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The edge-distinguishing chromatic number of petal graphs, chorded cycles, and spider graphs

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Abstract

The edge-distinguishing chromatic number (EDCN) of a graph G is the minimum positive integer k such that there exists a vertex coloring $c : V(G) \to \{1, 2, \ldots, k\}$ whose induced edge labels $\{c(u), c(v)\}$ are distinct for all edges uv. Previous work has determined the EDCN of paths, cycles, and spider graphs with three legs. In this paper, we determine the EDCN of petal graphs with two petals and a loop, cycles with one chord, and spider graphs with four legs. These are achieved by graph embedding into looped complete graphs.

Keywords: vertex coloring, edge-distinguishing, graph embedding, EDCN, caterpillars, petal graphs, chorded cycles, spiders
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1. Introduction

Let the graph G be composed of a simple graph together with at most one loop at each vertex, and let $V(G)$ and $E(G)$ be the vertex set and the edge set of G respectively. Let $[k] = \{1, 2, \ldots, k\}$ denote a set of k colors, and let $\binom{[k]}{i}$ denote the set of i-subsets of $[k]$. For every vertex coloring $c : V(G) \to [k]$, there is an induced edge coloring $c' : E(G) \to \binom{[k]}{1} \cup \binom{[k]}{2}$, defined by

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$c'(uv) = \{c(u), c(v)\}$ for each edge uv. A vertex coloring c is considered *edge-distinguishing* if c' is injective. The *edge-distinguishing chromatic number (EDCN)* of G, denoted by $\lambda(G)$, is the minimum integer k such that an edge-distinguishing vertex coloring with k colors exists for G.

The concept of coloring certain elements of a graph G to distinuguish another set of elements associated with G has many variations. For instance, the problem of vertex-distinguishing edge coloring has been studied in the literature [2, 4, 5], and the problem of distinguishing graph automorphisms through vertex coloring has also been studied [8, 15].

This notion of edge-distinguishing vertex coloring, also called a *line-distinguishing vertex coloring*, was first introduced by Frank et al. [12]. This problem was further studied by Al-Wahabi et al. [3], Zagaglia Salvi [16], Fickes and Wong [11], etc. Brunton et al. [7] considered the case when the edge coloring function c' is also surjective. It is worth mentioning that the more popular notion of harmonious chromatic number typically refers to a slight variation of the EDCN [14]. More specifically, if G is a simple graph, then the *harmonious chromatic number* of G, denoted by $h(G)$, is the minimum integer k such that an edge-distinguishing proper vertex coloring with k colors exists for G. It is obvious that $\lambda(G) \leq h(G)$.

Determining $\lambda(G)$ for a general graph G is NP-complete [13]. Consequently, most work in the literature focuses on providing bounds on $h(G)$ and thus $\lambda(G)$ [1] or studying the asymptotic behavior of $h(G)$ [6] for various families of graphs G. When trying to determine the exact formula for $\lambda(G)$ or $h(G)$, only very limited families have been tackled. For example, $\lambda(G)$ is determined for paths and cycles [3, 12], and $h(G)$ is determined for complete r-ary trees [10].

We are interested in determining the exact formula for the EDCN of other families of graphs; however, pushing the results from paths and cycles to other graphs seems formidable, and it has not been done since the 1980's until very recently. A path is a tree with maximum degree 2, so a natural extension of a path is a *spider graph*, which has a unique vertex with degree at least 3, often called the *central vertex*. Each path between the central vertex and a leaf (i.e., a degree 1 vertex) is called a *leg*. We denote a spider graph as $S_{\ell_1,\ell_2,\ldots,\ell_\Delta}$, where $\Delta \geq 3$ is the number of legs, and $1 \leq \ell_1 \leq \ell_2 \leq \ldots \leq \ell_\Delta$ represent the number of edges in each leg. Here are some published results on the EDCNs of paths, cycles, and spider graphs in the literature.

Theorem 1.1 ([3, 12])**.** *Let $G = P_n$ be a path with n vertices. Then*

$$\lambda(P_n) = \begin{cases} 1 & \text{if } n \leq 2; \text{ and} \\ \min\left\{2\left\lceil\sqrt{\frac{n-2}{2}}\right\rceil, 2\left\lceil\frac{1+\sqrt{8n-7}}{4}\right\rceil - 1\right\} & \text{if } n \geq 3. \end{cases}$$

Theorem 1.2 ([3, 12])**.** *Let $G = C_n$ be a cycle with n vertices, where $n \geq 3$. Then*

$$\lambda(C_n) = \min\left\{2\left\lceil\sqrt{\frac{n}{2}}\right\rceil, 2\left\lceil\frac{1+\sqrt{8n+1}}{4}\right\rceil - 1\right\}.$$

Theorem 1.3 ([11])**.** *Let S_{ℓ_1,ℓ_2,ℓ_3} be a spider graph with 3 legs, where $\ell_1 + \ell_2 + \ell_3 = L$. Then*

$$\lambda(S_{\ell_1,\ell_2,\ell_3}) = \begin{cases} 3 & \text{if } 3 \leq L \leq 6 \text{ and } \ell_1 = 1; \\ 4 & \text{if } L = 6 \text{ and } \ell_1 = 2; \\ \left\lceil\frac{-1+\sqrt{1+8L}}{2}\right\rceil & \text{if } L \geq 7 \text{ and } \left\lceil\frac{-1+\sqrt{1+8L}}{2}\right\rceil \text{ is odd; and} \\ \left\lceil\sqrt{2L-4}\right\rceil & \text{otherwise.} \end{cases}$$

Theorem 1.4 ([11]). *Let $S_{\ell_1,\ell_2,\ldots,\ell_\Delta}$ be a spider graph with Δ legs, where $2 \le \ell_i \le \frac{\Delta+3}{2}$ for each $1 \le i \le \Delta$. Then*

$$\lambda(S_{\ell_1,\ell_2,\ldots,\ell_\Delta}) = \Delta + 1.$$

Although the edge-distinguishing chromatic number was introduced almost 40 years ago, the above list seemingly exhausts all major progress on the EDCN of various families of graphs. In this paper, we extend this list by studying spider graphs with four legs. The technique used to obtain Theorem 1.3 was too cumbersome to apply, so we adopt a different approach here. We try to "fold up" or embed a spider graph with four legs into petal graphs and cycles with one chord to aid our discussions. Through this method, we determine the EDCN of three families of graphs: some petal graphs, all cycles with one chord, and all spider graphs with four legs.

To determine the EDCN of various graphs, let us first introduce a simple tool to establish lower bounds.

Proposition 1.1 ([11]). *Let G be a simple graph with maximum degree $\Delta(G)$. Then $\lambda(G) \ge \Delta(G)$. Furthermore, if there exists a vertex u of G such that $\deg(u) = \Delta(G)$ and every neighbor of u has degree > 1, then $\lambda(G) \ge \Delta(G) + 1$.*

Next, we present the main tool to determine the EDCN of a graph, namely Theorem 1.5. Let K_k denote the complete graph on k vertices $\{v_0, v_1, v_2, \ldots, v_{k-1}\}$, and let v_iv_j denote the edge between vertices v_i and v_j. If we attach a loop v_iv_i at every vertex v_i of K_k, then we obtain a new graph, which is denoted by K_k^*. In other words,

$$K_k^* = K_k \cup \{v_0v_0, v_1v_1, v_2v_2, \ldots, v_{k-1}v_{k-1}\}.$$

A *graph homomorphism* is a function from the vertex set of one graph to the vertex set of another that preserves edges. An *embedding* of a graph G in K_k^* refers to a graph homomorphism from G to K_k^* that induces an injection from $E(G)$ to $E(K_k^*)$.

Theorem 1.5 ([12]). *Let G be a simple graph, and let k be a positive integer. Then $\lambda(G) \le k$ if and only if there is an embedding of G in K_k^*.*

Another trivial but useful observation is the following.

Proposition 1.2. *Let H be a subgraph of G. If G can be embedded in K_k^*, then H can also be embedded in K_k^*, and hence, $\lambda(H) \le \lambda(G)$.*

In Section 2, we begin by proving the necessary and sufficient conditions to embed a subfamily of caterpillar trees in K_k^* when k is odd. Then, we proceed to prove the necessary and sufficient conditions to embed "petal graphs" with two petals and one loop, cycles with one chord, and finally spider graphs with four legs in K_k^*. For each of these families of graphs, the treatment for odd k is different from even k, so we separate our discussion on odd and even k into Sections 2 and 3 respectively. Lastly, in Section 4, we determine the EDCN of these petal graphs, chorded cycles, and spider graphs.

2. *k* is odd

2.1. Caterpillars

In the study of graph labelings, caterpillar trees are often the first object studied after paths. Naturally, this is where our discussion begins.

Definition 1. *Let ℓ be a positive integer and m be a nonnegative integer. Let $CP_\ell^{i_1,i_2,\ldots,i_m}$ denote a* caterpillar tree *with the* central path *$y_0y_1y_2\cdots y_\ell$ and m extra edges $y_{\ell+1}y_{i_1}$, $y_{\ell+2}y_{i_2},\ldots,y_{\ell+m}y_{i_m}$, where $0<i_1\le i_2\le\cdots\le i_m<\ell$.*

If $i_1,i_2,\ldots,i_m$ are distinct and $m\le k$, then we would like to embed such caterpillar trees in K_k^* with all the extra edges $y_{\ell+1}y_{i_1},y_{\ell+2}y_{i_2},\ldots,y_{\ell+m}y_{i_m}$ embedded as loops. Hence, the main question is whether we can embed the central path $y_0y_1y_2\cdots y_\ell$ in K_k^* such that $y_{i_1},y_{i_2},\ldots,y_{i_m}$ are mapped to distinct vertices. The following theorem by Dvořák et. al. answers our question positively with only a few exceptions.

Theorem 2.1 ([9])**.** *Let $k\ge 3$ be odd, let $\mathcal{C}=z_0z_1z_2\cdots z_{\binom{k}{2}-1}z_0$ be a cycle of length $\binom{k}{2}$ such that exactly k vertices, namely $z_{i_1},z_{i_2},\ldots,z_{i_k}$, are black, where $0\le i_1,i_2,\ldots,i_k\le\binom{k}{2}-1$. Then there exists an embedding of $\mathcal{C}$ in K_k that is injective on the black vertices if and only if either $k\ne 5$ or $k=5$ and $\{i_1,i_2,i_3,i_4,i_5\}\not\equiv\{0+c,3+c,4+c,6+c,7+c\}$ or $\{0+c,1+c,3+c,7+c,9+c\}$* (mod 10) *for every $c\in\mathbb{Z}_{10}$.*

This result is paramount to the proof of the following theorem.

