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Odd sum labeling of graphs obtained by duplicating any edge of some graphs

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Abstract

An injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is an odd sum labeling if the induced edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$, for all $uv \in E(G)$, is bijective and $f^*(E(G)) = \{1, 3, 5, \ldots, 2q - 1\}$. A graph is said to be an odd sum graph if it admits an odd sum labeling. In this paper we study the odd sum property of graphs obtained by duplicating any edge of some graphs.

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1. Introduction

Throughout this paper, by a graph we mean a finite, undirected simple graph. For notations and terminology, we follow [3].

A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. A *ladder* $L_n, n \geqslant 2$, is the graph $P_2 \square P_n$, the Cartesian product of the graphs P_2 and P_n. The graph $G \circ K_1$ is obtained from the graph G by attaching a new pendant vertex at each vertex of G. *Duplicating* of an edge $e = uv$ of a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$ [9].

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In [7], the concept of mean labeling was introduced and further studied in [8]. An injective function $f : V(G) \to \{0, 1, 2, \ldots, q\}$ is said to be a *mean labeling* if the induced edge labeling f^* defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd,} \end{cases}$$

is injective and $f^*(E(G)) = \{0, 1, 2, \ldots, q\}$. A graph G is said to be an *odd mean graph* if there exists an injective function f from $V(G)$ to $\{0, 1, 2, \ldots, 2q-1\}$ such that the induced map f^* from $E(G) \to \{1, 3, 5, \ldots, 2q-1\}$ defined by

$$f^*(uv) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u) + f(v) \text{ is odd} \end{cases}$$

is a bijection [6].

In [4], an odd edge labeling of a graph is defined as follows. A labeling $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ is called an *odd edge labeling* of G if the edge labeling f^* on $E(G)$ defined by $f^*(uv) = f(u) + f(v)$ for any edge $uv \in E(G)$ is such that the edge weights are odd. Here the edge labeling is not necessarily injective.

In [1], we introduced a new concept called *odd sum labeling* of graphs. The odd sum property of subdivision of some graphs have been studied in [2] and the same was referred in [5]. An injective function $f : VG) \to \{0, 1, 2, \ldots, q\}$ is an odd sum labeling if the induced edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$, for all $uv \in E(G)$, is bijective and $f^*(E(G)) = \{1, 3, 5, \ldots, 2q-1\}$. A graph is said to be an *odd sum graph* if it admits an odd sum labeling. In this paper, we study the odd sum property of graphs obtained by duplicating any edge of some graphs.

Theorem 1.1. [1] *Every graph having an odd cycle is not an odd sum graph.*

2. Main Results

Proposition 2.1. *Let G be a graph obtained by duplicating an edge e of a path $P_n, n \geqslant 3$. Then, G is not an odd sum graph except the case when e is a pendant edge of P_5.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on the path P_n. Let $e' = v_i'v_{i+1}'$ be the duplicating edge of $e = v_iv_{i+1}$, for some $i, 1 \leqslant i \leqslant n-1$.

Case 1. $i = 1$ or $i = n-1$.

Since the graph G is isomorphic when $i = 1$ or $i = n-1$, we may take $i = 1$.

Subcase (i). $n \equiv 0 (mod\, 4)$.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+1\}$ as follows:

$$f(v_j) = \begin{cases} 6-j, & j = 1, 2, 3 \\ j-2, & 4 \leqslant j \leqslant n \text{ and } j \equiv 0 (mod\, 4) \\ j+4, & 5 \leqslant j \leqslant n-3 \text{ and } j \equiv 1 (mod\, 4) \\ j+2, & 6 \leqslant j \leqslant n-2 \text{ and } j \equiv 2 (mod\, 4) \\ j, & 7 \leqslant j \leqslant n-1 \text{ and } j \equiv 3 (mod\, 4) \end{cases}$$

and $f(v_j') = 2 - j, \quad j = 1, 2.$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

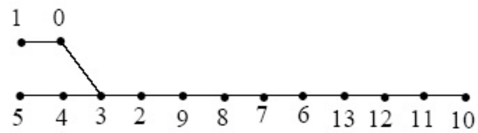


Figure 2.1. An odd sum labeling of G when $n = 12$ and $i = 1$.

