On 2-power unicyclic cubic graphs

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Abstract

In a graph, a cycle whose length is a power of two (that is, 2^k) is called a 2-power cycle. In this paper, we show that the existence of an infinite family of cubic graphs which contain only one cycle whose length is a power of 2. Such graphs are called as 2-power unicyclic cubic graphs. Further we observe that the only 2-power cycle in a cubic graph cannot be removed implying that there does not exist a counter example for Erdős-Gyárfás conjecture.

1. Introduction

In a graph G, a 2-power cycle is a cycle whose length is a power of 2. A graph which contains a unique 2-power cycle is called a 2-power unicyclic graph. In 1995, Erdős and Gyárfás [4] put forward a conjecture which states that every graph with minimum degree 3 contains a simple cycle whose length is a power of two. If the conjecture is false, a counter example would take the form of a graph with minimum degree three having no cycles whose length is a power of two. It is known through computer searches of Gordon Royle and Klas Markstrom that any counterexample must have at least 17 vertices, and any cubic counter example must have at least 30 vertices. Markstrom's searches found four graphs on 24 vertices in which the only power-of-two cycles have 16 vertices, one of these four graphs is planar. However, the Erdős-Gyárfás conjecture is now known to be true

for the special case of 3-connected cubic planar graphs, see Heckman and Krakovski [6]. Weaker results relating the degree of a graph to unavoidable sets of cycle lengths are known. Verstraete [12] showed that there is a set S of lengths, with $|S| = O(n^{0.99})$, such that every graph with average degree ten or more contains a cycle with its length in S. Sudakov and Verstraete [11] proved that every graph whose average degree is exponential in the iterated logarithm of n necessarily contains a cycle whose length is a power of two. Daniel and Shauger [2] proved the conjecture to be true for planar claw-free graphs and Shauger [10] showed the conjecture to be true for graphs that avoid large induced stars and satisfy additional constraints on their degrees. In 2004, Klas Markstrom [8] provided extremal graphs for some problems on cycles in graphs.

A graph is planar if it can be embedded in the plane without crossing edges. A plane graph is an embedded planar graph. A graph G is 3-connected if it becomes disconnected by removing at least 3 vertices. A graph G is cubic if every vertex of G is of degree three. More on graphs and cycles can be seen in [3, 7, 9].By computer searches, Markstrom [6] verified Erdős-Gyárfás conjecture for cubic graphs of order at most 29, and found that the smallest cubic planar graph with no 4- or 8-cycles has 24 vertices (see Figure 1). Note that this graph contains a 16-cycle. Shauger [8] proved the conjecture for $K_{1,m}$ free graphs of minimum degree at least $m + 1$ or maximum degree at least $2m - 1$. Every 3-connected cubic planar graph contains a 2^m- cycle, for some $2 \leq m \leq 7$, see [6].
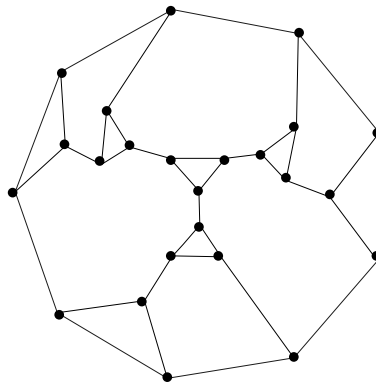

Figure 1. cubic graph with 24 vertices with no 4- or 8-cycle.

Markstrom's 24-vertex cubic planar graph with no 4 or 8 cycles, found in a computer search for counter example to the Erdős-Gyárfás conjecture, has cycles with 16 vertices. A recent study by Bensmail [1] showed that their exist arbitrarily large cubic graphs with no q-power cycle. Frank [5] showed that every k-regular k-connected graph on n vertices has k-diameter at most $\frac{n}{2}$ and this upper bound cannot be improved when $n = 4k - 6 + I(2k - 4)$. In particular, the maximal 3-diameter of 3-regular graphs with $2n$ vertices is equal to n, that is, the maximal 3-diameter of 3-regular graphs with n vertices is equal to $\frac{n}{2}$.