Theorem 2.2. *Let k be odd, and let $m\le k$. Consider caterpillar trees $CP_\ell^{i_1,i_2,\ldots,i_m}$ such that $0<i_1<i_2<\cdots<i_m<\ell$ and $CP_\ell^{i_1,i_2,\ldots,i_m}\ne CP_4^{1,3}$. Such caterpillar trees can be embedded in K_k^* if and only if $\ell+m\le\binom{k+1}{2}$.*

Proof. The number of edges in $CP_\ell^{i_1,i_2,\ldots,i_m}$ and K_k^* are $\ell+m$ and $\binom{k+1}{2}$ respectively, so $\ell+m\le\binom{k+1}{2}$ is a necessary condition for $CP_\ell^{i_1,i_2,\ldots,i_m}$ to be embedded in K_k^*.

To prove that the condition is sufficient, if $\ell'+m<\binom{k+1}{2}$, we can first consider $CP_{\ell'}^{i_1,i_2,\ldots,i_m}$ as a subgraph of $CP_\ell^{i_1,i_2,\ldots,i_m}$, where $\ell+m=\binom{k+1}{2}$. Due to Proposition 1.2, it suffices to show that $CP_\ell^{i_1,i_2,\ldots,i_m}$ can be embedded in K_k^* if $\ell+m=\binom{k+1}{2}$ and $CP_\ell^{i_1,i_2,\ldots,i_m}\ne CP_4^{1,3}$.

If $k=1$, then $\ell+m=\binom{k+1}{2}=1$, implying that $\ell=1$ and $m=0$. It is obvious that CP_1, which is a path with two vertices, can be embedded in K_1^*, which is a loop at one vertex. If $k=3$ or 5, we exhaust all possible caterpillar trees that satisfy the given conditions, and the only caterpillar tree that cannot be embedded in K_k^* is $CP_4^{1,3}$. If $k\ge 7$, we need the following claim so that we can apply Theorem 2.1 in our proof.

Claim: If $k\ge 7$ and $m<k$, then there exists a set

$$B=\{y_{j_1},y_{j_1+1},y_{j_2},y_{j_2+1},\ldots,y_{j_{k-m}},y_{j_{k-m}+1}\}$$

of $2(k-m)$ distinct vertices, where $j_1<j_2<\cdots<j_{k-m}<\ell$, such that $\{y_{i_1},y_{i_2},\ldots,y_{i_m}\}\cap B=\emptyset$.

Proof of Claim: Let

$$J = \big\{ j \in \mathbb{N} : j < \ell,\ j \text{ is odd, and } \{j, j+1\} \cap \{i_1, i_2, \ldots, i_m\} = \emptyset \big\}.$$

Note that $\lfloor \frac{\ell}{2} \rfloor \geq \frac{1}{2}(\ell - 1) = \frac{1}{2}\left(\binom{k+1}{2} - m - 1\right) \geq \frac{1}{2}\left(\binom{k+1}{2} - k\right) = \frac{1}{2}\binom{k}{2} \geq k$ when $k \geq 7$. Hence, $|J| \geq \lfloor \frac{\ell}{2} \rfloor - m \geq k - m$. As a result, our claim follows by picking distinct integers $j_1, j_2, \ldots, j_{k-m}$ from J.

Assume $k \geq 7$. Let $CP_\ell^{i_1, i_2, \ldots, i_m}$ be a caterpillar tree that satisfies the conditions given by the statement of the theorem. Let $j_0 = -1$ and $j_{k-m+1} = \ell$, and if $m < k$, then let B be the set of vertices as provided by the claim. Let

$$\phi : \{y_0, y_1, y_2, \ldots, y_{\ell+m}\} \to \{z_0 = z_{\binom{k}{2}}, z_1, z_2, \ldots, z_{\binom{k}{2}-1}\}$$

be defined such that

$$\phi(y_\alpha) = \begin{cases} z_{\alpha-\beta} & \text{if } j_\beta < \alpha \leq j_{\beta+1} \text{ for some } 0 \leq \beta \leq k-m; \text{ and} \\ \phi(y_{i_\gamma}) & \text{if } \alpha = \ell + \gamma \text{ for some } 1 \leq \gamma \leq m. \end{cases}$$

This function defines an embedding from $CP_\ell^{i_1, i_2, \ldots, i_m}$ to a graph $\mathcal{C}^*$ that consists of a cycle $z_0 z_1 z_2 \cdots z_{\binom{k}{2}-1} z_{\binom{k}{2}}$ of length $\binom{k}{2}$ together with the set of k loops

$$\{\phi(y_{i_\gamma})\phi(y_{\ell+\gamma}) : 1 \leq \gamma \leq m\} \cup \{\phi(y_{j_{\beta+1}})\phi(y_{j_{\beta+1}+1}) = z_{j_{\beta+1}-\beta} z_{j_{\beta+1}-\beta} : 0 \leq \beta \leq k-m-1\}.$$

Note that these k loops are at distinct vertices due to the constructions in our claim and our function ϕ.

Define the cycle $\mathcal{C}$ by removing all the loops from $\mathcal{C}^*$ and coloring the vertices

$$\phi(y_{i_1}), \phi(y_{i_2}), \ldots, \phi(y_{i_m}), \phi(y_{j_1}), \phi(y_{j_2}), \ldots, \phi(y_{j_{k-m}})$$

black. Note that these are all the vertices that have loops in $\mathcal{C}^*$. By Theorem 2.1, there exists an embedding of $\mathcal{C}$ in K_k that is injective on the black vertices. As a result, there is an embedding of $\mathcal{C}^*$ in K_k^* with the loops embedded as loops, which completes our proof. □

2.2. *Petal graphs*

A petal graph can be considered as a resultant graph by connecting the legs of a spider graph. It can also be considered as a resultant graph by identifying vertices of a chorded cycle. Hence, it is logical to study the embedding of petal graphs in K_k^* prior to the subsections on cycles with one chord and spider graphs.

Definition 2. *Let $m \in \mathbb{N}$ such that $m \geq 2$, and let $c_1, c_2, \ldots, c_m \in \mathbb{N}$ such that $c_1 \leq c_2 \leq \cdots \leq c_m$. Let $P_{c_1, c_2, \ldots, c_m}$ be the* petal graph *that has vertices $u_0, u_1^1, u_2^1, \ldots, u_{c_1-1}^1$, $u_1^2, u_2^2, \ldots, u_{c_2-1}^2, \ldots, u_1^m, u_2^m, \ldots, u_{c_m-1}^m$, and for each $i = 1, 2, \ldots, m$, the edges form a cycle $u_0 u_1^i u_2^i \ldots u_{c_i-1}^i u_0$ of length c_i, which is called the i-th* petal *of the petal graph. If $c_i = 1$, then the i-th petal of $P_{c_1, c_2, \ldots, c_m}$ is simply a loop.*

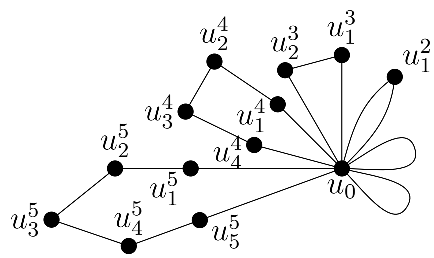


Figure 1. Petal graph $P_{1,1,2,3,5,6}$

We are going to show that petal graphs P_{1,c_2,c_3} can be embedded in K_k^* when k is odd as long as some trivial necessary conditions are satisfied.

Theorem 2.3. *If k is odd, then P_{1,c_2,c_3} can be embedded in K_k^* if and only if $c_2 \geq 3$ and $1+c_2+c_3 \leq \binom{k+1}{2}$.*

Proof. The only if direction is trivial. For the if direction, let $n = c_2 + c_3$. If $k \leq 3$, then $n \leq \binom{3+1}{2} - 1 = 5$, contradicting that $c_2 \geq 3$. Hence, it is only meaningful to consider $k \geq 5$. In all of the following cases, we embed the first petal u_0u_0 of P_{1,c_2,c_3} as the loop v_0v_0 in K_k^*.

Without loss of generality, we can assume that $n > \binom{k-1}{2} - 1$. Otherwise, if $n \leq \binom{k-1}{2} - 1$, we can consider embedding P_{1,c_2,c_3} in K_{k-2}^*, which can be further embedded in K_k^* trivially. Let $h = \binom{k+1}{2} - 1 - n$ be the gap between n and $\binom{k+1}{2} - 1$. Based on our assumption, $0 \leq h \leq \binom{k+1}{2} - 1 - \binom{k-1}{2} = 2k - 2$.

To embed P_{1,c_2,c_3} in K_k^*, after embedding the petal u_0u_0 as the loop v_0v_0 in K_k^*, it suffices to partition the edges of $K_k^* - v_0v_0$ into three subgraphs H_0, H_1, and H_2 that satisfy all the following conditions.

1. The number of edges in H_0, H_1, and H_2 are h, c_2, and c_3 respectively.
2. After removing isolated vertices, H_1 and H_2 are connected.
3. The degree of v_0 in each of H_1 and H_2 is positive.
4. All degrees in H_0, H_1, and H_2 are even.

This is because the petals $u_0u_1^2u_2^2 \cdots u_{c_2-1}^2u_0$ and $u_0u_1^3u_2^3 \cdots u_{c_3-1}^3u_0$ can then be embedded as H_1 and H_2 respectively, since H_1 and H_2 are Eulerian by conditions 2 and 4, and u_0 can be embedded as v_0 by condition 3.

Construction of H_1

Let ℓ be the unique odd integer such that $\binom{\ell-1}{2} - 1 < c_2 \leq \binom{\ell+1}{2} - 1$. Note that $\ell \geq 3$. Let $g = \binom{\ell+1}{2} - 1 - c_2$ be the gap between c_2 and $\binom{\ell+1}{2} - 1$, which satisfies $0 \leq g \leq \binom{\ell+1}{2} - 1 - \binom{\ell-1}{2} = 2\ell - 2$. Let S be the set of g edges in K_k^* defined as

$$S = \begin{cases} \{v_1v_1, v_2v_2, \ldots, v_gv_g\} & \text{if } 0 \leq g \leq \ell - 1; \text{ and} \\ \{v_0v_1v_2 \cdots v_{\ell-1}v_0, v_1v_1, v_2v_2, \ldots, v_{g-\ell}v_{g-\ell}\} & \text{if } \ell \leq g \leq 2\ell - 2. \end{cases}$$

Let $V_1 = \{v_0, v_1, v_2, \ldots, v_{\ell-1}\}$ and $V_2 = \{v_\ell, v_{\ell+1}, \ldots, v_{k-1}\}$. Let L_1 and L_2 be the induced subgraphs of K_k^* on V_1 and V_2 respectively, and let L_b be the complete bipartite graph between V_1 and V_2.

Define the subgraph H_1 as $L_1 - v_0v_0 - S$. It is clear from the construction that the following conditions hold.

1. The number of edges in H_1 is c_2.
2. After removing V_2, H_1 is connected since
   - if $\ell = 3$, then $g = \binom{\ell+1}{2} - 1 - c_2 = 5 - c_2 \leq 5 - 3 = 2$, so H_1 contains the cycle $v_0v_1v_2v_0$; and
   - if $\ell > 3$, then H_1 contains the cycle $v_0v_2v_4 \cdots v_{\ell-1}v_1v_3 \cdots v_{\ell-2}v_0$.
3. The degree of v_0 is positive since H_1 is connected after V_2 is removed.
4. All degrees in H_1 are even.