Subcase (ii). $n \equiv 2(mod\, 4)$.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+1\}$ as follows:

$$f(v_j) = \begin{cases} 6-j, & j = 1,3 \\ 2, & j = 2 \\ 4, & j = 6 \\ j+2, & j = 4,5 \\ j+2, & 8 \leqslant j \leqslant n-2 \text{ and } j \equiv 0(mod\, 4) \\ j, & 9 \leqslant j \leqslant n-1 \text{ and } j \equiv 1(mod\, 4) \\ j-2, & 10 \leqslant j \leqslant n \text{ and } j \equiv 2(mod\, 4) \\ j+4, & 7 \leqslant j \leqslant n-3 \text{ and } j \equiv 3(mod\, 4) \end{cases}$$

and $f(v'_j) = 2 - j, \quad j = 1, 2.$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

Subcase (iii). $n \equiv 1(mod\, 4)$ and $n \geqslant 9$.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+1\}$ as follows:

$$f(v_j) = \begin{cases} j-1, & 1 \leqslant j \leqslant 3 \\ j+3, & j = 4,7 \\ j+1, & j = 5,8 \\ j-1, & j = 6,9 \\ j-1, & 12 \leqslant j \leqslant n \text{ and } j \equiv 0,1(mod\, 4) \\ j+3, & 10 \leqslant j \leqslant n-2 \text{ and } j \equiv 2,3(mod\, 4) \end{cases}$$

and $f(v'_j) = 5 - j, \quad j = 1, 2.$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

When $n = 5$, the number of edges in G is 6. By assigning the label 0 (or 1, 2) to the vertex v_3, the labels 5 and 6 will be assigned to the end vertices of any one of the edges v_1v_2, v_4v_5 and $v'_1v'_2$ in order to get the edge label 11. If so, then 9 will not be an edge label. By assigning the label 4 (or 5, 6) to the vertex v_3, the labels 0 and 1 will be assigned to the end vertices of any one of the edges v_1v_2, v_4v_5 and $v'_1v'_2$ in order to get the edge label 1. If so, then 3 will not be an edge label. By assigning the label 3 to the vertex v_3, the pair of labels 0, 1 and 5, 6 will be assigned to the end vertices of any two of the edges v_1v_2, v_4v_5 and $v'_1v'_2$ in order to get the edge label 1 and 11 respectively. This implies that the labels 2 and 4 are to be assigned to the remaining vertices which gives an even edge label 6. Hence an odd sum labeling does not exist for G when $n = 5$.

Subcase (iv). $n \equiv 3(mod\, 4)$ and $n \geqslant 7$.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+1\}$ as follows:

$$f(v_j) = \begin{cases} j-1, & 1 \leqslant j \leqslant 3 \\ j+3, & 4 \leqslant j \leqslant n-2 \text{ and } j \equiv 0, 1(mod\, 4) \\ j-1, & 6 \leqslant j \leqslant n \text{ and } j \equiv 2, 3(mod\, 4) \end{cases}$$

and $f(v'_j) = 5 - j, \quad j = 1, 2.$

From this vertex labeling, the required induced edge labeling for G will be attained.

While $n = 3$, an odd sum labeling of G is given in Figure 2.2.

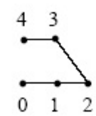


Figure 2.2. An odd sum labeling of G when $n = 3$ and

Case 2. $2 \leqslant i \leqslant n-2$.

Subcase (i). $(n-i) \equiv 0(mod\, 4)$ and $n - i \geqslant 12$.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+2\}$ as follows:

$$f(v_j) = \begin{cases} j-1, & 1 \leqslant j \leqslant i-1, j = i+1, i+6 \\ j+1, & j = i, i+2, i+3 \\ j+3, & j = i+4, i+5 \\ j+3, & i+7 \leqslant j \leqslant n-6 \text{ and } j-i \equiv 0(mod\, 4), j = n-2, n-1 \\ j+1, & i+7 \leqslant j \leqslant n-6 \text{ and } j-i \equiv 1(mod\, 4), j = n-4, n-3 \\ j-1, & i+7 \leqslant j \leqslant n-6 \text{ and } j-i \equiv 2(mod\, 4), j = n \\ j+5, & i+7 \leqslant j \leqslant n-6 \text{ and } j-i \equiv 3(mod\, 4), j = n-5, \end{cases}$$

$f(v'_i) = i - 1$ and $f(v'_{i+1}) = i + 6$.

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

Subcase (ii). $(n-i) \equiv 1(mod\, 4)$ and $n - i \geqslant 9$.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+2\}$ as follows:

$$f(v_j) = \begin{cases} j-1, & 1 \leqslant j \leqslant i-1, j = i+1, i+3 \\ j+1, & j = i \\ j+3, & j = i+2 \\ j+5, & i+4 \leqslant j \leqslant n-6 \text{ and } j-i \equiv 0(mod\, 4), j = n-5 \\ j+3, & i+4 \leqslant j \leqslant n-6 \text{ and } j-i \equiv 1(mod\, 4), j = n-2, n-1 \\ j+1, & i+4 \leqslant j \leqslant n-6 \text{ and } j-i \equiv 2(mod\, 4), j = n-4, n-3 \\ j-1, & i+4 \leqslant j \leqslant n-6 \text{ and } j-i \equiv 3(mod\, 4), j = n, \end{cases}$$

$f(v'_i) = i - 1$ and $f(v'_{i+1}) = i + 4$.