In a graph, the distance between the two vertices is the number of edges in a shortest path (also called a graph geodesic) connecting them. This is also known as the geodesic distance. Notice that there may be more than one shortest path between two vertices. If there is no path connecting the two vertices, that is, if they belong to different connected components, then conventionally the distance is defined as infinite.

2. On two power unicyclic cubic graphs

In a graph G, if the distance between the vertices u and v is $d(u, v) = r$, the number of vertices between u and v is $r + 1$. If G is a cubic graph of order n and e is any edge of G, then $G - e$ is the graph in which $n - 2$ vertices are of degree 3 and 2 vertices are of degree 2, such a graph is called an $(n - 2)$-cubic graph.
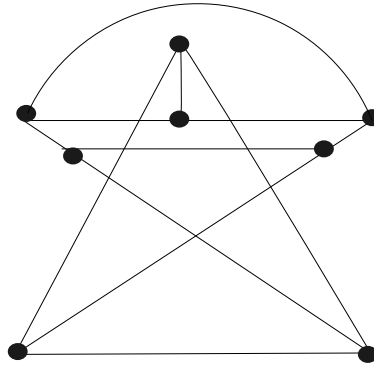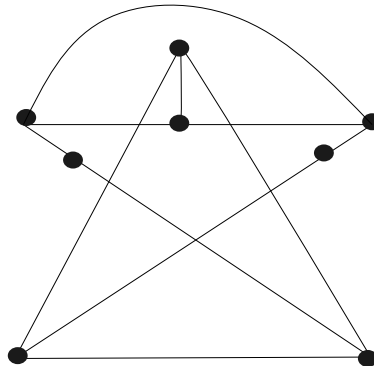

Figure 2. Cubic graph with eight vertices.

Figure 3. 6-cubic graph, that is, graph with eight vertices with six vertices of degree 3 and two vertices of degree 2.

The graph G, shown in Figure 2, has eight vertices and contains cycles of length 4, 5, 6, 7 and 8. Now remove the edge $e = uv$ from G, we get the graph $K = G - e$, as shown in Figure 3. Clearly K is $(n - 2)-$cubic and has no cycles of lengths $2^2, 2^3$. Further the distance between the vertices u and v in K is $d_K(u, v) = 3 < 4$.

We will show that the above example leads to the existence of a family of 2-power unicyclic cubic graphs, that is, a family of cubic graphs which contain only the cycle of length 2^k ($2^k < n$) but does not contain any cycle of length 2^t, $t \neq k$. We show the existence of this graph G by construction, the graph G constructed will have even number of vertices, say $n = 2s$, and further in G there is always a bridge which partitions $V(G)$ into two parts with each part having s vertices. So if G contains a cycle of length 2^k, then $2^k < s$, since the presence of bridge does not allow the cycle to have edges from both the parts, that is, contains edges from one of the two parts.

Theorem 2.1. *There exists a family of 2-power unicyclic cubic graphs, that is a family of cubic graphs with order say $n = 2s$, which contain a cycle of length 2^k ($2^k < s$) but does not contain any cycle of length 2^t, $t \neq k$.*

Proof. We show the existence of such a family by construction. We start with the construction of the smallest such graph.

Consider the graph G of order 8 as shown in Figure 2. We remove the edge $e = uv$ to get the graph $K = G - e$ as shown in Figure 3. We note that $d_K(u) = 2$ and $d_K(v) = 2$ so that K is (6)-cubic. Further $d_H(u, v) = 3 < 4$. Take 5 copies of K and name them as H_1, H_2, J_1, J_2 and H. Also we represent the vertices of degree 2 by $u_1, v_1 \in H_1$; $u_2, v_2 \in H_2$; $x_1, y_1 \in J_1$; $x_2, y_2 \in J_2$ and $u, v \in H$. Further, let $P_1 = H_1 \cup H_2$ and $Q_1 = J_1 \cup J_2$.