Construction of H_0

Let g' be the number of loops in S. If $h \leq g'$, then let H_0 be the subgraph that contains h loops in S. Otherwise, we have the following claim.

Claim 1: If $h > g'$, then $k > \ell$.

Proof of Claim 1: If $k = \ell$, then $\binom{k-1}{2} \leq c_2 \leq c_3$, thus $0 \leq h = \binom{k+1}{2} - 1 - n \leq \binom{k+1}{2} - 1 - 2c_2 \leq \binom{k+1}{2} - 1 - 2\binom{k-1}{2} = \frac{-(k-6)(k-1)}{2}$, which implies $k = 5$ and $h \leq 2$. Note that $\binom{5-1}{2} \leq c_2 \leq \frac{1}{2}\left(\binom{5+1}{2} - 1\right)$, so $c_2 = 6$ or 7. Hence, $g = \binom{5+1}{2} - 1 - c_2 = 8$ or 7, meaning that S has 8 or 7 edges. As a result, the number of loops in S, denoted by g', is 3 or 2. However, $h \leq 2$, contradicting the condition that $h > g'$.

Under the assumption that $h > g'$, we have $k - \ell \geq 2$. Index the vertices in $V_1 - \{v_0\}$ in pairs as v_{2s-1} and v_{2s}, where $1 \leq s \leq \frac{\ell-1}{2}$, and index the vertices in V_2 in pairs as $v_{\ell+2t-2}$ and $v_{\ell+2t-1}$, where $1 \leq t \leq \frac{k-\ell}{2}$. Since $\ell \geq 3$, the number of edges in the complete bipartite graph between $V_1 - \{v_0\}$ and V_2 is $(\ell-1)(k-\ell) \geq 2(k-3) = 2k-6$. Let $h' = h - g'$. We are going to define H_0 based on the following cases.

Case 1: $k - \ell \geq 4$. Let $\widetilde{h}$ be the smallest positive integer such that $h' \equiv \widetilde{h} \pmod 4$. Note that $h' - \widetilde{h} < h' \leq h \leq 2k-2$. Together with the fact that $h' - \widetilde{h} \equiv 2k - 2 \equiv 0 \pmod 4$, we have $h' - \widetilde{h} \leq 2k - 6$.

Define the subgraph H_0 as the disjoint union of

- $\frac{h'-\widetilde{h}}{4}$ copies of 4-cycles $v_{2s-1}v_{\ell+2t-2}v_{2s}v_{\ell+2t-1}v_{2s-1}$,
- g' loops in S, and
- $\widetilde{h}$ loops in L_2.

The number of edges in H_0 is $\frac{h'-\widetilde{h}}{4} \cdot 4 + g' + \widetilde{h} = h' + g' = h$, and it is clear that all degrees in H_0 are even.

Case 2: $k - \ell = 2$, i.e., $\ell = k - 2$. The number of edges in $K_k^* - v_0v_0 - H_0$ is at least $2c_2$, so $h' \leq h \leq \binom{k+1}{2} - 1 - 2c_2 \leq \binom{k+1}{2} - 1 - 2\binom{\ell-1}{2} = \binom{k+1}{2} - 1 - 2\binom{k-3}{2} = \frac{1}{2}(-k^2 + 15k - 26)$, in addition to the usual bound $h' \leq h \leq 2k-2$. In order for $\frac{1}{2}(-k^2+15k-26)$ to be nonnegative, we must have $k \leq 13$. Here is a list of values of $k \leq 13$ and the corresponding upper bounds on h.

k	5	7	9	11	13
$h \leq$	8	12	14	9	0

Claim 2: If $k \leq 13$, then $h' \leq 2k-4$.

Proof of Claim 2: Since $h' \leq h$, the only possible cases for $h' > 2k-4$ are when $(k,h) = (5,7)$, $(5,8)$, $(7,11)$, and $(7,12)$. Note that $\max\left\{3, \binom{k-3}{2}\right\} = \max\left\{3, \binom{\ell-1}{2}\right\} \leq c_2 \leq \frac{1}{2}\left(\binom{k+1}{2} - 1 - h\right)$. When $k=5$ and $h=7$ or 8, we have $c_2 = 3$, so H_1 is the cycle $v_0v_1v_2v_0$, and $g'=2$. Hence, $h' = h - g' = 5$ or 6. When $k=7$, we have

$$c_2 = \begin{cases} 6,\ 7,\ \text{or } 8 & \text{if } h = 11;\ \text{and} \\ 6 \text{ or } 7 & \text{if } h = 12. \end{cases}$$

As a result, H_1 is the union of the cycle $v_0v_2v_4v_1v_3v_0$ and $c_2 - 5$ loops in $L_1 - v_0v_0$. Hence,

$$g' = \begin{cases} 3,\ 2,\ \text{or } 1 & \text{if } h = 11;\ \text{and} \\ 3 \text{ or } 2 & \text{if } h = 12. \end{cases}$$

This implies

$$h' = h - g' = \begin{cases} 8,\ 9,\ \text{or } 10 & \text{if } h = 11;\ \text{and} \\ 9 \text{ or } 10 & \text{if } h = 12. \end{cases}$$

In all cases, $h' \leq 2k-4$. This completes the proof of Claim 2.

Next, let $\widetilde{h}$ be the smallest nonnegative integer such that $h' \equiv \widetilde{h} \pmod 4$, which is slightly different from Case 1. Note that Claim 2, together with the fact that $h' - \widetilde{h} \equiv 2k-2 \equiv 0 \pmod 4$, implies $h' - \widetilde{h} \leq 2k-6$. If $\widetilde{h} \leq 2$, define the subgraph H_0 as the union of

- 4-cycles $v_{2s-1}v_{k-2}v_{2s}v_{k-1}v_{2s-1}$, where $1 \leq s \leq \frac{h'-\widetilde{h}}{4}$,
- g' loops in S, and
- $\widetilde{h}$ loops in L_2.

If $\widetilde{h} = 3$, define the subgraph H_0 as the union of

- 4-cycles $v_{2s-1}v_{k-2}v_{2s}v_{k-1}v_{2s-1}$, where $1 \leq s \leq \frac{h'-\widetilde{h}}{4}$,
- g' loops in S, and
- the triangle $v_{k-3}v_{k-2}v_{k-1}v_{k-3}$.

The number of edges in H_0 is $4 \cdot \frac{h'-\widetilde{h}}{4} + g' + \widetilde{h} = h' + g' = h$, and it is clear that all degrees in H_0 are even.

Construction of H_2

Define the subgraph H_2 as $K_k^* - v_0v_0 - H_0 - H_1$. From the construction, the following conditions hold.

1. The number of edges in H_2 is $\binom{k+1}{2} - 1 - h - c_2 = n - c_2 = c_3$.
2. After removing isolated vertices in V_1, H_2 is connected due to the following analysis.
   - If $h \leq g'$ and $k = \ell$, then as provided in the proof of Claim 1, $k = 5$ and S contains the cycle $v_0v_1v_2v_3v_4v_0$, which is completely contained in H_2 since H_0 does not contain any edges in this cycle.
   - If $h \leq g'$ and $k > \ell$, then H_2 contains both L_2 and L_b.
   - If $h > g'$, then by Claim 1, $k > \ell$. Hence, in H_2, v_0 is connected with all vertices in V_2, and vertices $v_1, v_2, \ldots, v_{\ell-1}$ are either isolated vertices and thus removed, or are connected to v_0 through the cycle $v_0v_1v_2 \cdots v_{\ell-1}v_0$.
3. The degree of v_0 in H_2 is positive due to item 2.
4. All degrees in H_2 are even since all degrees in K_k^*, v_0v_0, H_0, and H_1 are even.

□

2.3. Cycles with one chord

Caterpillar trees are natural extensions of paths, while chorded cycles are natural extensions of cycles. In this section, we are going to study cycles with one extra chord.

Definition 3. *Let $C_n^{\{0,j\}}$ be a cycle on n vertices $w_0, w_1, w_2, \ldots, w_{n-1}$ with a chord between w_0 and w_j. Without loss of generality, let $2 \leq j \leq \frac{n}{2}$, which also implies $n \geq 4$. Note that $j \neq 1$, as w_0w_1 is already in the cycle C_n.*

Note that $C_n^{\{0,2\}}$ contains a triangle $w_0w_1w_2w_0$. When it is embedded in K_k^*, this triangle creates an obstruction if n is very close to $\binom{k+1}{2}$. This is reflected by the extra complication in the statement of the following theorem.

Theorem 2.4. *Let k be odd. Then $C_n^{\{0,2\}}$ can be embedded in K_k^* if and only if $n \leq \binom{k+1}{2} - 3$. When $j \geq 3$, then $C_n^{\{0,j\}}$ can be embedded in K_k^* if and only if $n \leq \binom{k+1}{2} - 1$.*

Proof. If $k \leq 3$, then it is obvious that $C_n^{\{0,j\}}$ cannot be embedded in K_k^*. Also, $k \leq 3$ is ruled out implicitly by $4 \leq n \leq \binom{k+1}{2} - 3$ as well as $6 \leq 2j \leq n \leq \binom{k+1}{2} - 1$. Hence, it is only meaningful to consider $k \geq 5$.

If $C_n^{\{0,2\}}$ can be embedded in K_k^*, then note that the vertices in the triangle $w_0w_1w_2w_0$ must be embedded as distinct vertices in K_k^*. Otherwise, we will create double edges or multiple loops at the same vertex, which do not exist in K_k^*. Without loss of generality, assume that w_0, w_1, w_2 are embedded as v_0, v_1, v_2 respectively. The degrees of w_0 and w_2 in the path $w_2w_3 \cdots w_{n-1}w_0$ are odd, while the degrees of v_0 and v_2 in $K_k^* - v_0v_1v_2v_0$ are even. To create odd degree vertices at v_0 and v_2 in $K_k^* - v_0v_1v_2v_0$, if we were to forgo only one edge from $K_k^* - v_0v_1v_2v_0$, that edge must be v_0v_2. However, v_0v_2 does not exist in $K_k^* - v_0v_1v_2v_0$, so at least two edges are forgone from $K_k^* - v_0v_1v_2v_0$ when the path $w_2w_3 \cdots w_{n-1}w_0$ is embedded in $K_k^* - v_0v_1v_2v_0$. Therefore, $n + 1 \leq \binom{k+1}{2} - 2$, or $n \leq \binom{k+1}{2} - 3$.

