(H)| - 1$.

Thus, in the rest of the paper, we will only deal with the case when $\mathrm{dis}(G \oplus H) < \infty$ and $\widehat{\mathrm{dis}}(G \oplus H) < \infty$.

We need to define some notations related to the non-inclusive distance irregularity strength of graphs as follows. Let G and H be graphs with $\mathrm{dis}(G) < \infty$ and $\mathrm{dis}(H) < \infty$. Let ϕ_G and ϕ_H be a non-inclusive distance vertex irregular $\mathrm{dis}(G)$-labeling of G and a non-inclusive distance vertex irregular $\mathrm{dis}(H)$-labeling of H, respectively. For a vertex $v \in V(G)$ and a non-negative integer α, we define an α-*weight* of v under a labeling ϕ_G of a graph G as

$$wt^{\alpha}_{\phi_G}(v) = wt_{\phi_G}(v) + \alpha \deg_G(v).$$

We denote by $v^{\alpha}_{\max}$ a vertex of G in such away that $wt^{\alpha}_{\phi_G}(v^{\alpha}_{\max}) = \max\left\{wt^{\alpha}_{\phi_G}(v) : v \in V(G)\right\}$. Analogously, we write $v^{\alpha}_{\min}$ to mean a vertex of G for which $wt^{\alpha}_{\phi_G}(v^{\alpha}_{\min}) = \min\left\{wt^{\alpha}_{\phi_G}(v) : v \in V(G)\right\}$. For a special $\alpha = 0$, we will use $wt_{\phi_G}(v)$, $wt_{\phi_G}(v_{\max})$ and $wt_{\phi_G}(v_{\min})$ instead of $wt^{0}_{\phi_G}(v)$, $wt^{0}_{\phi_G}(v^{0}_{\max})$ and $wt^{0}_{\phi_G}(v^{0}_{\min})$, respectively. Further, we also consider positive integers β_G and $\gamma_{G,H}$ such that

$$\beta_G = \max\left\{1, \max\left\{\left\lfloor \frac{wt_{\phi_G}(u_i) - wt_{\phi_G}(u_j)}{\deg_G(u_j) - \deg_G(u_i)} \right\rfloor + 1 : u_i, u_j \in V(G)\right\}\right\} \tag{1}$$

and

$$\gamma_{G,H} = \max\left\{\beta_G, \left\lfloor \frac{wt_{\phi_G}(u^{\beta_G}_{\max}) - wt_{\phi_H}(v_{\min}) + \sum_{v \in V(H)} \phi_H(v) - \sum_{u \in V(G)} \phi_G(u)}{|V(G)| - \Delta(G)} \right\rfloor + 1\right\}, \tag{2}$$

respectively.

With respect to the inclusive distance irregularity strength, we shall also define some notations as follows. Given two graphs G and H with $\widehat{\mathrm{dis}}(G) < \infty$ and $\widehat{\mathrm{dis}}(H) < \infty$, let $\widehat{\phi}_G$ and $\widehat{\phi}_H$ be an inclusive distance vertex irregular $\widehat{\mathrm{dis}}(G)$-labeling of G and an inclusive distance vertex irregular $\widehat{\mathrm{dis}}(H)$-labeling of H, respectively. Let $\widehat{\alpha}$ be a non-negative integer. We define an $\widehat{\alpha}$-*weight* of a vertex v of G under a labeling ϕ_G of a graph G as

$$wt^{\widehat{\alpha}}_{\widehat{\phi}_G}(v) = wt_{\widehat{\phi}_G}(v) + (\deg_G(v) + 1)\widehat{\alpha}.$$

Then we denote by $v_{\max}^{\widehat{\alpha}}$ a vertex of G in such away that $wt_{\widehat{\phi}_G}^{\widehat{\alpha}}(v_{\max}^{\widehat{\alpha}}) = \max\left\{wt_{\widehat{\phi}_G}^{\widehat{\alpha}}(v) : v \in V(G)\right\}$. Similarly, we also write $v_{\min}^{\widehat{\alpha}}$ to stand for a vertex of G in which $wt_{\widehat{\phi}_G}^{\widehat{\alpha}}(v_{\min}^{\widehat{\alpha}}) = \min\left\{wt_{\widehat{\phi}_G}^{\widehat{\alpha}}(v) : v \in V(G)\right\}$. In particular, when $\widehat{\alpha} = 0$, we will use, respectively, $wt_{\widehat{\phi}_G}(v)$, $wt_{\widehat{\phi}_G}(v_{\max})$ and $wt_{\widehat{\phi}_G}(v_{\min})$ instead of $wt_{\widehat{\phi}_G}^{0}(v)$, $wt_{\widehat{\phi}_G}^{0}(v_{\max}^{0})$ and $wt_{\widehat{\phi}_G}^{0}(v_{\min}^{0})$. Moreover, we also define positive integers $\widehat{\beta}_G$ and $\widehat{\gamma}_{G,H}$ such that

