Electronic Journal of Graph Theory and Applications **4** (2) (2016), 119–131

Electronic Journal of
Graph Theory and Applications

New attack on Kotzig's conjecture

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Abstract

In this paper we study a technique to transform α-labeled trees into ρ-labeled forests. We use this result to prove that the complete graph K_{2n+1} can be decomposed into these types of forests. In addition we show a robust family of trees that admit ρ-labelings, we use this result to describe the set of all trees for which a ρ-labeling must be found to completely solve Kotzig's conjecture about decomposing cyclically the complete graph K_{2n+1} into copies of any tree of size n.

Keywords: decomposition, α-labeling, ρ-labeling, caterpillar
Mathematics Subject Classification : Primary 05C70, 05C78; Secondary 05C05
DOI:10.5614/ejgta.2016.4.2.1

1. Motivation

For any graph G, an H-*decomposition* of G is a partition of $E(G)$ into edge-disjoint subgraphs isomorphic to H. If such a decomposition exists, G is said to be H-*decomposable*. A *decomposition* of K_n is an edge-disjoint decomposition, that is, a system R of subgraphs such that any edge of K_n belongs to exactly one of the subgraphs of R.

An open problem in this area comes from the conjecture proposed by Ringel [9] in 1963.

Conjecture 1. *(Ringel's Conjecture) If T is a given tree of size n, then the complete graph K_{2n+1} is edge-decomposable into n copies of T.*

Suppose that the vertices of K_n have been labeled with the integers $0, 1, ..., n-1$. Let $ij \in E(K_n)$, a *turning of the edge* ij is the increase of both labels by one, that is, the edge $(i+1)(j+1)$, where the addition is taken modulo n. A *turning of a subgraph* G of K_n is the simultaneous turning

Received: 14 February 2015, Revised: 10 March 2016, Accepted: 25 August 2016.

of all the edges in G. A decomposition R of K_n is said to be *cyclic* if the following holds: If R contains a graph G, then it also contains the graph G' obtained by turning G.

Using the concept of cyclic decomposition, in 1973, Kotzig [6] stated the following variation of Ringel's conjecture:

Conjecture 2. *(Kotzig's Conjecture) The complete graph K_{2n+1} can be cyclically decomposed into $2n+1$ subgraphs isomorphic to a given tree with n edges.*

Both conjectures are related to the Graceful Tree Conjecture (GTC) and to the ρ-Conjecture, which are as follows.

Conjecture 3. *(GTC) Every tree has a graceful labeling.*

Conjecture 4. *(ρ-Conjecture) Every tree has a ρ-labeling.*

It is well-known that the Graceful Tree Conjecture implies the ρ-Conjecture, which is equivalent to Kotzig's Conjecture, and this one implies Ringel's Conjecture.

The labeling-decomposition relationship is discussed in Section 2, where we introduce the notation and the known results that are used throughout the entire paper. In Section 3 we transform α-labeled graphs into ρ-labeled disconnected graphs, in particular we prove that when the k components of a forest are α-trees and one of these components is a caterpillar of size at least $k-2$, then the forest admits a ρ-labeling. Section 4 is dedicated to the study of ρ-labelings of trees, there we prove that almost all trees admit a ρ-labeling. Furthermore, we determine the exact characteristics of the graphs that need to be proven to admit ρ-labelings in order to prove Kotzig's Conjecture.

All graphs considered in this work are simple graphs, i.e., finite, with no loops or multiple edges. We mainly follow the notation used in [3] and [4].

2. The First Results

In 1966, Rosa [10] introduced, under the name "vertex valuations", a tool to attack Ringel's conjecture. Since then, a substantial amount of papers dealing with vertex valuations have been published, many of these works focus on specific families of trees; others are devoted to more general results. For a detailed and updated account of the results in the area of graph labeling, the reader is refered to Gallian' survey [4].

A *difference vertex labeling* of a graph G of size n is an injective mapping f from $V(G)$ into a set N of nonnegative integers, such that every edge uv of G has assigned a *weight* defined by $|f(u) - f(v)|$. All labelings considered in this work are difference vertex labelings. Rosa [10] defined four of these labelings. We present here three of them.

By $[a, b]$ we mean the set of all integers k such that $a \leq k \leq b$. A labeling f of G is said to be a *β-labeling* if $N = [0, n]$ and the set of weights is $[1, n]$. Golomb [5] used the word *graceful* to refer to a labeling of this type; this is the name more frequently used. A β-labeling of G is an *α-labeling* if there exists a number λ, called its *boundary value*, such that for an arbitrary edge uv of G, either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$. A labeling f of G is said to be a *ρ-labeling* if $N = [0, 2n]$ and the set of weights is $\{x_1, x_2, ..., x_n\}$ where $x_i = i$ or $x_i = 2n + 1 - i$. We say that G is an *α-graph* (or β- or ρ-) when G admits an α-labeling (or β- or ρ-).