Now, let X_1 be the graph consisting of P_1, Q_1, H, K_3 (vertices labeled as w_1, w_2, w_3) and $K_{1,3} + x$ (vertices labeled as z_1, z_2, z_3, z_4 with z_1 as the pendant vertex) together with the edges $v_1 u_2, y_1 x_2, v_2 w_1, y_2 w_2, w_3 v, u z_1, u_1 z_3, x_1 z_4$. Clearly degree of each vertex of X_1 is three except the vertex z_1 whose degree is two.

We now take two copies of X_1, one named as X_1 and another as X_1'. Label the vertices of X_1' as u_1', v_1' and so on. Consider the graph $G_1 = X_1 \cup X_1' \cup \{z_1 z_1'\}$. Clearly the graph G_1 is cubic.
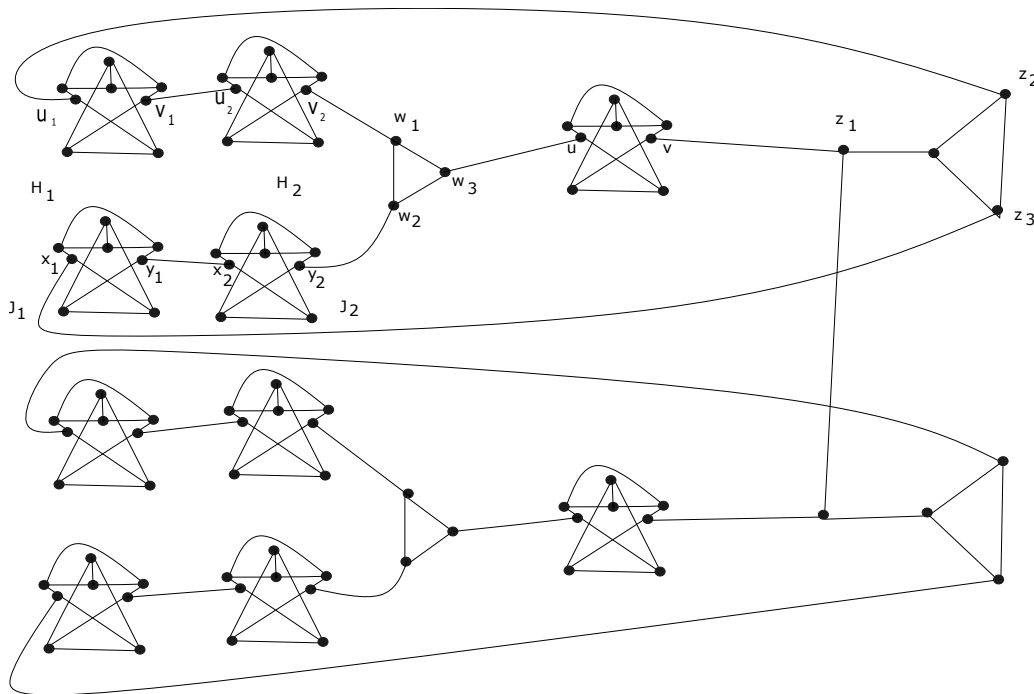

Figure 4. Cubic graph G_1 with 94 vertices, having no cycles of length 4, 8, 16 but has a cycle of length 32.

We observe that the edge $z_1 z_1'$ is a bridge in G_1 (joining X_1 and X_1'). Also the number of vertices in each of X_1 and X_1' is 47. So the maximum length of the cycle in G_1 has to be less or equal to 47 and thus G_1 can have only $2^2, 2^3, 2^4, 2^5$ cycles of the form 2^k. But as we see that G_1 does not contain the cycles of the lengths $2^2, 2^3, 2^4$. Evidently G_1 contains the cycle of length 2^5. Note that G_1 cannot have a cycle of length $2^6 = 64$, because of the presence of bridge $z_1 z_1'$

between X_1 and X'_1. Hence we have constructed a cubic graph G_1 of order 84 in which there are no cycles of lengths $2^2, 2^3, 2^4$ but it contains a cycle of length of 2^5. See Figure 4.