$$\widehat{\beta}_G = \max\left\{1, \max\left\{\left\lfloor \frac{wt_{\widehat{\phi}_G}(u_i) - wt_{\widehat{\phi}_G}(u_j)}{\deg_G(u_j) - \deg_G(u_i)} \right\rfloor + 1 : u_i, u_j \in V(G)\right\}\right\} \tag{3}$$

and

$$\widehat{\gamma}_{G,H} = \max\left\{\widehat{\beta}_G, \left\lfloor \frac{wt_{\widehat{\phi}_G}(u_{\max}^{\widehat{\beta}_G}) - wt_{\widehat{\phi}_H}(v_{\min}) + \sum_{v\in V(H)} \widehat{\phi}_H(v) - \sum_{u\in V(G)} \widehat{\phi}_G(u)}{|V(G)| - (\Delta(G)+1)} \right\rfloor + 1\right\}, \tag{4}$$

respectively.

Let x and y be two given integers. Then we define

$$\frac{x}{y} = \begin{cases} \frac{x}{y}, & \text{if } y \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. $\mathbf{dis}(\boldsymbol{G \oplus H})$ and $\widehat{\mathbf{dis}}(\boldsymbol{G \oplus H})$

In this section, we give the construction of the non-inclusive and inclusive distance vertex irregular labeling for the join product graphs. Our basic idea is to construct a new non-inclusive distance vertex irregular labeling for the join product graphs $G \oplus H$ from the described non-inclusive distance vertex irregular labeling of G and H. Similar ideas are then used to construct the inclusive distance vertex irregular labeling of the join product graphs $G \oplus H$.

Our first result below provides the lower bound of the non-inclusive distance irregularity strength for the join product of two graphs in terms of $\mathrm{dis}(G)$ and $\mathrm{dis}(H)$.

Lemma 2.1. *Let G and H be graphs such that $\mathrm{dis}(G \oplus H) < \infty$. Then*

$$\mathrm{dis}(G \oplus H) \geq \max\{\mathrm{dis}(G), \mathrm{dis}(H)\}.$$

Proof. We first show that there is no non-inclusive distance vertex irregular k-labeling of a graph $G \oplus H$ such that $k < \mathrm{dis}(G)$. Suppose to the contrary that such labeling ϕ exists, that is, a labeling $\phi : V(G \oplus H) \to \{1, 2, \dots, k\}$ is a non-inclusive distance vertex irregular k-labeling of $G \oplus H$. Since each vertex of G is adjacent to all the vertices of H and since all the vertices of G have distinct weights then if we subtract from all these weights the sum of labels of all vertices of H, it gives us a restriction of the labeling ϕ on the graph G which is a non-inclusive distance vertex irregular k'-labeling of G for some $k' \leq k$. But this gives a contradiction as $k' \leq k < \mathrm{dis}(G)$.

Next we prove that there is no non-inclusive distance vertex irregular k-labeling ϕ of a graph $G \oplus H$ such that $k < \mathrm{dis}(H)$. Using similar arguments with the previous case we can obtain a restriction of the labeling ϕ on the graph H which is a non-inclusive distance vertex irregular k''-labeling of H with $k'' \leq k$, giving a contradiction as $k'' \leq k < \mathrm{dis}(H)$. □

The following lemma gives the sufficient condition for α-weights of all vertices in a graph to be different.

Lemma 2.2. *Let G be a graph with $\operatorname{dis}(G) < \infty$ and let ϕ be a non-inclusive distance vertex irregular $\operatorname{dis}(G)$-labeling of G. Let β_G be an integer defined in* (1). *Then for any integer $\alpha \geq \beta_G$ and every two distinct vertices $u, v \in V(G)$, $wt_\phi^\alpha(u) \neq wt_\phi^\alpha(v)$. Moreover, if $\deg_G(u) < \deg_G(v)$ then $wt_\phi^\alpha(u) < wt_\phi^\alpha(v)$.*

Proof. For some α' and some $u', v' \in V(G)$, $u' \neq v'$, if $wt_\phi^{\alpha'}(u') = wt_\phi(u') + \alpha' \deg_G(u') = wt_\phi(v') + \alpha' \deg_G(v') = wt_\phi^{\alpha'}(v')$ then

$$\alpha' = \frac{wt_\phi(u') - wt_\phi(v')}{\deg_G(v') - \deg_G(u')}.$$

However, on the other hand, as $\alpha' \geq \beta_G$, we have

$$\begin{aligned}\alpha' &\geq \max\left\{\left\lfloor \frac{wt_\phi(u) - wt_\phi(v)}{\deg_G(v) - \deg_G(u)} \right\rfloor + 1 : u, v \in V(G)\right\} \\ &\geq \left\lfloor \frac{wt_\phi(u') - wt_\phi(v')}{\deg_G(v') - \deg_G(u')} \right\rfloor + 1 > \frac{wt_\phi(u') - wt_\phi(v')}{\deg_G(v') - \deg_G(u')},\end{aligned}$$

which gives us a contradiction. This proves the first part of the statement.