Rosa observed that there is a hierarchy within these labelings: α, β, ρ, with α being the most restrictive type, i.e., $\alpha \Rightarrow \beta \Rightarrow \rho$. He also observed that if G is an α-graph, then G is bipartite, and if G is a β-graph of order m and size n, then $m - n \leq 1$. As a consequence of this last observation we have that a forest with $k > 1$ components is not a β-graph. Hence, the study of ρ-labelings of forests is relevant in the context of the conjectures mentioned before.

Let G be a bipartite graph where $\{A, B\}$ is the bipartition of $V(G)$. A *bipartite labeling* of G is an injection $f : V(G) \to \{0, 1, ..., t\}$ for which there is an integer λ, named the *boundary value* of f, such that $f(u) \leq \lambda < f(v)$ for every $(u, v) \in A \times B$. This is just an extension of the definition given by Rosa and Širáň in [11]; there, they focussed on bipartite labelings of trees. From the definition we may conclude that $t \geq |V(G)| + 1$, furthermore, the labels assigned by f on the vertices of A and B are in the intervals $[0, \lambda]$ and $[\lambda + 1, t]$, respectively. The most studied labeling of this type is the α-labeling, where $t = |E(G)|$.

The following definition was introduced in 1982 independently by Slater [12] and Maheo and Thuillier [8]. Let $d \geq 1$ be an integer and G be a graph of size n. A labeling $\phi : V(G) \to [0, d+n-1]$ is a d-graceful labeling of G when the set of induced weights is $[d, d+n-1]$. Thus, graceful labelings are 1-graceful. They showed that any α-labeling of a graph G of size n can be transformed into a d-graceful labeling. In fact, let f be an α-labeling of G with boundary value λ, then for any $d \geq 1$, the labeling ψ given by

$$\psi(v) = \begin{cases} f(v) & \text{if } f(v) \leq \lambda, \\ f(v) + d - 1 & \text{if } f(v) > \lambda. \end{cases}$$

is a d-graceful labeling of G.

Suppose that we apply this procedure to an α-labeling f of a tree T of size n. Hence, the set of labels assigned by ψ on the vertices of T is $[0, \lambda] \cup [\lambda + d, d + n - 1]$, and the set of induced weights is $[d, d + n - 1]$. We refer to this procedure as amplification, i.e., ψ is an *amplification* of f.

Remark 2.1. The reason why this amplification works is the bipartite nature of α-labelings; in other terms, the argument presented above works on any bipartite labeling.

If f is a labeling of a graph G of size n, a *shifting* of ε units of f is the labeling g defined for every $v \in V(G)$ as $g(v) = f(v) + \varepsilon$. Since the constant ε is added to every vertex of G, the weight of $uv \in E(G)$ is the same under both labelings, f and g.

Therefore, when an α-labeling of a tree T of size n, with boundary value λ, is transformed into a d-graceful labeling shifted ε units, the set of labels is $[\varepsilon, \varepsilon + \lambda] \cup [\varepsilon + \lambda + d, \varepsilon + d + n - 1]$ and the set of weights is $[d, d + n - 1]$.

The following results (Theorems 7 and 8 in [10]) show the connection between these labelings and cyclic decompositions.

Theorem 2.1. *A cyclic decomposition of K_{2n+1} into subgraphs isomorphic to a given graph G of size n exists if and only if there exists a ρ-labeling of the graph G.*

Theorem 2.2. *If a graph G of size n has an α-labeling, then there exists a cyclic decomposition of K_{2kn+1} into subgraphs isomorphic to G, where k is an arbitrary natural number.*

In the following sections we use α-graphs to produce new ρ-graphs, which can be used together with Theorem 2.1 to prove the existence of cyclic decompositions of K_{2n+1}.

3. Using α-Graphs to Produce ρ-Graphs

Based on the hierarchy of Rosa's labelings, we know that every α-graph is automatically a ρ-graph. Kotzig [6] showed that nearly all trees are α-trees. However forests with more than one component fail to have α- or β-labelings, because they have "too many" vertices. In this section we explore a technique to produce ρ-labelings of forests whose components are α-trees.

Caro, Roditty, and Schőnheim [2] asked the following question: If H is a connected graph of size n having a ρ-labeling and G is a new graph of size n constructed by breaking H up into disconnected parts, does G also have a ρ-labeling? The results in this section partially answer this question.

Theorem 3.1. *Let T_1 and T_2 be two α-trees. If there exists an α-labeling of T_2 such that one of the end-vertices of the edge of weight 1 is a leaf, then $F = T_1 \cup T_2$ is a ρ-graph.*

Proof. Let T_1 and T_2 be two α-trees of sizes n_1 and n_2 which have α-labelings f_1 and f_2 with boundary values λ_1 and λ_2, respectively. Suppose that f_2 assigns the weight 1 on an edge $xy \in E(T_2)$ where at least one of x or y is a leaf.