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

Subcase (iii). $(n-i) \equiv 2(mod\,4)$ and $n-i \geqslant 6$.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+2\}$ as follows:

$$f(v_j) = \begin{cases} j-1, & 1 \leqslant j \leqslant i-1, j = i+1, i+6 \\ j+1, & j = i, i+2, i+3 \\ j+3, & j = i+4, i+5 \\ j+3, & i+7 \leqslant j \leqslant n \text{ and } j-i \equiv 0(mod\,4) \\ j+1, & i+7 \leqslant j \leqslant n \text{ and } j-i \equiv 1(mod\,4) \\ j-1, & i+7 \leqslant j \leqslant n \text{ and } j-i \equiv 2(mod\,4) \\ j+5, & i+7 \leqslant j \leqslant n \text{ and } j-i \equiv 3(mod\,4), \end{cases}$$

$$f(v'_i) = i-1 \text{ and } f(v'_{i+1}) = i+6.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

Subcase (iv). $(n-i) \equiv 3(mod\,4)$ and $n-i \geqslant 7$.

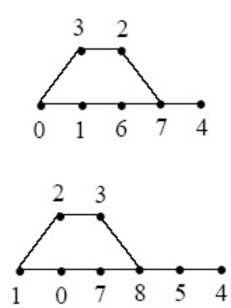
Define $f : V(G) \to \{0, 1, 2, \ldots, n+2\}$ as follows:

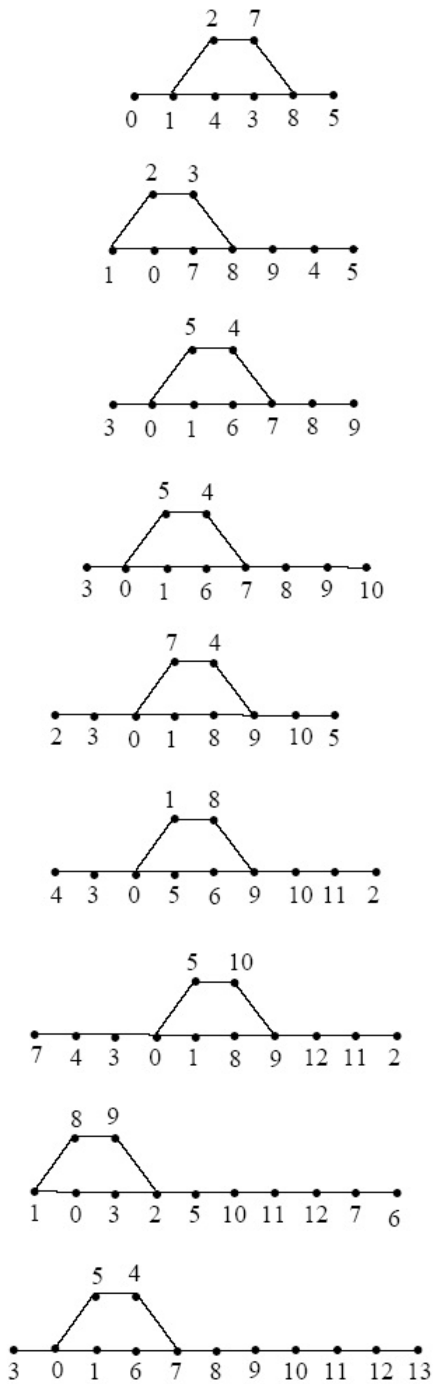
$$f(v_j) = \begin{cases} j-1, & 1 \leqslant j \leqslant i-1, j = i+1, i+3 \\ j+1, & j = i \\ j+3, & j = i+2 \\ j+5, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 0(mod\,4) \\ j+3, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 1(mod\,4) \\ j+1, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 2(mod\,4) \\ j-1, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 3(mod\,4), \end{cases}$$

$$f(v'_i) = i-1 \text{ and } f(v'_{i+1}) = i+4.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

The graphs which are not lying under any of the sub cases of Case 2 are given with their odd sum labeling in Figure 2.3.