The above construction can be extended to obtain a family of cubic graphs in the following way. Let P_i contain $\sum_{j=1}^{i} 2^j$ copies of K and name these copies as $H_1, H_2, H_3 \dots$. Let $P_i = H_1 \cup H_2 \cup H_3 \cup \dots$. Similarly let Q_i contain $\sum_{j=1}^{i} 2^j$ copies of K and name these copies as $J_1, J_2, J_3 \dots$. Let $Q_i = J_1 \cup J_2 \cup J_3 \cup \dots$. Label the vertices of degree 2 in $H_1, H_2, H_3 \dots$, $J_1, J_2, J_3 \dots$ respectively by $u_1, v_1 \in H_1$; $u_2, v_2 \in H_2$; $u_3, v_3 \in H_3$; $\dots$; $x_1, y_1 \in J_1$; $x_2, y_2 \in J_2$; $x_3, y_3 \in J_3 \dots$.

Let X_i be the graph containing P_i, Q_i, H (a copy of K), K_3 (vertices labeled as w_1, w_2, w_3) and $K_{1,3} + x$ (vertices labeled as z_1, z_2, z_3, z_4 with z_1 as the pendant vertex) together with the edges $v_1 u_2, v_2 u_3, \dots$; $y_1 x_2, y_2 x_3, \dots$; $w_3 v, u z_1, u_1 z_3, x_1 z_4$; the edge between the vertex w_1 and the second vertex of the last copy $K \in P_i$; the edge between w_2 and the second vertex of the last copy $K \in Q_i$. Clearly degree of each vertex of X_i is three except the vertex z_1 whose degree is two. Now, take two copies of X_i, represented as X_i and X'_i.

Consider the graph $G_i = X_i \cup X'_i \cup \{z_1 z'_1\}$, which is cubic. We observe that the edge $z_1 z'_1$ is a bridge in G_i (joining X_i and X'_i). Also the number of vertices in each of X_i and X'_i is $|X_i| = |P_i| + |Q_i| + |H| + |K_3| + |K_{1,3} + x| = 2|P_i| + 8 + 3 + 4 = 2(8 \sum_{j=1}^{i} 2^j) + 15$, since each copy of K contains eight vertices. Thus the length of the largest cycle has to be less or equal to $2(8 \sum_{j=1}^{i} 2^i) + 15 = 2^4(2 + 2^2 + 2^3 + \dots + 2^{i-1} + 2^i) + 15$. Further, we note that $|X_i| = |X_{i-1}| + 2^{i+1}$. If $|G_i|$ is the order of G_i, it can be easily seen that $|G_i| = |G_{i-1}| + 2^5$.

Evidently, G_i does not contain the cycles of lengths $2^2, 2^3, 2^4, \dots, 2^{i+3}$ but contains the cycle of length 2^{i+4}. Also G_i does not contain the cycles of length 2^t, $t \geq i + 5$, since the length of the largest cycle is less or equal to $2^4(2 + 2^2 + 2^3 + \dots + 2^{i-1} + 2^i) + 15$. Hence G_i contains only one cycle whose length is a power of 2. $\qquad \square$

For instance, G_2 is a cubic graph of order 222 having no cycles of lengths $2^2, 2^3, 2^4, 2^5$ but it contains a cycle of length 2^6, see Figure 5. This is because the graph has a bridge and each part has 111 vertices. The minimum length of the cycle is 33 and total vertices in each part is 111. Thus, the only cycle whose length is of the form 2^k with $33 \leq 2^k \leq 111$ is 64.