Next we prove the second part of the statement. Here we use the similar technique as the first part. Thus we suppose to the contrary that for some α' and some $u', v' \in V(G)$, $u' \neq v'$, with $\deg_G(u') < \deg_G(v')$, there is $wt_\phi^{\alpha'}(u') = wt_\phi(u') + \alpha' \deg_G(u') > wt_\phi(v') + \alpha' \deg_G(v') = wt_\phi^{\alpha'}(v')$. Then $wt_\phi(u') > wt_\phi(v')$ and

$$\alpha' < \frac{wt_\phi(u') - wt_\phi(v')}{\deg_G(v') - \deg_G(u')}.$$

However, on the other hand, as $\alpha' \geq \beta_G$, we obtain

$$\alpha' \geq \left\lfloor \frac{wt_{\phi_G}(u') - wt_{\phi_G}(v')}{\deg_G(v') - \deg_G(u')} \right\rfloor + 1 > \frac{wt_{\phi_G}(u') - wt_{\phi_G}(v')}{\deg_G(v') - \deg_G(u')},$$

again a contradiction. □

Notice that the property in Lemma 2.2 implies that for any integer $\alpha \geq \beta_G$, $wt_\phi(v_{\max}^\alpha) = wt_\phi(v_{\max}^{\beta_G})$. Next, as $\gamma_{G,H} \geq \beta_G$, the following property is satisfied according to Lemma 2.2.

Corollary 2.1. *Let G and H be graphs such that $\operatorname{dis}(G \oplus H) < \infty$, and let ϕ_G and ϕ_H be a non-inclusive distance vertex irregular $\operatorname{dis}(G)$-labeling of G and a non-inclusive distance vertex irregular $\operatorname{dis}(H)$-labeling of H, respectively. Let β_G and $\gamma_{G,H}$ be integers defined in* (1) *and* (2)*, respectively. Then for any two distinct vertices $u, v \in V(G)$, $wt_{\phi_G}^{\gamma_{G,H}}(u) \neq wt_{\phi_G}^{\gamma_{G,H}}(v)$.*

The value of the non-inclusive distance irregularity strength for $G \oplus H$ is given in the following theorem.

Theorem 2.1. *Let G and H be graphs such that $\operatorname{dis}(G \oplus H) < \infty$, and let ϕ_G and ϕ_H be a non-inclusive distance vertex irregular $\operatorname{dis}(G)$-labeling of G and a non-inclusive distance vertex irregular $\operatorname{dis}(H)$-labeling of H, respectively. If either*

(i) $\sum_{u\in V(G)} \phi_G(u) - \sum_{v\in V(H)} \phi_H(v) < wt_{\phi_G}(u_{\min}) - wt_{\phi_H}(v_{\max})$ *or*
(ii) $\sum_{u\in V(G)} \phi_G(u) - \sum_{v\in V(H)} \phi_H(v) > wt_{\phi_G}(u_{\max}) - wt_{\phi_H}(v_{\min})$,

then

$$\operatorname{dis}(G \oplus H) = \max\{\operatorname{dis}(G), \operatorname{dis}(H)\}.$$

Otherwise,

$$\operatorname{dis}(G \oplus H) \le \min\{\max\{\operatorname{dis}(G), \operatorname{dis}(H) + \gamma_{H,G}\}, \max\{\operatorname{dis}(H), \operatorname{dis}(G) + \gamma_{G,H}\}\}.$$

Proof. We distinguish our proof into two cases.

Case 1. $\sum_{u\in V(G)} \phi_G(u) - \sum_{v\in V(H)} \phi_H(v) < wt_{\phi_G}(u_{\min}) - wt_{\phi_H}(v_{\max})$ or $\sum_{u\in V(G)} \phi_G(u) - \sum_{v\in V(H)} \phi_H(v) > wt_{\phi_G}(u_{\max}) - wt_{\phi_H}(v_{\min})$.

Put $k = \max\{\operatorname{dis}(G), \operatorname{dis}(H)\}$. Due to Lemma 2.1 it is enough to show that there exists a non-inclusive distance vertex irregular k-labeling of $G \oplus H$. Let φ be a labeling on the vertices of $G \oplus H$ defined as follows.

$$\begin{aligned} \varphi(v) &= \phi_G(v) && \text{if } v \in V(G),\\ \varphi(v) &= \phi_H(v) && \text{if } v \in V(H). \end{aligned}$$

Obviously the largest label appearing on the vertices under the labeling φ is k and the weights of the vertices are given by

$$\begin{aligned} wt_\varphi(v) &= wt_{\phi_G}(v) + \sum_{u\in V(H)} \phi_H(u) && \text{if } v \in V(G),\\ wt_\varphi(v) &= wt_{\phi_H}(v) + \sum_{u\in V(G)} \phi_G(u) && \text{if } v \in V(H). \end{aligned}$$

We show that the vertex weights are distinct for every two vertices $u, v \in V(G \oplus H)$. If both u and v are in $V(G)$ (resp. $V(H)$) then $wt_\varphi(u) \neq wt_\varphi(v)$ as $wt_{\phi_G}(u) \neq wt_{\phi_G}(v)$ (resp. $wt_{\phi_H}(u) \neq wt_{\phi_H}(v)$).