If $n \le \binom{k+1}{2} - 3$, without loss of generality, we can assume that $n > \binom{k-1}{2} - 3$. Otherwise, if $n \le \binom{k-1}{2} - 3$, we can consider embedding $C_n^{\{0,2\}}$ in K_{k-2}^*, which can be further embedded in K_k^* trivially. Let $h = \binom{k+1}{2} - 3 - n$ be the gap between n and $\binom{k+1}{2} - 3$. Based on our assumption, $0 \le h \le \binom{k+1}{2} - 3 - \left(\binom{k-1}{2} - 2\right) = 2k - 2$. Let H be defined such that

$$H = \begin{cases} \{v_0v_3v_2, v_0v_0, v_1v_1, v_2v_2, \ldots, v_{h-1}v_{h-1}\} & \text{if } 0 \le h \le k; \\ \{v_0v_4v_1v_3v_4v_4v_2, v_0v_0, v_1v_1, \ldots, v_{h-5}v_{h-5}\} & \text{if } k = 5 \text{ and } 6 \le h \le 8; \text{ and} \\ \{v_0v_3v_2, v_{k-1}v_1v_3v_4v_5 \cdots v_{k-1}, & \\ \qquad v_0v_0, v_1v_1, v_2v_2, \ldots, v_{h-k+1}v_{h-k+1}\} & \text{if } k \ge 7 \text{ and } k+1 \le h \le 2k-2. \end{cases}$$

Note that the number of edges in H is $h + 2$ in all cases. Now, the chorded cycle $C_n^{\{0,2\}}$ can be embedded in K_k^* such that w_0, w_1, w_2 are embedded as v_0, v_1, and v_2 respectively, and the path $w_2w_3 \cdots w_{n-1}w_0$ is embedded as the Eulerian path in $K_k^* - v_0v_1v_2v_0 - H$. This is because the only odd degree vertices in the graph $K_k^* - v_0v_1v_2v_0 - H$ are v_0 and v_2, and this graph is connected after isolated vertices are removed.

When $j \ge 3$, the only if direction is trivial. For the if direction, the chorded cycle $C_n^{\{0,j\}}$ can be embedded in the petal graph $P_{1,j,n-j}$ as follows: the chord w_0w_j is embedded as the first petal u_0u_0, the path $w_0w_1w_2 \cdots w_j$ is embedded as the second petal $u_0u_1^2u_2^2 \cdots u_{j-1}^2u_0$, and the path $w_jw_{j+1} \cdots w_{n-1}w_0$ is embedded as the third petal $u_0u_1^3u_2^3 \cdots u_{n-j-1}^3u_0$. Theorem 2.3 completes the proof. □

2.4. Spider graphs with three or four legs

We have provided the definition of spider graphs in Section 1, but it is more convenient for our future discussions if we formalize notation.

Definition 4. *Let $\Delta \in \mathbb{N}$ such that $\Delta \ge 3$, and let $\ell_1, \ell_2, \ldots, \ell_\Delta \in \mathbb{N}$ such that $\ell_1 \le \ell_2 \le \ldots \le \ell_\Delta$. Let $S_{\ell_1,\ell_2,\ldots,\ell_\Delta}$ be the* spider graph *with vertices $x_0, x_1^1, x_2^1, \ldots, x_{\ell_1}^1, x_1^2, x_2^2, \ldots, x_{\ell_2}^2, \ldots, x_1^\Delta, x_2^\Delta, \ldots, x_{\ell_\Delta}^\Delta$, and for each $i = 1, 2, \ldots, \Delta$, the edges form a path $x_0x_1^i, x_2^i, \ldots, x_{\ell_i}^i$ between the* central vertex x_0 *and the leaf $x_{\ell_i}^i$, which is called the i-th* leg *of the spider graph.*

Although the necessary and sufficient conditions for embedding spider graphs with three legs in K_k^* were proved [11], we are going to provide a much simplified proof when k is odd, utilizing Theorem 2.3.

Theorem 2.5. *Let $n = \ell_1 + \ell_2 + \ell_3$, and let k be odd. If $n \ge 7$, then S_{ℓ_1,ℓ_2,ℓ_3} can be embedded in K_k^* if and only if $n \le \binom{k+1}{2}$.*

Proof. If $k \le 3$, then $n \le \binom{3+1}{2} = 6$, violating the condition that $n \ge 7$. Hence, it is only meaningful to consider $k \ge 5$. Furthermore, $n \ge 7$ implies $\ell_3 \ge \left\lceil \frac{1}{3}n \right\rceil \ge 3$.

The only if direction is trivial. For the if direction, we consider the following cases.

Case 1: $\ell_1 = 1$. Hence, the spider graph S_{1,ℓ_2,ℓ_3} can be embedded in the petal graph $P_{1,3,n-4}$ as follows: the vertices x_0 and x_1^1 of the first leg are both embedded as u_0; the vertices $x_1^2, x_2^2, \ldots, x_{\ell_2}^2$ of the second leg are embedded as $u_1^3, u_2^3, \ldots, u_{\ell_2}^3$; and the vertices $x_1^3, x_2^3, \ldots, x_{\ell_3}^3$ of the third leg are embedded as $u_1^2, u_2^2, u_0, u_{n-5}^3, u_{n-6}^3, \ldots, u_{\ell_2}^3$. Theorem 2.3 completes the proof.

Case 2: $\ell_1 \geq 2$. Let $(i,j) = (2,3)$ if $\ell_1 + \ell_2 - 1 \leq \ell_3$, and let $(i,j) = (3,2)$ otherwise. The spider graph S_{ℓ_1,ℓ_2,ℓ_3} can be embedded in the petal graph P_{1,c_2,c_3}, where $c_i = \ell_1+\ell_2-1$ and $c_j = \ell_3$, as follows: the vertices $x_0, x_1^1, x_2^1, \dots, x_{\ell_1}^1$ of the first leg are embedded as $u_0, u_0, u_1^i, u_2^i, \dots, u_{\ell_1-1}^i$; the vertices $x_1^2, x_2^2, \dots, x_{\ell_2}^2$ of the second leg are embedded as $u_{c_i-1}^i, u_{c_i-2}^i, \dots, u_{\ell_1-1}^i$; and the vertices $x_1^3, x_2^3, \dots, x_{\ell_3}^3$ of the third leg are embedded as $u_1^j, u_2^j, \dots, u_{c_j-1}^j, u_0$. Again, Theorem 2.3 completes the proof. □

The last theorem of this section provides the necessary and sufficient conditions for embedding spider graphs with four legs, the family of graphs that originally drew our interest, in K_k^* when k is odd.

Theorem 2.6. *Let $n = \ell_1+\ell_2+\ell_3+\ell_4$, and let k be odd. If $n \geq 7$, then $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ can be embedded in K_k^* if and only if*

(a) $\ell_3 = 1$ *and* $n \leq \binom{k+1}{2} - 1$, *or*

(b) $\ell_3 \geq 2$ *and* $n \leq \binom{k+1}{2}$.

Proof. If $k \leq 3$, then $n \leq \binom{3+1}{2} = 6$, violating that $n \geq 7$. Hence, it is only meaningful to consider $k \geq 5$.

Assume $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ can be embedded in K_k^*. It is trivial that $n \leq \binom{k+1}{2}$. If $n = \binom{k+1}{2}$, then since $x_{\ell_1}^1$, $x_{\ell_2}^2$, $x_{\ell_3}^3$, and $x_{\ell_4}^4$ are of odd degree in $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ and all vertices in K_k^* are of even degree, at least two of $x_{\ell_1}^1$, $x_{\ell_2}^2$, and $x_{\ell_3}^3$ must be embedded as the same vertex. If $\ell_3 = 1$, then this is impossible since there are no double edges in K_k^*, and hence $n \leq \binom{k+1}{2} - 1$.

For the if direction, due to Proposition 1.2, it suffices to show that $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ can be embedded in K_k^* if $n = \binom{k+1}{2} - 1$ when $\ell_3 = 1$ and $n = \binom{k+1}{2}$ when $\ell_3 \geq 2$.

Case 1: $\ell_3 = 1$. The spider graph $S_{1,1,1,\ell_4}$ can be embedded in K_k^* as follows: the vertices x_0, x_1^1, x_1^2, and x_1^3 are embedded as v_0, v_0, v_1, and v_2 respectively; the remaining graph $K_k^* - v_0v_0 - v_0v_1v_2v_0$ is an Eulerian graph, so the fourth leg can be embedded as an Eulerian cycle.

Case 2: $\ell_3 \geq 2$. Let $(i,j) = (2,3)$ if $\ell_1 + \ell_3 \leq \ell_2 + \ell_4 - 1$, and let $(i,j) = (3,2)$ otherwise. Define $c_i = \ell_1 + \ell_3$ and $c_j = \ell_2 + \ell_4 - 1$. Note that $c_i \geq 1 + 2 = 3$ and $c_j \geq \max\{n - \ell_1 - \ell_3 - 1, \ell_1 + \ell_3 - 1\} \geq 3$ since $n \geq 7$. The spider graph $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ can be embedded in the petal graph P_{1,c_2,c_3}, where $c_i = \ell_1+\ell_3$ and $c_j = \ell_2+\ell_4-1$ as follows: the vertices $x_0, x_1^1, x_2^1, \dots, x_{\ell_1}^1$ of the first leg are embedded as $u_0, u_1^i, u_2^i, \dots, u_{\ell_1}^i$; the vertices $x_1^2, x_2^2, \dots, x_{\ell_2}^2$ of the second leg are embedded as $u_0, u_1^j, u_2^j, \dots, u_{\ell_2-1}^j$; the vertices $x_1^3, x_2^3, \dots, x_{\ell_3}^3$ of the third leg are embedded as $u_{c_i-1}^i, u_{c_i-2}^i, \dots, u_{c_i-\ell_3}^i$; and the vertices $x_1^4, x_2^4, \dots, x_{\ell_4}^4$ of the fourth leg are embedded as $u_{c_j-1}^j, u_{c_j-2}^j, \dots, u_{c_j-\ell_4}^j$. Theorem 2.3 completes the proof. □

3. *k* is even

In Section 2, we restrict our consideration to k being odd. In this section, except for caterpillar trees, we prove the results parallel to the theorems in the previous section for even k.

To aid our discussions in this section, here are some notations related to an edge decomposition of K_k^*. For each $j = 0, 1, 2, \dots, \frac{k-2}{2}$, let D_j be the subgraph of K_k^* such that the edge set of D_j is

$$\{v_pv_q : p \leq q, \text{ either } q - p = j \text{ or } q - p = k - j\}.$$

Note that each D_j is a regular degree 2 graph and has exactly k edges. Furthermore, let I be the perfect matching subgraph of K_k^* such that the edge set of I is

$$\left\{v_i v_{\frac{k}{2}+i} : i = 0, 1, 2, \ldots, \frac{k-2}{2}\right\}.$$

Figure 2 illustrates how K_6^* is decomposed into D_0, D_1, D_2, and I.