Let g be the labeling of the forest $F = T_1 \cup T_2$ obtained by modifying f_1 and f_2 in the following form:

$$g(v) = \begin{cases} f_1(v) & \text{if } v \in V(T_1) \text{ and } f_1(v) \leq \lambda_1, \\ f_1(v) + n_1 + 2n_2 & \text{if } v \in V(T_1) \text{ and } f_1(v) > \lambda_1, \\ f_2(v) + \lambda_1 + 1 & \text{if } v \in V(T_2) \text{ and } f_2(v) \leq \lambda_2, \\ f_2(v) + n_1 + n_2 + \lambda_1 & \text{if } v \in V(T_2) \text{ and } f_2(v) > \lambda_2. \end{cases}$$

The labeling g restricted to T_1 is just an amplification of f_1, that is, g_{T_1} is a $(n_1 + 2n_2 + 1)$-graceful labeling of T_1. Thus, the labels assigned by g on the vertices of T_1 form the set $[0, \lambda_1] \cup [n_1 + 2n_2 + \lambda_1 + 1, 2n_1 + 2n_2]$, and the induced weights form the set $[n_1 + 2n_2 + 1, 2n_1 + 2n_2]$.

On the other side, when g is restricted to T_2, the situation is a little more complex, g_{T_2} is a $(n_1 + n_2)$-graceful labeling of T_2 shifted $\lambda_1 + 1$ units. Hence, the labels assigned by g on the vertices of T_2 form the set $[\lambda_1 + 1, \lambda_1 + \lambda_2 + 1] \cup [n_1 + n_2 + \lambda_1 + \lambda_2 + 1, n_1 + 2n_2 + \lambda_1]$, and the weights induced form the set $[n_1 + n_2, n_1 + 2n_2 - 1]$.

Summarizing, the labels assigned by g on the vertices of F form the set $[0, \lambda_1 + \lambda_2 + 1] \cup [n_1 + n_2 + \lambda_1 + \lambda_2 + 1, 2n_1 + 2n_2]$, while the induced weights form the set $[n_1 + n_2, 2n_1 + 2n_2] - \{n_1 + 2n_2\}$. When these weights are considered under the definition of a ρ-labeling, we obtain the weights $1, 2, ..., n_1, n_1 + 2, n_1 + 3, ..., n_1 + n_2, n_1 + n_2$. So, the weight $n_1 + n_2$ is obtained twice while the number $n_1 + 1$ is not in this list, unless that $n_2 = 1$; if that is the case, g is a ρ-labeling of F. If $n_2 > 1$, let $xy \in E(F)$ such that $g(x) = \lambda_1 + \lambda_2 + 1$ and $g(y) = n_1 + n_2 + \lambda_1 + \lambda_2 + 1$. When $\deg(x) = 1$, $g(x)$ is redefined to be $g(x) = \lambda_1 + \lambda_2 + n_2$. When $\deg(y) = 1$, $g(y)$ is redefined to be $g(y) = \lambda_1 + \lambda_2 + n_1 + 2$. In either case, the new weight of xy is $n_1 + 1$. Since none of the possible new labels have been assigned before, the labeling g is a ρ-labeling of F. $\square$

In Figure 1 we show a complete example, exhibiting the α-labelings of T_1 and T_2, together with the corresponding ρ-labeling of $F = T_1 \cup T_2$.

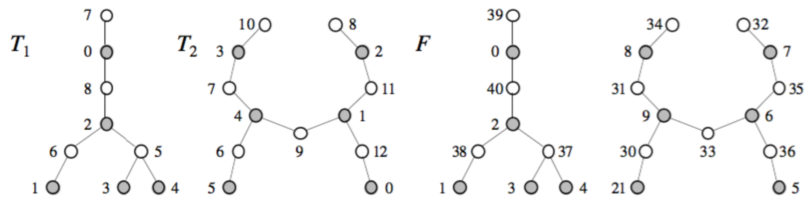

Figure 1. ρ-labeling of a forest with two α-components

Note that in the previous theorem, we can change the trees for two α-graphs, where G_2 has at least one leaf and an α-labeling that assigns λ_2 or $\lambda_2 + 1$ on a leaf.

As a consequence of this last theorem and Theorem 2.1 we have the following corollary.