We now suppose that $u \in V(G)$ and $v \in V(H)$. The condition (i) implies that $wt_\varphi(v_{\max}) < wt_\varphi(u_{\min})$ which means that $wt_\varphi(u) \neq wt_\varphi(v)$. Similarly, the restriction (ii) implies that $wt_\varphi(u_{\max}) < wt_\varphi(v_{\min})$ meaning that $wt_\varphi(u) \neq wt_\varphi(v)$.

Case 2. $wt_{\phi_G}(u_{\min}) - wt_{\phi_H}(v_{\max}) \le \sum_{u\in V(G)} \phi_G(u) - \sum_{v\in V(H)} \phi_H(v) \le wt_{\phi_G}(u_{\max}) - wt_{\phi_H}(v_{\min})$.

Put $k = \min\{k_1, k_2\}$ where $k_1 = \max\{\operatorname{dis}(G), \operatorname{dis}(H)+\gamma_{H,G}\}$ and $k_2 = \max\{\operatorname{dis}(H), \operatorname{dis}(G)+\gamma_{G,H}\}$. We define a vertex k_1-labeling φ_1 of $G \oplus H$ as follows.

$$\begin{aligned} \varphi_1(v) &= \phi_G(v) && \text{if } v \in V(G),\\ \varphi_1(v) &= \phi_H(v) + \gamma_{H,G} && \text{if } v \in V(H). \end{aligned}$$

Clearly the labels used on the labeling φ_1 are at most k_1. For the vertex weights we have

$$wt_{\varphi_1}(v) = wt_{\phi_G}(v) + \sum_{u\in V(H)} \phi_H(u) + |V(H)|\gamma_{H,G} \qquad \text{if } v \in V(G),$$
$$wt_{\varphi_1}(v) = wt_{\phi_H}(v) + \sum_{u\in V(G)} \phi_G(u) + \gamma_{H,G}\deg_H(v) \qquad \text{if } v \in V(H).$$

We show that for every two distinct vertices u and v of $G \oplus H$, $wt_{\varphi_1}(u) \neq wt_{\varphi_1}(v)$. If $u, v \in V(G)$, clearly, $wt_{\varphi_1}(u) \neq wt_{\varphi_1}(v)$ as $wt_{\phi_G}(u) \neq wt_{\phi_G}(v)$. Assume $u, v \in V(H)$. Applying β_H and $\gamma_{H,G}$ to Corollary 2.1, we can obtain that $wt_{\phi_H}(u)+\gamma_{H,G}\deg_H(u) \neq wt_{\phi_H}(v)+\gamma_{H,G}\deg_H(v)$ meaning that $wt_{\varphi_1}(u) \neq wt_{\varphi_1}(v)$. We now consider $u \in V(G)$ and $v \in V(H)$. It suffices for us to show that $wt_{\varphi_1}(u_{\min}) > wt_{\varphi_1}(v_{\max})$. As $\gamma_{H,G} \geq \beta_H$, by Lemma 2.2, $wt_{\phi_H}(v_{\max}^{\gamma_{H,G}}) = wt_{\phi_H}(v_{\max}^{\beta_H})$. Using these informations together with the facts that

$$\gamma_{H,G} \geq \left\lfloor \frac{wt_{\phi_H}(v_{\max}^{\beta_H}) - wt_{\phi_G}(u_{\min}) + \sum_{u\in V(G)} \phi_G(u) - \sum_{v\in V(H)} \phi_H(v)}{|V(H)| - \Delta(H)} \right\rfloor + 1$$

and $y\big(\lfloor \frac{x}{y} \rfloor + 1\big) > x$, we get

$$\begin{aligned}
wt_{\varphi_1}(u_{\min}) - wt_{\varphi_1}(v_{\max}) = & \left(wt_{\phi_G}(u_{\min}) + \sum_{v\in V(H)} \phi_H(v) + |V(H)|\gamma_{H,G} \right) \\
& - \left(wt_{\phi_H}(v_{\max}^{\gamma_{H,G}}) + \sum_{u\in V(G)} \phi_G(u) + \gamma_{H,G}\Delta(H) \right) \\
= & wt_{\phi_G}(u_{\min}) - wt_{\phi_H}(v_{\max}^{\beta_H}) + \sum_{v\in V(H)} \phi_H(v) - \sum_{u\in V(G)} \phi_G(u) + (|V(H)| - \Delta(H))\gamma_{H,G} \\
\geq & wt_{\phi_G}(u_{\min}) - wt_{\phi_H}(v_{\max}^{\beta_H}) + \sum_{v\in V(H)} \phi_H(v) - \sum_{u\in V(G)} \phi_G(u) \\
& + (|V(H)| - \Delta(H)) \left(\left\lfloor \frac{wt_{\phi_H}(v_{\max}^{\beta_H}) - wt_{\phi_G}(u_{\min}) + \sum_{u\in V(G)} \phi_G(u) - \sum_{v\in V(H)} \phi_H(v)}{|V(H)| - \Delta(H)} \right\rfloor + 1 \right) \\
> & wt_{\phi_G}(u_{\min}) - wt_{\phi_H}(v_{\max}^{\beta_H}) + \sum_{v\in V(H)} \phi_H(v) - \sum_{u\in V(G)} \phi_G(u) \\
& + \left(wt_{\phi_H}(v_{\max}^{\beta_H}) - wt_{\phi_G}(u_{\min}) + \sum_{u\in V(G)} \phi_G(u) - \sum_{v\in V(H)} \phi_H(v) \right) = 0,
\end{aligned}$$