Similarly, G_3 is a cubic graph of order 478 having no cycles of lengths $2^2, 2^3, 2^4, 2^5, 2^6$ but it contains a cycle of length 2^7. This is illustrated in Figure 6. This graph has a bridge, so each part has 239 vertices and minimum length of a cycle is 65 and total vertices in each part is 239. Therefore, the only cycle of the form 2^k with $65 \leq 2^k \leq 239$ is 128.

From the above discussion and examples, we get a recurrence relation of these cubic graphs as follows.

Let us assume that there exists a cubic graph G which contains a cycle of length 2^k only but does not contain cycles of length 2^t, $t \neq k$. In particular, G will have no cycles of length 2^t, $2 \leq t \leq k - 1$. Thus the number of vertices in this graph is at least 2^k, that is, $n \geq 2^k$, where n is the total number of vertices in the graph. Since it has no cycle of length 2^{k-1}, this implies that an edge was removed in the graph between vertices u and v (say) which has removed all the cycles of length 2^t, $2 \leq t \leq k - 1$. Thus $d(u, v) = 2^{k-1} - 1$. As $2^{k-1} < 2^k$, implies that $2(2^{k-1}) = 2^k$, or $2(2^{k-1} - 1) < 2^k$, or $2^{k-1} - 1 < \frac{2^k}{2}$ or $2^{k-1} - 1 < \frac{n}{2}$, where $n \geq 2^k$. Thus, $d(u, v) < \frac{n}{2}$, which negates the above observation that each cubic graph with degree greater or equal to 3 has one cycle
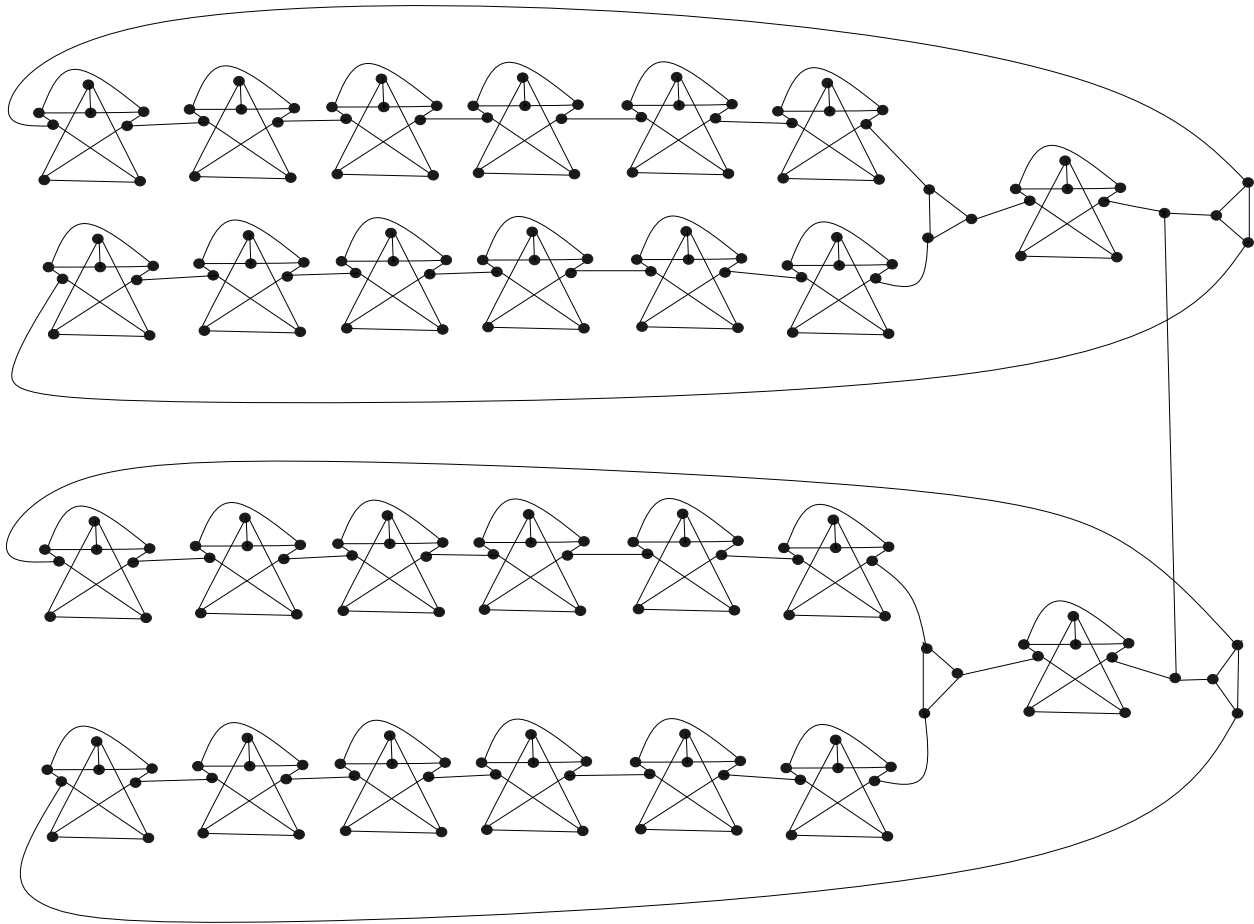
Figure 5. Cubic graph G_2 with 222 vertices, having no cycles of length 4,8,16,32 but has a cycle of length 64.