Corollary 3.1. *Let $F = T_1 \cup T_2$, where T_1 and T_2 are two α-trees. If there exists an α-labeling of T_2 such that one of the end-vertices of the edge of weight 1 is a leaf, then K_{2n+1} is F-decomposable, where n is the size of F.*

Within the proof of the next result we use what we call the *two row planar representation* of caterpillars. In this representation the vertices of each stable set of the caterpillar are placed in a row in such a way that there is no edge crossings. In Figure 2 we show an example of a caterpillar of size 13 represented in this way. In addition, we introduce an order in the set of edges, in such a way that the edge i is placed to the left of the edge $i+1$. This idea is strongly related to the π-representation of α-graphs given by Kotzig [6].

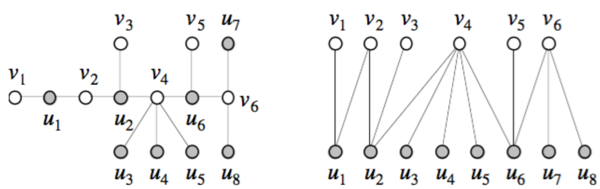

Figure 2. Two row representation of a caterpillar

Lemma 3.1. *Let $W = \{w_1, w_2, ..., w_n\}$ be a set of n positive integers, with $w_i > w_{i+1}$, for every $1 \le i \le n-1$. If T is a caterpillar of size n, then there exists a bipartite labeling $f : V(T) \to [0, w_1]$ such that the set of induced weights is W.*

Proof. Suppose that T is drawn using the two row planar representation. We define $f(u_1) = 0$ and $f(v_1) = w_1$, thus the edge 1 has weight w_1. Because T is connected, for every $2 \le i \le n$, the edge i shares a vertex with the edge $i-1$. Let u and v be the end vertices of the edge i; if u is the vertex shared with the edge $i-1$, then $f(u)$ is known and $f(v) = f(u) + w_i$; if v is the vertex shared with the edge $i-1$, then $f(v)$ is known and $f(u) = f(v) - w_i$. Either way, the weight of the edge i is w_i. Since the sequence $\{w_i\}_{i=1}^n$ is strictly decreasing, the sequence formed by the labels assigned to the u vertices is strictly increasing while the sequence formed by the labels assigned to the v vertices is strictly decreasing. Given the nature of this labeling, the labels used on the u vertices are always smaller than the labels used on the v vertices. Thus, f is a bipartite labeling of the vertices of T, that assigns labels from $[0, w_1]$ and induces the weights $w_1, w_2, ..., w_n$. □

In Figure 3 we show an example of this labeling for the caterpillar in Figure 2, where the set of prescribed weights is $W = \{29, 27, 24, 23, 20, 15, 12, 11, 8, 6, 5, 2, 1\}$.

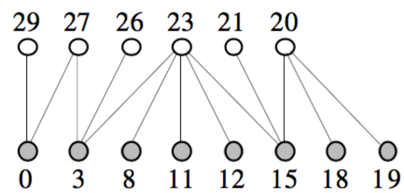

Figure 3. Labeled caterpillar with prescribed weights

Lemma 3.1 can be used to extend Theorem 3.1 by proving the existence of a ρ-labeling of any disconnected graph formed by $k-1$ components that admit α-labelings and a caterpillar of size at least $k-2$. In particular, this theorem proves that any forest of α-trees admits a ρ-labeling when one component is a caterpillar with a suitable size.

Theorem 3.2. *Let $G_1, G_2, ..., G_k$ be a collection of α-graphs where G_k is a caterpillar of size at least $k-2$. Then, the graph $G = \bigcup_{i=1}^k G_i$ admits a ρ-labeling.*

Proof. For every $1 \leq i \leq k-1$, let G_i be an α-graph of size n_i and f_i an α-labeling of G_i with boundary value λ_i. Let G_k be a caterpillar of size $n_k \geq k-2$.

Suppose that $N_0 = 0$ and $N_i = \sum_{j=1}^i n_j$, $N = N_k$, $\Lambda_0 = 0$ and $\Lambda_i = \sum_{j=1}^i \lambda_j$, and $d_i = 2N + 2 - N_i - i$, for every $1 \leq i \leq k$. When $k = 1$, $G = G_1$ which is an α-graph, therefore G is a ρ-graph. Suppose that $k \geq 2$; for every $1 \leq i \leq k-1$, the labeling f_i of G_i is transformed into a d_i-graceful labeling, denoted by g_i, shifted $\Lambda_{i-1} + i - 1$ units. In this way, the labels assigned by g_i to the vertices of G_i are in the set

$$[\Lambda_{i-1} + i - 1, \Lambda_i + i - 1] \cup [2N - N_i + \Lambda_i + 1, 2N - N_{i-1} + \Lambda_{i-1}],$$

while the set of weights induced by g_i on the edges of G_i is

$$[d_i, d_i + n_i - 1].$$

Thus, the labels used on $G - G_k$ form the set

$$\bigcup_{i=1}^{k-1} \left([\Lambda_{i-1} + i - 1, \Lambda_i + i - 1] \cup [2N - N_i + \Lambda_i + 1, 2N - N_{i-1} + \Lambda_{i-1}] \right).$$