or equivalently $wt_{\varphi_1}(u_{\min}) > wt_{\varphi_1}(v_{\max})$. Thus φ_1 is a non-inclusive distance vertex irregular k_1-labeling of $G \oplus H$ and hence $\mathrm{dis}(G \oplus H) \leq k_1$.

Analogously, we define another vertex k_2-labeling φ_2 of $G \oplus H$ as follows.

$$\varphi_2(v) = \phi_H(v) \qquad \text{if } v \in V(H),$$

$$\varphi_2(v) = \phi_G(v) + \gamma_{G,H} \qquad \text{if } v \in V(G).$$

Using similar arguments with the previous one we can obtain that φ_2 is a non-inclusive distance vertex irregular k_2-labeling of $G \oplus H$ and hence $\mathrm{dis}(G \oplus H) \leq k_2$. Taking the minimum from both k_1 and k_2, it brings us to the desired result. $\square$

The following results related to the inclusive distance irregularity strength are presented. The proofs are omitted since ideas similar with Lemmas 2.1 and 2.2, Corollary 2.1 and Theorem 2.1, respectively, are used as arguments.

Lemma 2.3. *Let G and H be graphs such that $\widehat{\mathrm{dis}}(G \oplus H) < \infty$. Then*

$$\widehat{\mathrm{dis}}(G \oplus H) \geq \max\left\{\widehat{\mathrm{dis}}(G), \widehat{\mathrm{dis}}(H)\right\}.$$

Lemma 2.4. *Let G be a graph with $\widehat{\mathrm{dis}}(G) < \infty$ and let $\widehat{\phi}$ be an inclusive distance vertex irregular $\widehat{\mathrm{dis}}(G)$-labeling of G. Let $\widehat{\beta}_G$ be an integer defined in* (3)*. Then for any integer $\widehat{\alpha} \geq \widehat{\beta}_G$ and every two distinct vertices $u, v \in V(G)$, $wt_{\widehat{\phi}}^{\widehat{\alpha}}(u) \neq wt_{\widehat{\phi}}^{\widehat{\alpha}}(v)$. Moreover, if $\deg_G(u) < \deg_G(v)$ then $wt_{\widehat{\phi}}^{\widehat{\alpha}}(u) < wt_{\widehat{\phi}}^{\widehat{\alpha}}(v)$.*

Corollary 2.2. *Let G and H be graphs such that $\widehat{\mathrm{dis}}(G \oplus H) < \infty$, and let $\widehat{\phi}_G$ and $\widehat{\phi}_H$ be an inclusive distance vertex irregular $\widehat{\mathrm{dis}}(G)$-labeling of G and an inclusive distance vertex irregular $\widehat{\mathrm{dis}}(H)$-labeling of H, respectively. Let $\widehat{\beta}_G$ and $\widehat{\gamma}_{G,H}$ be integers defined in* (3) *and* (4)*, respectively. Then for any two distinct vertices $u, v \in V(G)$, $wt_{\widehat{\phi}_G}^{\widehat{\gamma}_{G,H}}(u) \neq wt_{\widehat{\phi}_G}^{\widehat{\gamma}_{G,H}}(v)$.*

Theorem 2.2. *Let G and H be graphs such that $\widehat{\mathrm{dis}}(G \oplus H) < \infty$, and let $\widehat{\phi}_G$ and $\widehat{\phi}_H$ be an inclusive distance vertex irregular $\widehat{\mathrm{dis}}(G)$-labeling of G and an inclusive distance vertex irregular $\widehat{\mathrm{dis}}(H)$-labeling of H, respectively. If either*

(i) $\sum_{u \in V(G)} \widehat{\phi}_G(u) - \sum_{v \in V(H)} \widehat{\phi}_H(v) < wt_{\widehat{\phi}_G}(u_{\min}) - wt_{\widehat{\phi}_H}(v_{\max})$ *or*
(ii) $\sum_{u \in V(G)} \widehat{\phi}_G(u) - \sum_{v \in V(H)} \widehat{\phi}_H(v) > wt_{\widehat{\phi}_G}(u_{\max}) - wt_{\widehat{\phi}_H}(v_{\min})$,

then

$$\widehat{\mathrm{dis}}(G \oplus H) = \max\left\{\widehat{\mathrm{dis}}(G), \widehat{\mathrm{dis}}(H)\right\}.$$