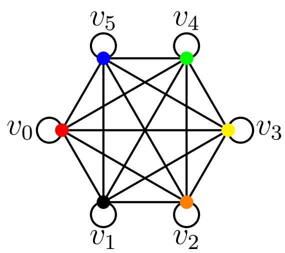
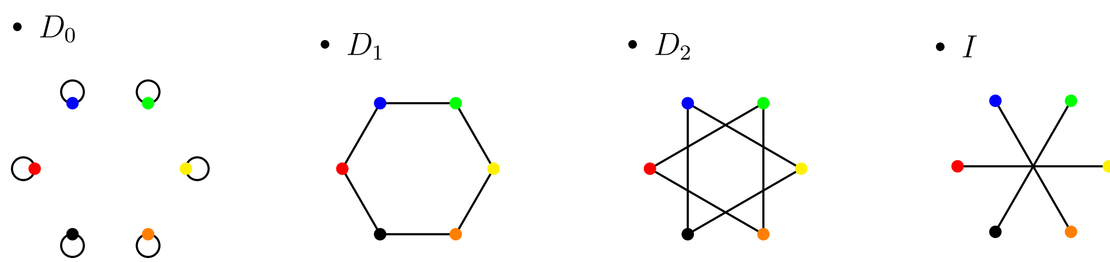


Figure 2. K_6^* contains D_0, D_1, and D_2 as shown below



3.1. Petal graphs

Similar to the case when k is odd, we are going to show that petal graphs P_{1,c_2,c_3} can be embedded in K_k^* when k is even as long as some trivial necessary conditions are satisfied.

Theorem 3.1. *If k is even, then P_{1,c_2,c_3} can be embedded in K_k^* if and only if $k \geq 6$, $c_2 \geq 3$, and $1 + c_2 + c_3 \leq \frac{k^2}{2}$.*

Proof. For the only if direction, the condition $c_2 \geq 3$ is trivial. If $k \leq 4$, then the maximum degree in $K_k^* - I$ is at most 4, which is less than the maximum degree in P_{1,c_2,c_3}. Hence, we must have $k \geq 6$. Also, the petal graph P_{1,c_2,c_3} has only even degrees, so when it is embedded in K_k^*, the image must also have only even degrees. Since all the k vertices in K_k^* have odd degrees, at least one edge will be missing from each vertex after embedding. In other words, there are at least $\frac{k}{2}$ edges missing, implying that the number of edges in P_{1,c_2,c_3} is at most $\binom{k+1}{2} - \frac{k}{2} = \frac{k^2}{2}$.

For the if direction, let $n = c_2 + c_3$. We are going to show that if $k \geq 6$, $c_2 \geq 3$, and $1 + n \leq \frac{k^2}{2}$, then P_{1,c_2,c_3} can be embedded in $K_k^* - I$. In all of the following cases, we embed the petal u_0u_0 as the loop v_0v_0 in K_k^*.

Without loss of generality, we can assume that $n > \binom{k}{2} - 1$. Otherwise, if $n \leq \binom{k}{2} - 1$, we can embed P_{1,c_2,c_3} in K_{k-1}^* by Theorem 2.3, which can be further embedded in K_k^* trivially. Let

$h = \frac{k^2}{2} - 1 - n$ be the gap between n and $\frac{k^2}{2} - 1$. Based on our assumption, $0 \le h \le \frac{k^2}{2} - 1 - \binom{k}{2} = \frac{k}{2} - 1$.

Case 1: $3 \le c_2 < k$. Let H_0, H_1, and H_2 be subgraphs of $K_k^* - I$ such that $H_0 = \{v_2v_2, v_3v_3, \ldots, v_{h+1}v_{h+1}\}$,

$$H_1 = \begin{cases} v_0v_1v_2v_3\cdots v_{c_2-1}v_0 & \text{if } c_2 \ne \frac{k}{2}+1; \text{ and} \\ v_0v_1v_1v_2v_3\cdots v_{\frac{k}{2}-1}v_0 & \text{if } c_2 = \frac{k}{2}+1, \end{cases}$$

and $H_2 = K_k^* - I - v_0v_0 - H_0 - H_1$. Note that H_0 is well-defined since $h \le \frac{k}{2} - 1 \le k - 2$. With a simple count, we see that H_0 has h edges, H_1 has c_2 edges, and H_2 has $\binom{k+1}{2} - \frac{k}{2} - 1 - h - c_2 = n - c_2 = c_3$ edges. It is clear that all vertices in H_1 and H_2 are of even degree. Furthermore, H_2 is connected since H_2 contains $D_2 \cup \{v_{k-1}v_0\} - \{v_{c_2-1}v_0\}$. Therefore, H_1 and H_2 are both Eulerian, so we can embed the petals $u_0u_1^2u_2^2\cdots u_{c_2-1}^2u_0$ as H_1 and $u_0u_1^3u_2^3\cdots u_{c_3-1}^3u_0$ as H_2.

Case 2: $c_2 \ge k$ and $k = 6$. Note that $c_2 < \frac{1}{2}(1+n) \le \frac{1}{2}\cdot\frac{k^2}{2} = 9$, i.e., $c_2 = 6$, 7, or 8. We can embed the petal $u_0u_1^2u_2^2\cdots u_{c_2-1}^2u_0$ as $v_0v_1v_3v_5v_4v_2v_0$ along with $c_2 - 6$ additional loops, and embed the petal $u_0u_1^3u_2^3\cdots u_{c_3-1}^3u_0$ as $v_0v_5v_1v_2v_3v_4v_0$ along with $c_3 - 6$ additional loops. Since $c_2 - 6 + c_3 - 6 = n - 12 \le 5$, we can ensure that the two petals are embedded using different loops in $K_k^* - I - v_0v_0$.

Case 3: $c_2 \ge k$ and $k \ge 8$. Let z be the odd integer in $\{\frac{k-2}{2}, \frac{k-4}{2}\}$. Note that $\gcd(k, z) = 1$. Since $k \ge 8$, $z \ne 1$. In other words, D_1 and D_z are two distinct Hamiltonian cycles in K_k^*. Let c_2' and c_3' be the smallest nonnegative integers such that $c_2 \equiv c_2' \pmod{k}$ and $c_3 \equiv c_3' \pmod{k}$.

Case 3.1: $c_2' + c_3' < k$. We can embed the petal $u_0u_1^2u_2^2\cdots u_{c_2-1}^2u_0$ as the subgraph composed of D_1, $\frac{c_2-c_2'}{k} - 1$ elements of $\{D_2, D_3, \ldots, D_{\frac{k-2}{2}}\} \setminus \{D_z\}$, and c_2' loops in D_0. Next, we can embed the petal $u_0u_1^3u_2^3\cdots u_{c_3-1}^3u_0$ as the subgraph composed of D_z, $\frac{c_3-c_3'}{k} - 1$ elements of $\{D_2, D_3, \ldots, D_{\frac{k-2}{2}}\} \setminus \{D_z\}$, and c_3' loops in D_0.

Case 3.2: $c_2' + c_3' \ge k$. Since $\frac{k^2}{2} - \frac{k}{2} \le n \le \frac{k^2}{2} - 1$ and $n \equiv c_2' + c_3' \pmod{k}$, we have $\frac{k}{2} \le c_2' + c_3' - k \le k - 1$, which implies $\frac{3k}{2} \le c_2' + c_3' \le 2k - 1$. Hence, $\min\{c_2', c_3'\} \ge \frac{3k}{2} - \max\{c_2', c_3'\} \ge \frac{3k}{2} - (k-1) > \frac{k}{2}$. Let H_1' be the subgraph that contains $c_2' - \frac{k}{2}$ loops in D_0 together with the cycle $v_0v_2v_4\cdots v_{k-2}v_0$ in D_2, and let H_2' be the subgraph that contains $c_3' - \frac{k}{2}$ loops in D_0 together with the cycle $v_1v_3v_5\cdots v_{k-1}v_1$ in D_2. Let H_1 be the subgraph that contains D_1, $\frac{c_2-c_2'}{k} - 1$ elements of $\{D_3, \ldots, D_{\frac{k-2}{2}}\} \setminus \{D_z\}$, and H_1'; let H_2 be the subgraph that contains D_z, $\frac{c_3-c_3'}{k} - 1$ elements of $\{D_3, \ldots, D_{\frac{k-2}{2}}\} \setminus \{D_z\}$, and H_2'. It is easy to see that we can embed the petals $u_0u_1^2u_2^2\cdots u_{c_2-1}^2u_0$ as H_1 and $u_0u_1^3u_2^3\cdots u_{c_3-1}^3u_0$ as H_2. □

3.2. Cycles with one chord

Unlike the case when k is odd, the obstruction of the triangle in $C_n^{\{0,2\}}$ when embedded in K_k^* only occurs when $k = 4$. This is because when k is even, the chord w_0w_j in $C_n^{\{0,j\}}$ can be embedded as a diagonal $v_0v_{\frac{k}{2}}$, instead of as a loop in K_k^* when k is odd. Hence, with the exception of $(k, n, j) = (4, 8, 2)$, the trivial necessary conditions are also sufficient for embedding $C_n^{\{0,j\}}$ in K_k^*.

Theorem 3.2. *Let k be even. Then $C_n^{\{0,j\}}$ can be embedded in K_k^* if and only if $k \geq 4$, $n \leq \frac{k^2}{2}$, and $(k,n,j) \neq (4,8,2)$.*

Proof. For the only if direction, if $k \leq 2$, then $n \leq \frac{2^2}{2} = 2$, which is impossible according to Definition 3. Hence, $k \geq 4$. Since the chorded cycle $C_n^{\{0,j\}}$ has only two vertices with odd degrees, when it is embedded in K_k^*, the image will have at most two vertices with odd degrees. Since all the k vertices in K_k^* have odd degrees, after embedding, at least one edge will be missing per vertex from at least $k-2$ vertices. In other words, there are at least $\frac{k-2}{2}$ edges missing, implying that the number of edges in $C_n^{\{0,j\}}$ is at most $\binom{k+1}{2} - \frac{k-2}{2} = \frac{k^2+2}{2}$. Since $C_n^{\{0,j\}}$ has $n+1$ edges, we have $n \leq \frac{k^2}{2}$.

If $(k,n,j) = (4,8,2)$, when $C_8^{\{0,2\}}$ is embedded in K_4^*, at least one edge that connects two vertices will be missing as discussed. Without loss of generality, let this edge be v_1v_3. Note that the subgraph given by the cycle $w_0w_1w_2w_0$ in $C_8^{\{0,2\}}$ must be embedded as a triangle in $K_4^* - v_1v_3$. Without loss of generality, let $w_0w_1w_2w_0$ be embedded as $v_0v_1v_2v_0$. The remaining edges of K_4^*, namely $K_4^* - v_1v_3 - v_0v_1v_2v_0$, form a disconnected graph. As a result, it is impossible to embed the subgraph $w_2w_3\cdots w_7w_0$ of $C_8^{\{0,2\}}$. Therefore, if $C_n^{\{0,j\}}$ can be embedded in K_k^* , then $(k,n,j) \neq (4,8,2)$.

For the if direction, we are going to show that if $k \geq 4$, $n \leq \frac{k^2}{2}$, and $(k,n,j) \neq (4,8,2)$, then $C_n^{\{0,j\}}$ can be embedded in $(K_k^* - I) \cup \{v_0v_{\frac{k}{2}}\}$, which contains $\frac{k^2}{2}+1$ edges. In all of the following cases, we embed the chord w_0w_j as $v_0v_{\frac{k}{2}}$ in K_k^*.