Since

$$\begin{aligned}\bigcup_{i=1}^{k-1}[\Lambda_{i-1}+i-1,\Lambda_i+i-1] &= [0,\Lambda_{k-1}+k-2],\\ \bigcup_{i=1}^{k-1}[2N-N_i+\Lambda_i+1,2N-N_{i-1}+\Lambda_{i-1}] &= [2N-N_{k-1}+\Lambda_{k-1}+1,2N],\end{aligned}$$

and

$$\Lambda_{k-1}+k-2<2N-N_{k-1}+\Lambda_{k-1}$$

the labels used on $G-G_k$ are in the set

$$[0,\Lambda_{k-1}+k-2]\cup[2N-N_{k-1}+\Lambda_{k-1}+1,2N].$$

Now we determine the set of weights induced on the edges of $G-G_k$. First, note that for $1\leq i\leq k-2$

$$\begin{aligned}& [d_{i+1},d_{i+1}+n_{i+1}-1]\cup[d_i,d_i+n_i-1]\\ =\ & [2N-N_{i+1}+1-i,2N-N_i-i]\cup[2N-N_i+2-i,2N-N_{i-1}+1-i]\\ =\ & [2N-N_{i+1}+1-i,2N-N_{i-1}+1-i]-\{2N-N_i+1-i:1\leq i\leq k-2\}\end{aligned}$$

In order to label the remaining component of G, we use Lemma 3.1. First, observe that the labeling of $G-G_k$ has not used the labels in the interval $[\Lambda_{k-1}+k-1,2N-N_{k-1}+\Lambda_{k-1}]=[\Lambda_{k-1}+k-1,N+n_k+\Lambda_{k-1}]$, which is an interval of length $N+n_k+2-k$. Since $\{d_i-1:1\leq i\leq k-2\}$ is equivalent to $\{N_i+i:1\leq i\leq k-2\}$ under a ρ-labeling of G, the set W of prescribed weights in Lemma 3.1 is defined to be:

$$W=\begin{cases}\{N_i+i:1\leq i\leq k-2\} & \text{if } n_k=k-2,\\ \{N_i+i:1\leq i\leq k-1\}\cup[N+1,N+n_k+2-k] & \text{if } n_k>k-2.\end{cases}$$

Thus, $\max W=N_{k-2}+k-2$ when $n_k=k-2$ or $\max W=N+n_k+2-k$. Either way, there are enough labels, not used yet, to apply Lemma 3.1.

Once G_k has been labeled, using Lemma 3.1, we shift the vertex labels by adding the constant $\varepsilon=\Lambda_{k-1}+k-1$; in this way the labels used on G_k have not been used in any G_i for $1\leq i\leq k-1$, and the induced weights form the set W.

Summarizing, every component of G has been labeled, the labels used are in $[0,2N]$ and the set of weights is $\{x_1,x_2,...,x_N\}$ where $x_i=i$ or $x_i=2N+1-i$ for every $1\leq i\leq N$. Therefore, this labeling of G is a ρ-labeling and G is a ρ-graph. □

In Figure 4 we show an example of this labeling for a forest, of size 38, formed as the union of four α-trees.

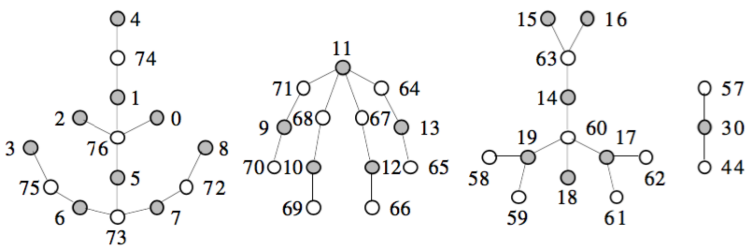

Figure 4. ρ-labeling of a forest

Corollary 3.2. *If $G_1, G_2, ..., G_k$ is a collection of α-graphs where G_k is a caterpillar of size at least $k-2$ and $G = \bigcup_{i=1}^{k} G_i$, then there is a cyclic decomposition of K_{2n+1} into copies of G, where n is the size of G.*

Corollary 3.3. *If each of the k components of a forest F is an α-tree, then F is a ρ-graph when one of its components is a caterpillar of size at least $k-2$.*

Corollary 3.4. *There is a cyclic decomposition of K_{2n+1} into copies of any forest of size n which k trees admit α-labelings and one component is a caterpillar of size at least $k-2$.*

4. Rho-Labeling of Trees

In this section we focus on ρ-labelings of trees. In particular we use a labeling technique similar to the one used by Van Bussel [13] and recently by Barrientos and Krop [1]. There, the authors prove that for every tree T of size n, there exists a labeling of T such that the labels used are in the interval $[0, n+\varepsilon(T)]$. Before explaining the parameter $\varepsilon(T)$ some definitions and conventions are needed.