Otherwise,

$$\widehat{\mathrm{dis}}(G \oplus H) \leq \min\left\{\max\left\{\widehat{\mathrm{dis}}(G), \widehat{\mathrm{dis}}(H) + \widehat{\gamma}_{H,G}\right\}, \max\left\{\widehat{\mathrm{dis}}(H), \widehat{\mathrm{dis}}(G) + \widehat{\gamma}_{G,H}\right\}\right\}.$$

If we take $H \cong K_1$ then from Theorem 2.2 we obtain the inclusive distance irregularity strength for the graph $G \oplus K_1$ which was proved by Bača *et al.* [3].

Corollary 2.3. *[3] Let G be a graph such that $\widehat{\mathrm{dis}}(G \oplus K_1) < \infty$. Then $\widehat{\mathrm{dis}}(G \oplus K_1) = \widehat{\mathrm{dis}}(G)$.*

3. $\mathrm{dis}(G \oplus K_1)$

In [4], Bong *et al.* showed that the non-inclusive distance irregularity strength of $G \oplus K_1$ and G is equal as stated in the following theorem.

Theorem 3.1. *[4] Let G be a connected graph with $\mathrm{dis}(G) < \infty$. Then $\mathrm{dis}(G \oplus K_1) = \mathrm{dis}(G)$.*

However, the above assertion is not true as we can easily see a counter example namely the complete graph $K_n \cong K_{n-1} \oplus K_1$ of Slamin [13] which showed that $\mathrm{dis}(K_n) = \mathrm{dis}(K_{n-1} \oplus K_1) = n \neq n - 1 = \mathrm{dis}(K_{n-1})$.

In this section, we provide a correction for Theorem 3.1. We prove that $\mathrm{dis}(G \oplus K_1)$ can be either $\mathrm{dis}(G)$ or $\mathrm{dis}(G) + 1$. We will need the following lemma in order to prove our theorem.

Lemma 3.1. *Let G be a graph with $\mathrm{dis}(G) < \infty$. If $\sum_{u \in V(G)} \phi_G(u) = wt_{\phi_G}(u_{\max}) + 1$ for every non-inclusive distance vertex irregular $\mathrm{dis}(G)$-labeling ϕ_G of G then $\Delta(G) = |V(G)| - 1$. Moreover, if G is not a complete graph then $G \cong G^* \oplus K_m$ for some graph G^* with $\Delta(G^*) < |V(G^*)| - 1$ and $m = \mathrm{dis}(G)$.*

Proof. Let $\sum_{u \in V(G)} \phi_G(u) = wt_{\phi_G}(u_{\max}) + 1$ for each non-inclusive distance vertex irregular $\mathrm{dis}(G)$-labeling ϕ_G of G. On contrary, assume that $\Delta(G) < |V(G)| - 1$. Then $wt_{\phi_G}(u_{\max}) < \sum_{u \in V(G)} \phi_G(u) - 1$ or $\sum_{u \in V(G)} \phi_G(u) > wt_{\phi_G}(u_{\max}) + 1$, a contradiction. Thus $\Delta(G) = |V(G)| - 1$. Let $G \ncong K_n$. Then we may write $G \cong G^* \oplus K_m$ for some graph G^* with $\Delta(G^*) < |V(G^*)| - 1$ and some positive integer m. For each $x, y \in V(G) \backslash V(G^*)$, $\phi_G(x) \neq \phi_G(y)$. Clearly $u_{\max} \in V(G) \backslash V(G^*)$ and $\phi_G(u_{\max}) = 1$.

Next we show that $m = \mathrm{dis}(G)$. By Lemma 2.1, $m \leq \mathrm{dis}(G)$. Now assume that $m < \mathrm{dis}(G)$. Then a labeling ϕ'_G on the vertices of G defined as

$$\begin{aligned} \phi'_G(u) &= \phi_G(u) && \text{if } u \in V(G) \backslash \{u_{\max}\}, \\ \phi'_G(u) &= p && \text{if } u = u_{\max}, \end{aligned}$$

where $p \in \{1, 2, \ldots, \mathrm{dis}(G)\} \backslash \{\phi_G(u) : u \in V(G) \backslash V(G^*)\}$, is a non-inclusive distance vertex irregular $\mathrm{dis}(G)$-labeling of G. Next let $u'_{\max} \in V(G)$ (possibly $u'_{\max} = u_{\max}$) such that $wt_{\phi'_G}(u'_{\max}) = \max\{wt_{\phi'_G}(u) : u \in V(G)\}$. In fact, we have $\phi'_G(u'_{\max}) > 1$ and

$$\sum_{u \in V(G)} \phi'_G(u) = wt_{\phi'_G}(u'_{\max}) + \phi'_G(u'_{\max}) > wt_{\phi_G}(u'_{\max}) + 1,$$