Table 1. Recurrence relation for the cubic graphs

Graph	Cycles of the form 2^t removed	Cycle of the form 2^k present.
G_1 with $|G_1| = 94$	$2^2, 2^3, 2^4$	$2^{4+1} = 2^5$
G_2 with $|G_2| = |G_1| + 2^6$	$2^2, 2^3, 2^4, 2^5$	$2^{4+2} = 2^6$
G_3 with $|G_3| = |G_2| + 2^7$	$2^2, 2^3, 2^4, 2^5, 2^6$	$2^{4+3} = 2^7$
G_4 with $|G_4| = |G_3| + 2^8$	$2^2, 2^3, 2^4, 2^5, 2^6, 2^7$	$2^{4+4} = 2^8$
G_5 with $|G_5| = |G_4| + 2^9$	$2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8$	$2^{4+5} = 2^9$
G_6 with $|G_6| = |G_5| + 2^{10}$	$2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9$	$2^{4+5} = 2^{10}$
G_i with $|G_i| = |G_{i-1}| + 2^{i+4}$	$2^2, 2^3, 2^4, 2^5, \cdots, 2^{i+2}, 2^{i+3}$	2^{i+4}

of length 2^k. Accordingly, there is no cubic graph in which we can develop a counter case with no cycle of power 2^k. This way we conclude that every graph with minimum degree 3 contains a cycle whose length is a power of two.
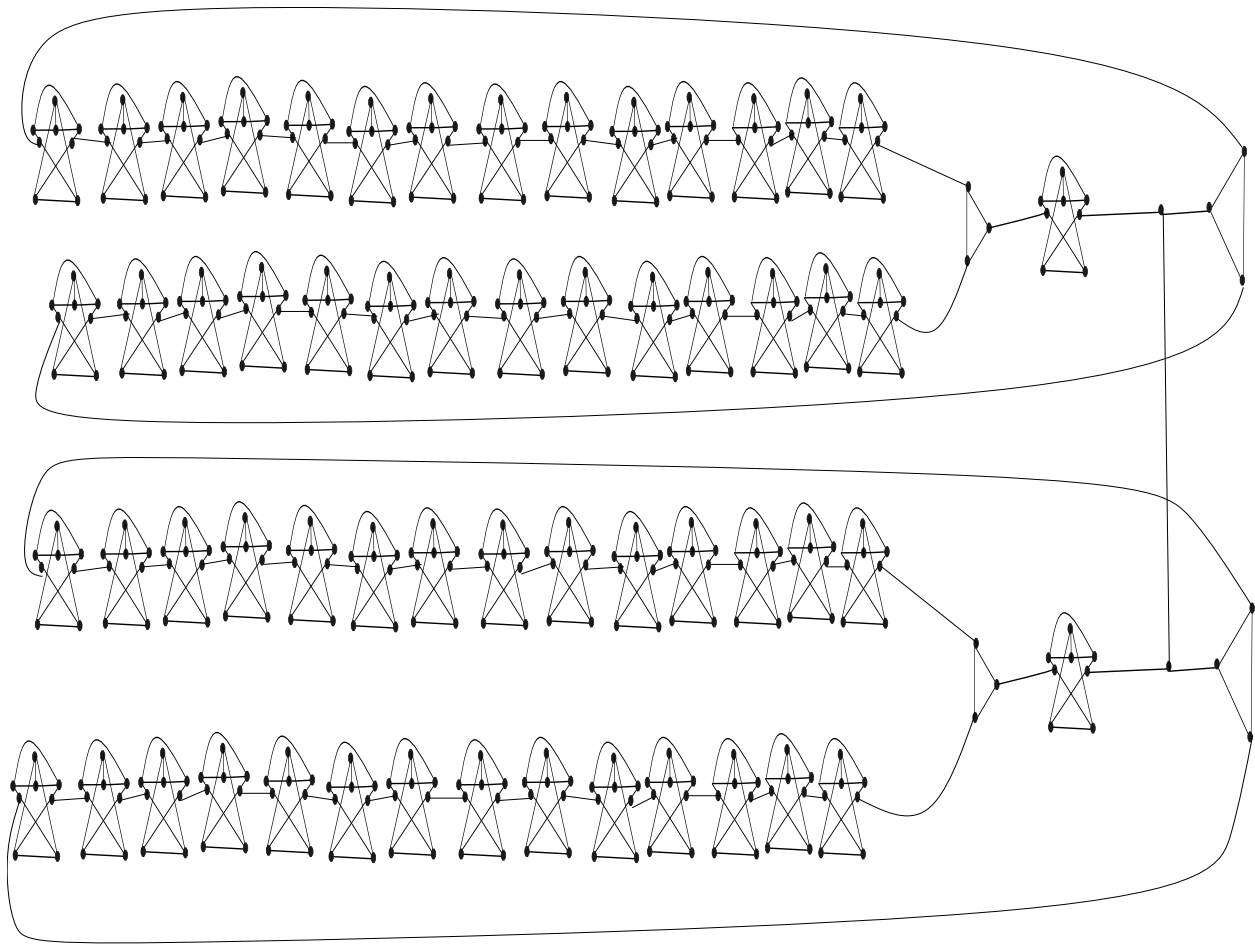
Figure 6. Cubic graph G_3 with 478 vertices, having no cycles of length 4, 8, 16, 32, 64 but has a cycle of length 128.

3. Conclusion

The Erdős-Gyárfás conjecture is one of the important problems in graph theory. The problem has been partially proved to be true. Daniel and Shauger proved the conjecture to be true for planar claw-free graphs and Shauger showed the conjecture to be true for graphs that avoid large induced stars and satisfy additional constraints on their degrees.

The main aim of this paper is to show that counter example for the conjecture is not possible. For this, we first prove the existence of an infinite family of cubic graphs which contain only one cycle whose length is a power of 2. We show the existence of a family of cubic graphs in which all the cycles of the form 2^t, where $2 \leq t \leq k - 1$, can be removed and does contain only a cycle of order 2^k, $2^k < n$. Such graphs are called as 2-power unicyclic cubic graphs. Further, we observe that the only 2-power cycle in a cubic graph cannot be removed implying that there does not exist a counter example for Erdős-Gyárfás conjecture.

Acknowledgement

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