Let T be a tree, when T is represented as a rooted tree, with root r, this representation T_r induces an order of the vertices within every level $L_k, 0 \leq k \leq h$, where h is the height of T_r. Recall that L_k is formed by all the vertices of T at distance k from r. Let $L_k = \{v_j^k : 1 \leq j \leq n_k\}$ where $n_k = |L_k|$. We assume that v_j^k is placed to the left of v_{j+1}^k, for all $1 \leq j \leq n_{k-1}$.

The *excess of* L_k, denoted Ω_k, is defined to be

$$\Omega_k = \begin{cases} 0 & \text{if } k = 0, h, \\ n_k - n_{k,0} - 1 & \text{if } 1 \leq k \leq h-1, \end{cases}$$

where $n_{k,0}$ is the number of vertices in L_k with no children.

The *excess of* T_r, denoted $ex(T_r)$, is given by

$$ex(T_r) = \sum_{k=0}^{h} \Omega_k.$$

The *excess* of T, denoted by $\varepsilon(T)$, is defined to be

$$\varepsilon(T) = \min\{ex(T_r) : \text{for all } r \in V(T)\}.$$

For example, if $v_1, v_2, ..., v_n$ are the consecutive vertices of the path $P_n = T$, then for every $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, $ex(T_{v_i}) = ex(T_{v_{n+1-i}})$, where $ex(T_{v_i}) = 0$ when $i \in [1, 2]$ and $ex(T_{v_i}) = i - 2$ when $i \in [3, \left\lceil \frac{n}{2} \right\rceil]$. Thus $\varepsilon(P_n) = 0$. In general, if T is a caterpillar, $\varepsilon(T) = 0$. Suppose now that T is a lobster. Consider any path of maximum length in T. The excess of T is given by the number of vertices of degree larger than one that are not in this path of maximum length.

Let G be a tree and H be a connected subgraph of G. We say that H is a *branch* of G if all but one of the leaves of H are leaves of G and $G - H$ is a tree. Let r be the vertex shared by H and $G - H$, and S be the set of all leaves of $G - H$ that are incident to r. We are interested in all those branches H of G that are caterpillars. Let H' be the caterpillar induced by $V(H) \cup S$. We define $\Re$ to be the family of all trees G such that there exists a branch H of G, that is a caterpillar, and $ex(T_r) \leq |E(H')|$ when r is taken as the root of $T = G - H'$. We claim that all the elements of $\Re$ admit ρ-labelings. Before proving this claim, let us note that all caterpillars are members of $\Re$; moreover, every tree is an induced subgraph of a tree in $\Re$. In fact, for any tree T, there is a vertex $r \in V(T)$, such that $ex(T_r) = \varepsilon(T)$. Let H be a caterpillar of size at least $\varepsilon(T)$, identifying r with any of the vertices of maximum eccentricity in H, we obtain a tree T' that is in $\Re$. Thus, for any given tree T, T is in $\Re$ or T is an induced subgraph of a tree in $\Re$.

The labeling used in our next theorem works with ordered rooted trees. Let T be a rooted tree. For any vertex $v \in V(T)$, let $\gamma(v)$ denote the number of levels in T where v has at least one descendent. We order the vertices within each level according to their degrees and the associated parameter γ so that edges do not cross. We denote by $u \prec v$ the placement of u to the left of v. With this notation, we define the order on each level.

(i) If u and v are siblings of degree one, the order of u and v is arbitrary.
(ii) If u and v are siblings and $\gamma(u) < \gamma(v)$, then $u \prec v$.
(iii) If u and v are siblings and $\gamma(u) = \gamma(v)$, and deg$(u) \geq$deg(v), then $u \prec v$.
(iv) If u and v are siblings and $u \prec v$, u' and v' descendants of u and v respectively, on the same level, then $u' \prec v'$.

We call a rooted tree so represented a *left-layered tree*.

Theorem 4.1. *If $G \in \Re$, then G is a ρ-tree.*

Proof. Let $G \in \Re$. Suppose the size of G is n, the size of H' is t, and r is the vertex shared by H' and $T = G - H'$. We assume that T is represented as a left-layered tree, with root r and height h; in addition we assume that $ex(T_r) \leq t$.

The vertices of T are labeled per level, starting with level L_h. Let g be the labeling of G defined on the vertices of T, in a recursive way, as follows.