Without loss of generality, we can assume that $n > \binom{k}{2} - 3$. Otherwise, if $4 \leq n \leq \binom{k}{2} - 3$, we can embed $C_n^{\{0,j\}}$ in K_{k-1}^* by Theorem 2.4, which can be further embedded in K_k^* trivially. Let $h = \frac{k^2}{2} - n$ be the gap between n and $\frac{k^2}{2}$. Based on our assumption, $0 \leq h \leq \frac{k^2}{2} - \left(\binom{k}{2} - 2\right) = \frac{k}{2}+2$.

Case 1: $j \leq \frac{k}{2}$. Let H_0, H_1, and H_2 be subgraphs of $K_k^* - I$ such that $H_0 = \{v_1v_1, v_2v_2, \ldots, v_hv_h\}$, $H_1 = v_0v_1v_2\cdots v_{j-1}v_{\frac{k}{2}}$, and $H_2 = K_k^* - I - H_0 - H_1$. Note that H_0 is well-defined since $h \leq \frac{k}{2} + 2 \leq \frac{k}{2} + \frac{k}{2} = k$, and v_kv_k represents the loop v_0v_0. With a simple count, we see that H_0 has h edges, H_1 has j edges, and H_2 has $\binom{k+1}{2} - \frac{k}{2} - h - j = \frac{k^2}{2} - h - j = n - j$ edges. It is clear that the only two odd degree vertices in H_1 and H_2 are v_0 and $v_{\frac{k}{2}}$. Furthermore, we claim that H_2 is connected: if $k = 4$, then $(k,n,j) \neq (4,8,2)$ implies that $h \neq \frac{4^2}{2} - 8 = 0$, so H_2 contains the path $v_0v_3v_2$ but not the loop v_1v_1; if $k \geq 6$, then H_2 contains $(D_2 \cup \{v_{k-1}v_0\}) - \{v_{j-1}v_{\frac{k}{2}}\}$, so H_2 is connected. Therefore, H_1 and H_2 both contain an Eulerian trail, and we can embed the paths $w_0w_1w_2\cdots w_j$ and $w_0w_{n-1}w_{n-2}\cdots w_j$ as H_1 and H_2 respectively.

Case 2: $j > \frac{k}{2}$. Let j' and j'' be the smallest positive integers such that $j - \frac{k}{2} \equiv j' \pmod{k}$ and $j' \equiv j'' \pmod{\frac{k}{2}}$. Let

$$H_0 = \begin{cases} \{v_{j''+1}v_{j''+1}, v_{j''+2}v_{j''+2}, \ldots, v_{j''+h}v_{j''+h}\} & \text{if } h \leq \frac{k}{2}; \\ \{v_{j''+1}v_{j''+1}, v_{j''+2}v_{j''+2}, \ldots, v_{j''+h-\frac{k}{2}}v_{j''+h-\frac{k}{2}}\} \\ \qquad \cup\{v_0v_2v_4\cdots v_{k-2}v_0\} & \text{if } h > \frac{k}{2}, \end{cases}$$

H_1 contain the path $v_0v_1v_2\cdots v_{\frac{k}{2}}$ along with $\frac{j-\frac{k}{2}-j'}{k}$ elements of $\{D_3, D_4, \ldots, D_{\frac{k-2}{2}}\}$ and the edges

$$\begin{cases}\{v_1v_1, v_2v_2, \ldots, v_{j'}v_{j'}\} & \text{if } j' \leq \frac{k}{2};\ \text{and}\\ \{v_1v_1, v_2v_2, \ldots, v_{j''}v_{j''}\} \cup \{v_1v_3v_5\cdots v_{k-1}v_1\} & \text{if } j' > \frac{k}{2},\end{cases}$$

and $H_2 = K_k^* - I - H_0 - H_1$. Note that v_kv_k represents the loop v_0v_0. With a simple count, we see that H_0 has h edges, H_1 has j edges, and H_2 has $\binom{k+1}{2} - \frac{k}{2} - h - j = \frac{k^2}{2} - h - j = n - j$ edges. It is clear that the only two odd degree vertices in H_1 and H_2 are v_0 and $v_{\frac{k}{2}}$. It is also clear that H_1 is connected. We will now show that H_2 is connected.

- If $k = 4$, then H_2 is connected since it contains the path $v_0v_3v_2$ and does not contain the loop v_1v_1.
- If $k = 6$, then H_2 contains the path $v_0v_5v_4v_3$. Hence, the only possible disconnected edges in H_2 are v_1v_1, v_2v_2, and v_1v_2. However, v_1v_1 and v_1v_2 are always contained in H_1. Moreover, H_2 contains v_2v_2 if and only if $(j', h) \in \{(1, 0), (4, 0)\}$, and in both cases, H_2 also contains the cycle $v_0v_2v_4v_0$, so H_2 is connected.
- If $k = 8$, then H_2 contains the path $v_0v_7v_6v_5v_4$. Hence, the only possible disconnected edges in H_2 are v_1v_1, v_2v_2, v_3v_3, v_1v_2, v_2v_3, and v_1v_3. However, v_1v_1, v_1v_2, and v_2v_3 are always contained in H_1. The loop v_2v_2 is contained in H_2 if and only if $(j', h) \in \{(1, 0), (5, 0)\}$, and in both cases, H_2 also contains the cycle $v_0v_2v_4v_6v_0$. Finally, note that if $j' \leq \frac{k}{2}$, then H_2 contains the cycle $v_1v_3v_5v_7v_1$; if $j' > \frac{k}{2} = 4$, then $\frac{j-\frac{k}{2}-j'}{k} < \frac{16-4-4}{8} \leq 1$, so H_2 contains D_3. Hence, vertex v_3 is always connected to v_0 in H_2, thus the edges v_1v_3 and v_3v_3 will never be isolated edges in H_2. Therefore, H_2 is always connected.
- If $k \geq 10$, then H_2 contains $\{v_0v_{k-1}v_{k-2}\cdots v_{\frac{k}{2}}\} \cup D_i$ for some $i \in \{3, 4, \ldots, \frac{k-2}{2}\}$. To demonstrate this, let us first assume the contrary, i.e., every D_i, $i \in \{3, 4, \ldots, \frac{k-2}{2}\}$, is in H_1. Hence, $\frac{j-\frac{k}{2}-j'}{k} = \frac{k-2}{2} - 2$, which implies $j - j' = \frac{k^2-5k}{2}$. Since $j \leq \frac{n}{2} \leq \frac{1}{2} \cdot \frac{k^2}{2} = \frac{k^2}{4}$ and $j' > 0$, we have $\frac{k^2-5k}{2} < \frac{k^2}{4}$, which occurs if and only if $0 < k < 10$, contradicting that $k \geq 10$. Therefore, H_2 is connected.

Therefore, H_1 and H_2 both contain an Eulerian trail, and we can embed the paths $w_0w_1w_2\cdots w_j$ and $w_0w_{n-1}w_{n-2}\cdots w_j$ as H_1 and H_2 respectively. □

3.3. Spider graphs with four legs

Theorem 3.3. *Let k be even, and let $n = \ell_1 + \ell_2 + \ell_3 + \ell_4$. Then $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ can be embedded in K_k^* if and only if $k \geq 4$, $n \leq \frac{k^2+4}{2}$, and $(k, \ell_1) \neq (4, 2)$.*

Proof. For the only if direction, if $k \leq 2$, then the number of edges in K_k^* is at most $\binom{2+1}{2} = 3$, which is less than the number of edges in $S_{\ell_1,\ell_2,\ell_3,\ell_4}$, so $k \geq 4$ is a necessary condition.

Since the spider graph $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ has only four vertices with odd degrees, when it is embedded in K_k^*, the image will have at most four vertices with odd degrees. Since all the k vertices in K_k^*

have odd degrees, after embedding, at least one edge will be missing per vertex from at least $k-4$ vertices. In other words, there are at least $\frac{k-4}{2}$ edges missing, implying that the number of edges in $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ is at most $\binom{k+1}{2} - \frac{k-4}{2} = \frac{k^2+4}{2}$, i.e., $n \leq \frac{k^2+4}{2}$.

Finally, embedding $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ in K_k^* when $(k,\ell_1) = (4,2)$ is impossible due to Proposition 1.1, since $\Delta(S_{\ell_1,\ell_2,\ell_3,\ell_4}) = 4$. Therefore, $(k,\ell_1) \neq (4,2)$.

For the if direction, by Proposition 1.2, we only need to consider $n = \frac{k^2+4}{2}$. We are going to define the embedding $\phi : V(S_{\ell_1,\ell_2,\ell_3,\ell_4}) \to V(K_4^*)$ explicitly.

When $k = 4$, since $n = 10$ and $\ell_1 \neq 2$,

$$(\ell_1,\ell_2,\ell_3,\ell_4) \in \{(1,1,1,7),(1,1,2,6),(1,1,3,5),(1,1,4,4),(1,2,2,5),(1,2,3,4),(1,3,3,3)\}.$$

For all these instances, we embed the first leg $x_0x_1^1$ of $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ in K_4^* as the loop v_0v_0, i.e., $\phi(x_0) = v_0$ and $\phi(x_1^1) = v_0$. The embedding of other legs are given by the following table.

$(\ell_1,\ell_2,\ell_3,\ell_4)$	$(\phi(x_1^2),\ldots,\phi(x_{\ell_2}^2))$	$(\phi(x_1^3),\ldots,\phi(x_{\ell_3}^3))$	$(\phi(x_1^4),\ldots,\phi(x_{\ell_4}^4))$
$(1,1,1,7)$	(v_1)	(v_2)	$(v_3,v_3,v_2,v_2,v_1,v_1,v_3)$
$(1,1,2,6)$	(v_1)	(v_2,v_2)	$(v_3,v_3,v_2,v_1,v_1,v_3)$
$(1,1,3,5)$	(v_1)	(v_2,v_2,v_3)	(v_3,v_3,v_1,v_1,v_2)
$(1,1,4,4)$	(v_1)	(v_2,v_2,v_3,v_3)	(v_3,v_1,v_1,v_2)
$(1,2,2,5)$	(v_1,v_1)	(v_2,v_2)	(v_3,v_3,v_1,v_2,v_3)
$(1,2,3,4)$	(v_1,v_1)	(v_2,v_2,v_3)	(v_3,v_3,v_1,v_2)
$(1,3,3,3)$	(v_1,v_1,v_2)	(v_2,v_2,v_3)	(v_3,v_3,v_1)

When $k \geq 6$, we proceed by considering the following cases.