yielding a contradiction. Hence $m = \mathrm{dis}(G)$. □

Now we are ready to prove the main result of this section. Note that for each graph G with $\mathrm{dis}(G) < \infty$ and non-inclusive distance vertex irregular labeling ϕ_G, it holds that

$$\sum_{u \in V(G)} \phi_G(u) \geq wt_{\phi_G}(u_{\max}) + 1. \tag{5}$$

Theorem 3.2. *Let G be a graph with $\operatorname{dis}(G) < \infty$. If there exists a non-inclusive distance vertex irregular $\operatorname{dis}(G)$-labeling ϕ_G of G such that $\sum_{u \in V(G)} \phi_G(u) > wt_{\phi_G}(u_{\max})+1$ then $\operatorname{dis}(G \oplus K_1) = \operatorname{dis}(G)$. Otherwise $\operatorname{dis}(G \oplus K_1) = \operatorname{dis}(G) + 1$.*

Proof. The first case follows from Theorem 2.1. Now we consider the second case, i.e., for every non-inclusive distance vertex irregular $\operatorname{dis}(G)$-labeling ϕ_G of G there is $\sum_{u \in V(G)} \phi_G(u) \leq wt_{\phi_G}(u_{\max})+1$. Combining this inequality with (5), we have that $\sum_{u \in V(G)} \phi_G(u) = wt_{\phi_G}(u_{\max})+1$ for each non-inclusive distance vertex irregular $\operatorname{dis}(G)$-labeling ϕ_G of G.

Evidently $\operatorname{dis}(G \oplus K_1) = \operatorname{dis}(G) + 1$ if $G \cong K_n$. Suppose that $G \ncong K_n$. From Lemma 3.1, $\Delta(G) = |V(G)| - 1$ and $G \cong G^* \oplus K_m$ for some graph G^* with $\Delta(G^*) < |V(G^*)| - 1$ and $m = \operatorname{dis}(G)$.

Now let $H \cong G \oplus K_1 \cong G^* \oplus K_{m+1}$. By Lemma 2.1, $\operatorname{dis}(H) \geq m + 1 = \operatorname{dis}(G) + 1$. On the other hand, the labeling φ defined below is a non-inclusive distance vertex irregular $(\operatorname{dis}(G) + 1)$-labeling of H,

$$
\begin{aligned}
\varphi(u) &= \phi_G(u) + \operatorname{dis}(G) + 1 - q && \text{if } u \in V(G^*),\\
\varphi(u) &= \phi_G(u) && \text{if } u \in V(K_m),\\
\varphi(u) &= \operatorname{dis}(G) + 1 && \text{if } u \in V(K_1),
\end{aligned}
$$

where $q = \max\{\phi_G(u) : u \in V(G^*)\}$. □

4. Inclusive distance irregularity strength of complete multipartite graphs

In this part, we deal with the inclusive distance vertex irregular labeling of complete multipartite graphs. Let us denote the complete multipartite graphs with $\sum_{i=1}^{r} p_i$ partite sets, $r \geq 2$, $p_i \geq 1$, by $G \cong K_{\underbrace{n_1, n_1, \ldots, n_1}_{p_1 \text{ times}}, \underbrace{n_2, n_2, \ldots, n_2}_{p_2 \text{ times}}, \ldots, \underbrace{n_r, n_r, \ldots, n_r}_{p_r \text{ times}}}$ where $1 \leq n_1 < n_2 < \cdots < n_r$.

We begin with the following observation which is easy to prove.

Observation 4.1. *Let $n \geq 1$. Then $\widehat{\operatorname{dis}}(nK_1) = n$.*

The next lemma presents the upper bound for the inclusive distance irregularity strength of complete multipartite graphs with same size of partite sets.

Lemma 4.1. *Let $G \cong K_{\underbrace{n, n, \ldots, n}_{p \text{ times}}}$ where $n, p \geq 2$. Then $\widehat{\operatorname{dis}}(G) \leq n + 2(p - 1)$.*

Proof. By labeling n vertices in the i-th partite of G with $2(i-1)+1, 2(i-1)+2, \ldots, 2(i-1)+n$, it is not difficult to see that the vertex weights are all distinct. □

Complete multipartite graphs with infinite inclusive distance irregularity strength are given in the following result.

Observation 4.2. *Let $G \cong K_{\underbrace{n_1, n_1, \ldots, n_1}_{p_1 \text{ times}}, \underbrace{n_2, n_2, \ldots, n_2}_{p_2 \text{ times}}, \ldots, \underbrace{n_r, n_r, \ldots, n_r}_{p_r \text{ times}}}$ where $r \geq 2$, $p_1, p_2, \ldots, p_r \geq 1$ and $1 \leq n_1 < n_2 < \cdots < n_r$. If $n_1 = 1$ and $p_1 \geq 2$ then $\widehat{\operatorname{dis}}(G) = \infty$.*

In the following, an algorithm for determining the upper bound for the inclusive distance irregularity strength of complete multipartite graphs for other cases is provided. Note that $K_{\underbrace{n_1, n_1, \ldots, n_1}_{p_1 \text{ times}}, \underbrace{n_2, n_2, \ldots, n_2}_{p_2 \text{ times}}, \ldots, \underbrace{n_r, n_r, \ldots, n_r}_{p_r \text{ times}}} \cong K_{\underbrace{n_1, n_1, \ldots, n_1}_{p_1 \text{ times}}} \oplus K_{\underbrace{n_2, n_2, \ldots, n_2}_{p_2 \text{ times}}} \oplus \ldots \oplus K_{\underbrace{n_r, n_r, \ldots, n_r}_{p_r \text{ times}}}$.