- $g(v_j^h) = n_h - j, 1 \leq j \leq n_h$.
- When h is odd, for every $1 \leq i \leq \frac{h-1}{2}$,

$$g(v_j^{h-2i}) = n_{h-2i} + g(v_1^{h-2i+2}) + \Omega_{h-2i+1} + 1 - j \text{ where } 1 \leq j \leq n_{h-2i},$$

$$g(v_1^0) = n + t - ex(r) + g(v_1^1) + 1,$$

and for every $1 \leq i \leq \frac{h-1}{2}$,

$$g(v_j^{2i}) = g(v_{n_{2i-2}}^{2i-2}) + \Omega_{2i-1} + j, 1 \leq j \leq n_{2i}.$$

- When h is even, for every $1 \leq i \leq \frac{h}{2}$,

$$g(v_j^{h-2i}) = n_{h-2i} + g(v_1^{h-2i+2}) + \Omega_{h-2i+1} + 1 - j \text{ where } 1 \leq j \leq n_{h-2i},$$

$$g(v_j^1) = n + t - ex(r) + g(v_1^0) + j,$$

and for every $2 \leq i \leq \frac{h}{2}$,

$$g(v_j^{2i-1}) = g(v_{n_{2i-3}}^{2i-3}) + \Omega_{2i-2} + j, 1 \leq j \leq n_{2i-1.}$$

Thus, regardless the parity of h, the labels assigned on the levels L_{h-2i} are, from left to right, in descending order, for every $0 \leq i \leq \left\lfloor \frac{h}{2} \right\rfloor$, and in ascending order on the other levels. Given the bipartite nature of this partial labeling, the bipartite set of $V(T)$ containing the vertices on the levels which indices have the same parity than h receive labels from the set

$$\left[0, \left(\sum_{i=0}^{\frac{h-1}{2}} n_{h-2i} + \sum_{i=1}^{\frac{h-1}{2}} \Omega_{2i}\right) - 1\right]$$

when h is odd, and from the set

$$\left[0, \left(\sum_{i=0}^{\frac{h}{2}} n_{h-2i} + \sum_{i=1}^{\frac{h}{2}} \Omega_{2i-1}\right) - 1\right]$$

when h is even. The vertices on the other bipartite set receive labels from the set

$$\left[\left(\sum_{i=0}^{\frac{h-1}{2}} n_{h-2i} + \sum_{i=1}^{\frac{h-1}{2}} \Omega_{2i}\right) + n + t - ex(T_r), 2n\right]$$

when h is odd, and from the set

$$\left[\left(\sum_{i=0}^{\frac{h}{2}} n_{h-2i} + \sum_{i=1}^{\frac{h}{2}} \Omega_{2i-1}\right) + n + t - ex(T_r), 2n\right]$$

when h is even.

Note that in both cases there are $n+t-ex(T_r)$ consecutive integers that have not been used as labels of T.

Similarly, the weights of the edges connecting the vertices of L_k and L_{k+1} are also in increasing order from left to right. In order to see that all the induced weights are distinct, consider the first edge between L_{k+1} and L_k, and the last edge between L_{k-1} and L_k. Suppose that $v_1^{k+1}v_{n_k-\Omega_k}^k$ and $v_{n_k}^k v_{n_{k-1}}^{k-1}$ are these edges:

When k and h have the same parity $g(v_1^{k+1}) = g(v_{n_{k-1}}^{k-1})+\Omega_k+1$ and $g(v_{n_k-\Omega_k}^k) = g(v_{n_k}^k)+\Omega_k$. Thus,

$$\begin{aligned}\left|g(v_1^{k+1}) - g(v_{n_k-\Omega_k}^k)\right| &= g(v_{n_{k-1}}^{k-1}) + \Omega_k + 1 - g(v_{n_k}^k) - \Omega_k \\ &= \left(g(v_{n_{k-1}}^{k-1}) - g(v_{n_k}^k)\right) + 1.\end{aligned}$$

When k and h have different parity $g(v_1^{k+1}) = g(v_{n_{k-1}}^{k-1})-\Omega_k-1$ and $g(v_{n_k-\Omega_k}^k) = g(v_{n_k}^k)-\Omega_k$. Then,

$$\begin{aligned}\left|g(v_1^{k+1}) - g(v_{n_k-\Omega_k}^k)\right| &= g(v_{n_k}^k) - \Omega_k - g(v_{n_{k-1}}^{k-1}) + \Omega_k + 1 \\ &= \left(g(v_{n_k}^k) - g(v_{n_{k-1}}^{k-1})\right) + 1.\end{aligned}$$

Hence, the weights of these two edges are consecutive integers. Furthermore, the weights of the edges of T are in the interval $[n+1+t-ex(T_r), 2n]$.

Let $W^* = \{w_1^*, w_2^*, ..., w_{ex(r)}^*\}$ be the set formed by all the integers in this last interval that are not weights of T. We assume that the elements of W^* are written in ascending order. Given the way that H' is chosen, we know that $\min W^* \geq n+4+t-ex(T_r)$ and $\max W^* \leq 2n-1$.