Case 1: $\ell_2+\ell_3 \geq 4$. Let $(i,j) = (2,3)$ if $\ell_2+\ell_3-1 \leq \ell_1+\ell_4-2$, and let $(i,j) = (3,2)$ otherwise. Furthermore, let $c_i = \ell_2+\ell_3-1$ and $c_j = \ell_1+\ell_4-2$. We claim that the petal graph P_{1,c_2,c_3} can be embedded in $K_k^* - I$. Since $\ell_2+\ell_3 \geq 4$, we have $\ell_2+\ell_3-1 \geq 3$. Also, since $\ell_4 \geq \frac{k^2+4}{8}$ and $k \geq 6$, we have $\ell_4 \geq 5$, so $\ell_1+\ell_4-2 \geq 4 > 3$. Finally, $\ell_1+\ell_2+\ell_3+\ell_4 = \frac{k^2+4}{2}$, which implies $1+c_2+c_3 = 1+(\ell_2+\ell_3-1)+(\ell_3+\ell_4-2) = \frac{k^2}{2}$. Therefore, there exists an embedding $\psi : V(P_{1,c_2,c_3}) \to V(K_k^*-I)$ by the argument provided in the proof of Theorem 3.1.

Now, we start embedding $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ in K_4^* as follows. We embed the second leg $x_0x_1^2x_2^2\cdots x_{\ell_2}^2$ of $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ as the path $\psi(u_0)\psi(u_1^i)\psi(u_2^i)\cdots\psi(u_{\ell_2}^i)$ in K_k^*. Let $v_a = \psi(u_{\ell_2}^i)$ be a vertex in K_k^*. Then we embed the third leg $x_0x_1^3x_2^3\cdots x_{\ell_3}^3$ as the path $\psi(u_0)\psi(u_{c_i-1}^i)\psi(u_{c_i-2}^i)\cdots\psi(u_{\ell_2}^i)v_{\frac{k}{2}+a}$, where $\frac{k}{2}+a$ is performed under modulo k. Note that the edge $\psi(u_{\ell_2}^i)v_{\frac{k}{2}+a}$ is in the perfect matching I.

Let $v_{b_2} = \psi(u_{\ell_4-2}^j)$ and $v_{b_1} = \psi(u_{\ell_4-1}^j)$ be vertices in K_k^*. Here, if $\ell_4-1 = c_j$, then $u_{\ell_4-1}^j = u_0$.

Case 1.1: $v_{b_2} \notin \{v_a, v_{\frac{k}{2}+a}\}$. In this case, we embed the fourth leg $x_0x_1^4x_2^4\cdots x_{\ell_4}^4$ as the path $\psi(u_0)\psi(u_0)\psi(u_1^j)\psi(u_2^j)\cdots\psi(u_{\ell_4-2}^j)v_{\frac{k}{2}+b_2}$, where $\frac{k}{2}+b_2$ is performed under modulo k. Note that the edge $\psi(u_{\ell_4-2}^j)v_{\frac{k}{2}+b_2}$ is in the perfect matching I and is distinct from the edge $\psi(u_{\ell_2}^i)v_{\frac{k}{2}+a}$. Finally, we embed the first leg $x_0x_1^1x_2^1\cdots x_{\ell_1}^1$ as the path $\psi(u_0)\psi(u_{c_j-1}^j)\psi(u_{c_j-2}^j)\cdots\psi(u_{\ell_4-2}^j)$.

Case 1.2: $v_{b_2} \in \{v_a, v_{\frac{k}{2}+a}\}$ and $v_{b_1} \notin \{v_a, v_{\frac{k}{2}+a}\}$. In this case, we embed the fourth leg $x_0x_1^4x_2^4\cdots x_{\ell_4}^4$ as the path $\psi(u_0)\psi(u_0)\psi(u_1^j)\psi(u_2^j)\cdots\psi(u_{\ell_4-1}^j)$, and we embed the first leg

$x_0x_1^1x_2^1\cdots x_{\ell_1}^1$ as the path $\psi(u_0)\psi(u_{c_j-1}^j)\psi(u_{c_j-2}^j)\cdots\psi(u_{\ell_4-1}^j)v_{\frac{k}{2}+b_1}$, where $\frac{k}{2}+b_1$ is performed under modulo k. Note that the edge $\psi(u_{\ell_4-1}^j)v_{\frac{k}{2}+b_1}$ is in the perfect matching I and is distinct from the edge $\psi(u_{\ell_2}^i)v_{\frac{k}{2}+a}$.

Case 1.3: $v_{b_2}, v_{b_1} \in \{v_a, v_{\frac{k}{2}+a}\}$. In this case, we claim that $\ell_1 > 1$. To demonstrate this, let us first assume the contrary, i.e., $\ell_1 = 1$. As a result, $c_j = \ell_4 - 1$ and $v_{b_1} = \psi(u_{c_j}^j) = \psi(u_0)$, which is the vertex v_0 due to the proof of Theorem 3.1. Therefore, we have $v_{b_2}, v_{b_1} \in \{v_0, v_{\frac{k}{2}}\}$. However, $v_{b_2} = \psi(u_{c_j-1}^j)$ cannot be the vertex v_0, or else $\psi(u_0)\psi(u_{c_j-1}^j) = v_0v_0 = \psi(u_0)\psi(u_0)$, contradicting that ψ is an embedding. The vertex v_{b_2} cannot be the vertex $v_{\frac{k}{2}}$ either, or else $\psi(u_0)\psi(u_{c_j-1}^j) = v_0v_{\frac{k}{2}}$ is an edge in the perfect matching I, contradicting that ψ is an embedding of P_{1,c_2,c_3} in $K_k^* - I$. Hence, $\ell_1 > 1$.

Note that $v_{b_2} = v_{b_1}$, or else $\psi(u_{\ell_4-2}^j)\psi(u_{\ell_4-1}^j) = v_{b_2}v_{b_1}$ is an edge in the perfect matching I. Let $v_{b_0} = \psi(u_{\ell_4}^j)$. Here, if $\ell_4 = c_j$, then $u_{\ell_4}^j = u_0$. Since $v_{b_1}v_{b_0} = \psi(u_{\ell_4-1}^j)\psi(u_{\ell_4}^j) \neq \psi(u_{\ell_4-2}^j)\psi(u_{\ell_4-1}^j) = v_{b_2}v_{b_1}$ and $v_{b_1}v_{b_0} = \psi(u_{\ell_4-1}^j)\psi(u_{\ell_4}^j)$ is not in I, we conclude that $v_{b_0} \notin \{v_a, v_{\frac{k}{2}+a}\}$. As a result, we can embed the fourth leg $x_0x_1^4x_2^4\cdots x_{\ell_4}^4$ as the path $\psi(u_0)\psi(u_1^j)\psi(u_2^j)\cdots\psi(u_{\ell_4}^j)$, and we embed the first leg $x_0x_1^1x_2^1\cdots x_{\ell_1}^1$ as the path $\psi(u_0)\psi(u_0)\psi(u_{c_j-1}^j)\psi(u_{c_j-2}^j)\cdots\psi(u_{\ell_4}^j)v_{\frac{k}{2}+b_0}$, where $\frac{k}{2}+b_0$ is performed under modulo k. Note that the edge $\psi(u_{\ell_4}^j)v_{\frac{k}{2}+b_0}$ is in the perfect matching I and is distinct from the edge $\psi(u_{\ell_2}^i)v_{\frac{k}{2}+a}$.

Case 2: $\ell_2+\ell_3 < 4$. This means that $(\ell_2, \ell_3) \in \{(1,1),(1,2)\}$, and consequently $(\ell_1,\ell_2,\ell_3,\ell_4) \in \{(1,1,1,n-3),(1,1,2,n-4)\}$. In both cases, we are going to embed $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ in K_k^* such that the image of the embedding is $(K_k^* - I) \cup \{v_0v_{\frac{k}{2}}, v_1v_{\frac{k}{2}+1}\}$: the first leg $x_0x_1^1$ is embedded as the loop v_0v_0, and the second leg $x_0x_1^2$ is embedded as the diagonal $v_0v_{\frac{k}{2}}$. If $\ell_3 = 1$, then the third leg $x_0x_1^3$ is embedded as the edge v_0v_1; if $\ell_3 = 2$, then the third leg $x_0x_1^3x_2^3$ is embedded as the path $v_0v_1v_1$. Note that v_0, v_1, $v_{\frac{k}{2}}$, and $v_{\frac{k}{2}+1}$ are all the odd degree vertices in $(K_k^* - I) \cup \{v_0v_{\frac{k}{2}}, v_1v_{\frac{k}{2}+1}\}$, while v_1 and $v_{\frac{k}{2}}$ are all the odd degree vertices in the image graph of the first three legs. Hence, if we remove the image graph of the first three legs from $(K_k^* - I) \cup \{v_0v_{\frac{k}{2}}, v_1v_{\frac{k}{2}+1}\}$, we have an Eulerian graph with ℓ_4 edges and exactly two odd degree vertices, namely v_0 and $v_{\frac{k}{2}+1}$. Therefore, we can embed the fourth leg $x_0x_1^4x_2^4\cdots x_{\ell_4}^4$ as the remaining graph of $(K_k^* - I) \cup \{v_0v_{\frac{k}{2}}, v_1v_{\frac{k}{2}+1}\}$. □

4. Edge-distinguishing chromatic numbers

4.1. Petal graphs

Theorem 4.1. *Let $c_2, c_3 \in \mathbb{N}$ such that $3 \le c_2 \le c_3$. Let $e = 1 + c_2 + c_3$ be the number of edges of P_{1,c_2,c_3}. The edge-distinguishing chromatic number of P_{1,c_2,c_3} is given by*

$$\lambda(P_{1,c_2,c_3}) = \begin{cases} 5 & \text{if } e \le 10; \\ \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil & \text{if } e \ge 11 \text{ and } \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil \text{ is odd; and} \\ \lceil \sqrt{2e} \rceil & \text{otherwise.} \end{cases}$$

Proof. Since the maximum degree of P_{1,c_2,c_3} is 6, we must have $k \geq 5$ in order to embed P_{1,c_2,c_3} in K_k^*. On the other hand, by Theorem 2.3, P_{1,c_2,c_3} can be embedded in K_5^* if $e \leq 10$. Hence, $\lambda(P_{1,c_2,c_3}) = 5$ if $e \leq 10$.

If $e \geq 11$, note that

$$5 \leq \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil \leq \left\lceil \sqrt{2e} \right\rceil \leq \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil + 1.$$

Also note that the minimum positive integer k that satisfies $\binom{k+1}{2} \geq e$ is $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$, and the minimum positive integer k that satisfies $\frac{k^2}{2} \geq e$ is $\lceil \sqrt{2e} \rceil$.

If $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$ is odd, then $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil - 1$ is even but strictly less than $\lceil \sqrt{2e} \rceil$, so $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil - 1$ does not satisfy $\frac{k^2}{2} \geq e$. Hence, by Theorems 2.3 and 3.1, $k = \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$ is the minimum positive integer such that P_{1,c_2,c_3} can be embedded in K_k^*.

If $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$ and $\lceil \sqrt{2e} \rceil$ are both even, then $\lceil \sqrt{2e} \rceil = \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$, so $\lceil \sqrt{2e} \rceil - 1$ does not satisfy $\binom{k+1}{2} \geq e$. Hence, by Theorems 2.3 and 3.1, $k = \lceil \sqrt{2e} \rceil$ is the minimum positive integer such that P_{1,c_2,c_3} can be embedded in K_k^*.