Algorithm 1 Calculating an upper bound for the inclusive distance irregularity strength of complete multipartite graphs

---

Input: $r, p_1, p_2, \ldots, p_r, n_1, n_2, \ldots, n_r$: positive integers where $r \geq 2$, $p_1, p_2, \ldots, p_r \geq 1$ and $1 \leq n_1 < n_2 < \cdots < n_r$, $(n_1, p_1) \neq (1, s)$, $s \geq 2$;

Output: k, i.e. an upper bound for $\widehat{\text{dis}}(K_{\underbrace{n_1, n_1, \ldots, n_1}_{p_1 \text{ times}}, \underbrace{n_2, n_2, \ldots, n_2}_{p_2 \text{ times}}, \ldots, \underbrace{n_r, n_r, \ldots, n_r}_{p_r \text{ times}}})$;

$G \leftarrow K_{\underbrace{n_1, n_1, \ldots, n_1}_{p_1 \text{ times}}}$;

if $p_1 = 1$ **then**

  $G \leftarrow n_1K_1$;

  Construct an inclusive distance vertex irregular $\widehat{\text{dis}}(G)$-labeling of G by using Observation 4.1;

else

  Construct an inclusive distance vertex irregular $\widehat{\text{dis}}(G)$-labeling of G by using Lemma 4.1;

end if

for $i \leftarrow 2$ **to** r **do**

  $H \leftarrow K_{\underbrace{n_i, n_i, \ldots, n_i}_{p_i \text{ times}}}$;

  if $p_i = 1$ **then**

    $H \leftarrow n_iK_1$;

    Construct an inclusive distance vertex irregular $\widehat{\text{dis}}(H)$-labeling of H by using Observation 4.1;

  else

    Construct an inclusive distance vertex irregular $\widehat{\text{dis}}(H)$-labeling of H by using Lemma 4.1;

  end if

  Construct an inclusive distance vertex irregular $\widehat{\text{dis}}(G \oplus H)$-labeling of $G \oplus H$ by using Theorem 2.2;

  $G \leftarrow G \oplus H$;

  $\widehat{\text{dis}}(G) \leftarrow \widehat{\text{dis}}(G \oplus H)$;

end for

$k \leftarrow \widehat{\text{dis}}(G \oplus H)$;

return k;

From Algorithm 1 we immediately get the following.

Theorem 4.1. *Let* $G \cong K_{\underbrace{n_1, n_1, \ldots, n_1}_{p_1 \text{ times}}, \underbrace{n_2, n_2, \ldots, n_2}_{p_2 \text{ times}}, \ldots, \underbrace{n_r, n_r, \ldots, n_r}_{p_r \text{ times}}}$ *where* $r \geq 2$, $p_1, p_2, \ldots,$

$p_r \geq 1$ *and* $1 \leq n_1 < n_2 < \cdots < n_r$, $(n_1, p_1) \neq (1, s)$, $s \geq 2$. *Then* $\widehat{\mathrm{dis}}(G) \leq k$ *where* k *is an integer which is the output of Algorithm 1.*

Observe that in our construction of the inclusive distance vertex irregular labeling for the complete multipartite graphs in Algorithm 1, vertices with smaller degree receive smaller weights. From this observation, we then conjecture that the upper bound in Theorem 4.1 is tight.

Conjecture 1. *Let* $G \cong K_{\underbrace{n_1, n_1, \ldots, n_1}_{p_1 \text{ times}}, \underbrace{n_2, n_2, \ldots, n_2}_{p_2 \text{ times}}, \ldots, \underbrace{n_r, n_r, \ldots, n_r}_{p_r \text{ times}}}$ *where* $r \geq 2$, $p_1, p_2, \ldots,$ $p_r \geq 1$ *and* $1 \leq n_1 < n_2 < \cdots < n_r$, $(n_1, p_1) \neq (1, s)$, $s \geq 2$. *Then* $\widehat{\mathrm{dis}}(G) = k$ *where* k *is an integer which is the output of Algorithm 1.*

The following result supports Conjecture 1.

Corollary 4.1. *Let* $r \geq 2$ *and* $1 \leq n_1 < n_2 < \cdots < n_r$. *Then* $\widehat{\mathrm{dis}}(K_{n_1, n_2, \ldots, n_r}) = n_r$.

Proof. The upper bound follows from Theorem 4.1 and the lower bound is obtained from Lemma 2.3. □

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