Now we proceed to label the vertices of H'; recall that the vertex v_1^0 is shared by H' and T. By Lemma 3.1, there exists a labeling of H', where the vertex u_1 in Lemma 3.1 is the vertex v_1^0, such that induces any set W of weights, in particular, we take $W = [n+1, n+t-ex(T_r)] \cup \{2n+1-w_i^* : 1 \leq i \leq ex(T_r)\}$, assuming that its elements are previously ordered in descending order. Note that when $t = ex(T_r)$, the interval $[n+1, n+t-ex(T_r)]$ is empty, because $n+1 > n$. This labeling of H' uses labels from 0 to $\max W$. Suppose H' has been labeled using Lemma 3.1 in such a way that u_1, from Lemma 3.1, is v_1^0, the vertex shared by H' and T. If h is even, the labeling of H' is shifted $\varepsilon = g(v_1^0)$ units; if h is odd, the complementary labeling of H' is shifted $\varepsilon = g(v_1^0)$ units. Recall that the complementary labeling of f is obtained by subtracting from the largest label assigned, every vertex label. Thus, the resulting labeling of G would be a ρ-labeling, because it uses labels in $[0, 2n]$ and induces the weights in $[n+1, 2n] - W^*$ and the complements of the elements in W^* with respect to $2n+1$.

Thus, to prove that G is a ρ-graph we just need to show that there are enough consecutive integers to label to vertices of H' in order to achieve the prescribed weights. In other terms, we need to calculate $\max W$. If $t > ex(T_r)$, $\max W = n+t-ex(T_r)$, that is, the exact length of the interval of integers not assigned to the vertices of T. If $t = ex(T_r)$, $\max W = \max\{2n+1-w_i^* :$

$1 \leq i \leq ex(T_r)\}$, this maximum is achieved when $i = 1$, since $\min W^* \geq n + 4 + t - ex(T_r)$, $\max W \leq n - 3$. On the other side, $\min W = \min\{2n+1-w_i^* : 1 \leq i \leq ex(T_r)\}$, this minimum is achieved when $i = ex(T_r)$, since $\max W^* \geq 2n-1$, $\min W \geq 2$; thus $W \subseteq [2, n-3]$. In either case, we have enough consecutive integers to apply Lemma 3.1.

Therefore, the labeling of G is a ρ-labeling and G is a ρ-graph. □

In Figure 5 we show an example of this theorem for a tree G of size 22, H' is a caterpillar of size 7, $ex(r) = 6$, and $W = \{22, 16, 14, 12, 9, 7, 2\}$.

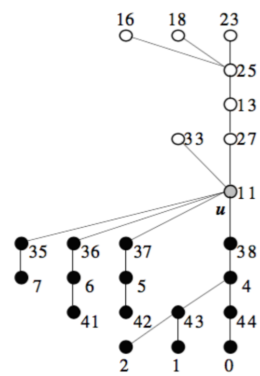

Figure 5. ρ-labeling of a tree

5. Conclusions

Let G be a tree with excess $\varepsilon(G)$ such that $G \notin \Re$. For every vertex u of G, there are infinitely many caterpillars that we can attach, via vertex amalgamation, to u in order to produce a graph G' in $\Re$. In fact, we can pick any caterpillar C, of size at least equal to $\varepsilon(G)$, and identify with u, any vertex of maximum eccentricity in C or any of its neighbors, to produce a tree in $\Re$. Therefore $\Re$ is a quite robust family inside the family $\mathcal{F}$ of all trees.

These results resemble some of the ones given by Kotzig [6]. In his Theorem 6, Kotzig showed that nearly all trees are α-trees. Lemma 8, in [6], says that the vertex amalgamation of a leaf of a "sufficiently long" path and an arbitrary vertex of an arbitrary α-graph results in a new graph that admits an α-labeling. A similar result can be found in the work of Lladó and López [7]; there the authors investigate bigraceful labelings of trees and use them to find isomorphic decompositions of the complete bipartite graph $K_{n,n}$. Theorem 2.2 in [7] says that for an arbitrary tree T, there is a vertex v in T and a positive integer k such that the amalgamation of k pendant vertices to v gives a bigraceful tree. In our case, we have determined exactly the minimum size of the caterpillar used in the vertex amalgamation and changed "α-graph" for "tree". In other terms, this is a more general construction and reduces considerably the number of trees that need to be analyzed to verify the ρ-Conjecture.

In conclusion, in order to prove the ρ-Conjecture we just need to find ρ-labelings for all trees in $\mathcal{F} - \Re$.

Acknowledgement

We want to thank Professor Joseph Gallian for facilitating to us a copy of reference [2], which we could not find without his help.

We want to thank the referee for his or her valuable comments and suggestions that helped us improve the quality of the present work.

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