If $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$ is even but $\lceil \sqrt{2e} \rceil$ is odd, then $\lceil \sqrt{2e} \rceil = \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil + 1$ is odd and satisfies $\binom{k+1}{2} \geq e$, but $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$ does not satisfy $\frac{k^2}{2} \geq e$. Hence, by Theorems 2.3 and 3.1, $k = \lceil \sqrt{2e} \rceil$ is the minimum positive integer such that P_{1,c_2,c_3} can be embedded in K_k^*.

Finally, we finish by applying Theorem 1.5. □

4.2. Cycles with one chord

Theorem 4.2. *Let $j, n \in \mathbb{N}$ such that $2 \leq j \leq \frac{n}{2}$. Let $e = n + 1$ be the number of edges of $C_n^{\{0,j\}}$. The edge-distinguishing chromatic number of $C_n^{\{0,j\}}$ is given by*

$$\lambda(C_n^{\{0,2\}}) = \begin{cases} 4 & \text{if } e \leq 8; \\ 5 & \text{if } e = 9; \\ \left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil & \text{if } e \geq 10 \text{ and } \left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil \text{ is odd; and} \\ \left\lceil \sqrt{2e-2} \right\rceil & \text{otherwise,} \end{cases}$$

and if $j \geq 3$, then

$$\lambda(C_n^{\{0,j\}}) = \begin{cases} 4 & \text{if } e \leq 6; \\ \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil & \text{if } e \geq 7 \text{ and } \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil \text{ is odd; and} \\ \left\lceil \sqrt{2e-2} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. It is obvious that no chorded cycle can be embedded in K_3^*, so $\lambda(C_n^{\{0,j\}}) \geq 4$ for all $j, n \in \mathbb{N}$ such that $2 \leq j \leq \frac{n}{2}$.

When $j = 2$, by Theorem 3.2, $C_n^{\{0,2\}}$ can be embedded in K_4^* if and only if $e \leq 8$. Hence, $\lambda(C_n^{\{0,2\}}) = 4$ if $e \leq 8$. Also, by Theorem 2.4, $C_n^{\{0,2\}}$ can be embedded in K_5^* when $e = 9$. Hence, $\lambda(C_n^{\{0,2\}}) = 5$ if $e = 9$.

If $e \geq 10$, note that

$$5 \leq \left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil \leq \left\lceil \sqrt{2e-2} \right\rceil \leq \left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil + 1.$$

Also note that the minimum positive integer k that satisfies $\binom{k+1}{2} - 2 \geq e$ is $\left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil$, and the minimum positive integer k that satisfies $\frac{k^2}{2} + 1 \geq e$ is $\left\lceil \sqrt{2e-2} \right\rceil$.

If $\left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil$ is odd, then $\left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil - 1$ is even but strictly less than $\left\lceil \sqrt{2e-2} \right\rceil$, so $\left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil - 1$ does not satisfy $\frac{k^2}{2} + 1 \geq e$. Hence, by Theorems 2.4 and 3.2, $k = \left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil$ is the minimum positive integer such that $C_n^{\{0,2\}}$ can be embedded in K_k^*.

If $\left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil$ and $\left\lceil \sqrt{2e-2} \right\rceil$ are both even, then $\left\lceil \sqrt{2e-2} \right\rceil = \left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil$, so $\left\lceil \sqrt{2e-2} \right\rceil - 1$ does not satisfy $\binom{k+1}{2} - 2 \geq e$. Hence, by Theorems 2.4 and 3.2, $k = \left\lceil \sqrt{2e-2} \right\rceil$ is the minimum positive integer such that $C_n^{\{0,2\}}$ can be embedded in K_k^*.

If $\left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil$ is even but $\left\lceil \sqrt{2e-2} \right\rceil$ is odd, then $\left\lceil \sqrt{2e-2} \right\rceil = \left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil + 1$ is odd and satisfies $\binom{k+1}{2} - 2 \geq e$, but $\left\lceil \frac{-1+\sqrt{8e+17}}{2} \right\rceil$ does not satisfy $\frac{k^2}{2} + 1 \geq e$. Hence, by Theorems 2.4 and 3.2, $k = \left\lceil \sqrt{2e-2} \right\rceil$ is the minimum positive integer such that $C_n^{\{0,2\}}$ can be embedded in K_k^*.

Our proof of the formula for $\lambda(C_n^{\{0,2\}})$ is finished by applying Theorem 1.5.

When $j \geq 3$, by Theorem 3.2, $C_n^{\{0,j\}}$ can be embedded in K_4^* if $e \leq 6$. Hence, $\lambda(C_n^{\{0,j\}}) = 4$ if $e \leq 6$. If $e \geq 6$, note that

$$4 \leq \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil \leq \left\lceil \sqrt{2e-2} \right\rceil \leq \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil + 1.$$

Also note that the minimum positive integer k that satisfies $\binom{k+1}{2} \geq e$ is $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$, and the minimum positive integer k that satisfies $\frac{k^2}{2} + 1 \geq e$ is $\left\lceil \sqrt{2e-2} \right\rceil$. The rest of the proof is analogous to the case when $j = 2$. □

4.3. Spider graphs with four legs

Theorem 4.3. *Let $\ell_1, \ell_2, \ell_3, \ell_4 \in \mathbb{N}$ such that $\ell_1 \leq \ell_2 \leq \ell_3 \leq \ell_4$. Let $e = \ell_1 + \ell_2 + \ell_3 + \ell_4$ be the number of edges of $S_{\ell_1,\ell_2,\ell_3,\ell_4}$. The edge-distinguishing chromatic number of $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ is given by*

$$\lambda(S_{1,1,1,\ell_4}) = \begin{cases} 4 & \text{if } e \leq 10; \\ \left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil & \text{if } e \geq 11 \text{ and } \left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil \text{ is odd; and} \\ \left\lceil \sqrt{2e-4} \right\rceil & \text{otherwise,} \end{cases}$$

and if $\ell_3 \geq 2$*, then*

$$\lambda(S_{\ell_1,\ell_2,\ell_3,\ell_4}) = \begin{cases} 4 & \text{if } e \leq 10 \text{ and } \ell_1 = 1; \\ 5 & \text{if } e \leq 10 \text{ and } \ell_1 = 2; \\ \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil & \text{if } e \geq 11 \text{ and } \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil \text{ is odd; and} \\ \left\lceil \sqrt{2e-4} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. Since the maximum degree of $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ is 4, we must have $\lambda(S_{\ell_1,\ell_2,\ell_3,\ell_4}) \geq 4$ by Proposition 1.1. In other words, $k \geq 4$ is necessary to embed $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ in K_k^*.

When $\ell_3 = 1$, by Theorem 3.3, $S_{1,1,1,\ell_4}$ can be embedded in K_4^* if and only if $e \leq 10$. Hence, $\lambda(S_{1,1,1,\ell_4}) = 4$ if $e \leq 10$.

If $e \geq 11$, note that

$$5 \leq \left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil \leq \left\lceil \sqrt{2e-4} \right\rceil \leq \left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil + 1.$$

Also note that the minimum positive integer k that satisfies $\binom{k+1}{2} - 1 \geq e$ is $\left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil$, and the minimum positive integer k that satisfies $\frac{k^2}{2} + 2 \geq e$ is $\left\lceil \sqrt{2e-4} \right\rceil$.

If $\left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil$ is odd, then $\left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil - 1$ is even but strictly less than $\left\lceil \sqrt{2e-4} \right\rceil$, so $\left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil - 1$ does not satisfy $\frac{k^2}{2} + 2 \geq e$. Hence, by Theorems 2.6 and 3.3, $k = \left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil$ is the minimum positive integer such that $S_{1,1,1,\ell_4}$ can be embedded in K_k^*.

If $\left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil$ and $\left\lceil \sqrt{2e-4} \right\rceil$ are both even, then $\left\lceil \sqrt{2e-4} \right\rceil = \left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil$, so $\left\lceil \sqrt{2e-4} \right\rceil - 1$ does not satisfy $\binom{k+1}{2} - 1 \geq e$. Hence, by Theorems 2.6 and 3.3, $k = \left\lceil \sqrt{2e-4} \right\rceil$ is the minimum positive integer such that $S_{1,1,1,\ell_4}$ can be embedded in K_k^*.

If $\left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil$ is even but $\left\lceil \sqrt{2e-4} \right\rceil$ is odd, then $\left\lceil \sqrt{2e-4} \right\rceil = \left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil + 1$ is odd and satisfies $\binom{k+1}{2} - 1 \geq e$, but $\left\lceil \frac{-1+\sqrt{8e+9}}{2} \right\rceil$ does not satisfy $\frac{k^2}{2} + 2 \geq e$. Hence, by Theorems 2.6 and 3.3, $k = \left\lceil \sqrt{2e-4} \right\rceil$ is the minimum positive integer such that $S_{1,1,1,\ell_4}$ can be embedded in K_k^*.

Our proof of the formula for $\lambda(S_{1,1,1,\ell_4})$ is finished by applying Theorem 1.5.

When $\ell_3 \geq 2$, by Theorem 3.3, $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ can be embedded in K_4^* if and only if $e \leq 10$ and $\ell_1 = 1$. Hence, $\lambda(S_{\ell_1,\ell_2,\ell_3,\ell_4}) = 4$ if $e \leq 10$ and $\ell_1 = 1$. If $\ell_1 = 2$, then $e \geq 8 \geq 7$, and by Theorem 2.6, $S_{\ell_1,\ell_2,\ell_3,\ell_4}$ can be embedded in K_5^* if $e \leq 10$. Hence, $\lambda(S_{\ell_1,\ell_2,\ell_3,\ell_4}) = 5$ if $e \leq 10$ and $\ell_1 = 2$.

If $e \geq 11$, note that

$$5 \leq \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil \leq \left\lceil \sqrt{2e-4} \right\rceil \leq \left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil + 1.$$

Also note that the minimum positive integer k that satisfies $\binom{k+1}{2} \geq e$ is $\left\lceil \frac{-1+\sqrt{8e+1}}{2} \right\rceil$, and the minimum positive integer k that satisfies $\frac{k^2}{2} + 2 \geq e$ is $\left\lceil \sqrt{2e-4} \right\rceil$. The rest of the proof is analogous to the case when $\ell_3 = 1$. □

5. Concluding remarks and future work

When the authors determined the EDCN of spider graphs with three legs [11], the focus was only on one graph, making the approach restrictive. In this paper, we consider a variety of graphs; in particular, petal graphs serve as an intermediate step when chorded cycles and spider graphs are embedded in K_k^*, which makes the proof easier to navigate. Moreover, this new approach allows us to solve all suggested problems listed in the aforementioned paper except the conjecture concerning caterpillar trees.

Although the embedding of caterpillar trees in K_k^* still eludes us when k is even, we are able to show that the trivial necessary conditions for embedding certain caterpillar trees in K_k^* are also sufficient when k is odd. Hence, one major future work direction is to continue our study of caterpillar trees.

Here is a list of other potential future projects on EDCN.

1. General petal graphs with three or four petals;
2. Spider graphs with more than 4 legs;
3. Ladder graphs.

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