2 7
0 1 4 3 8 5
2 3
1 0 7 8 9 4 5
5 4
3 0 1 6 7 8 9
5 4
3 0 1 6 7 8 9 10
7 4
2 3 0 1 8 9 10 5
1 8
4 3 0 5 6 9 10 11 2
5 10
7 4 3 0 1 8 9 12 11 2
8 9
1 0 3 2 5 10 11 12 7 6
5 4
3 0 1 6 7 8 9 10 11 12 13

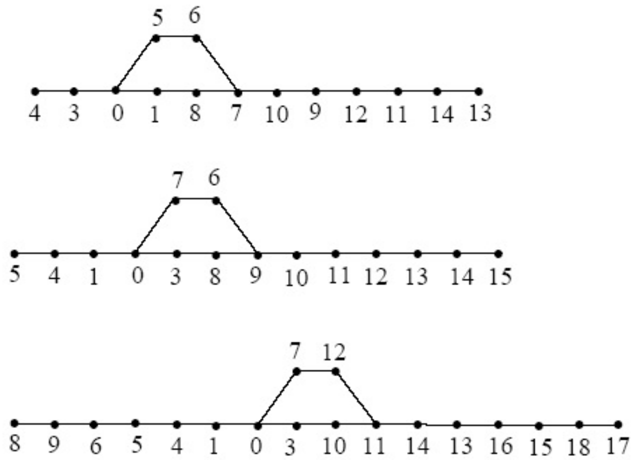


Figure 2.3. An odd sum labeling of G for various values of n.

□

Proposition 2.2. *Let G be the graph obtained from $P_n \circ K_1$ by duplicating an edge, for $n \geqslant 2$. Then, G is not an odd sum graph only when $n = 3$ and the pendant edge attached at the pendent vertex of P_n is duplicated.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path of length $n-1$ and $v_i, 1 \leqslant i \leqslant n$ be the pendant vertices of $u_i, 1 \leqslant i \leqslant n$. Let G be the graph obtained from $P_n \circ K_1$ by duplicating an edge e (other than u_1v_1 and u_nv_n) by e'.

Case 1. $e = u_iv_i$, for some $i, 2 \leqslant i \leqslant n-1$.

Let its duplication be $e' = u_i'v_i'$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 2n+2\}$ as follows:

For i is odd, $f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \end{cases}$ and $f(v_j) = \begin{cases} 1, & j = 1, \\ 2, & j = 2. \end{cases}$

For i is even, $f(u_j) = \begin{cases} 1, & j = 1 \\ 2, & j = 2 \end{cases}$ and $f(v_j) = \begin{cases} 0, & j = 1, \\ 3, & j = 2. \end{cases}$

$$f(u_j) = \begin{cases} f(u_{j-2}) + 4, & 3 \leqslant j \leqslant i-1 \\ f(u_{j-1}) + 5, & j = i \\ f(u_{j-1}) + 1, & j = i+1 \\ f(u_{j-2}) + 4, & i+2 \leqslant j \leqslant n, \end{cases}$$

$$f(v_j) = \begin{cases} f(v_{j-2}) + 4, & 3 \leqslant j \leqslant i-1 \\ f(v_{j-1}) + 5, & j = i \\ f(v_{j-1}) + 3, & j = i+1 \\ f(v_{j-2}) + 4, & i+2 \leqslant j \leqslant n, \end{cases}$$

$$f(u_i') = f(u_i) - 4 \text{ and } f(v_i') = f(u_i) - 3.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

Case 2. $e = u_1v_1$ (or u_nv_n).

Let its duplication be $e' = u_1'v_1'$. Assume that $n \geqslant 4$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 2n+1\}$ as follows:

$f(u_1) = 1, f(u_2) = 2, f(u_3) = 9, f(u_i) = \begin{cases} 2i, & 4 \leqslant i \leqslant n \text{ and } i \text{ is even} \\ 2i+1, & 5 \leqslant i \leqslant n \text{ and } i \text{ is odd,} \end{cases}$

$f(v_1) = 0, f(v_2) = 5, f(u_3) = 4, f(v_4) = 7,$

$f(v_i) = \begin{cases} 2i, & 5 \leqslant i \leqslant n \text{ and } i \text{ is odd} \\ 2i+1, & 6 \leqslant i \leqslant n \text{ and } i \text{ is even,} \end{cases}$

$f(u_1') = 3$ and $f(v_1') = 6$.

From this vertex labeling, the required induced edge labeling for G will be attained.

Thus f is an odd sum labeling of G.

When $n = 3$, the number of edges in G is 7. By assigning the label 0 (or 1, 2, 3) to the vertex v_2, the labels 6 and 7 will be assigned to the end vertices of any one of the edges u_1v_1, u_3v_3 and $u_1'v_1'$ in order to get the edge label 13. If so, then 11 will not be an edge label. By assigning the label 4 (or 5, 6, 7) to the vertex v_2, the labels 0 and 1 will be assigned to the end vertices of any one of the edges u_1v_1, u_3v_3 and $u_1'v_1'$ in order to get the edge label 1. If so, then 3 will not be an edge label. So an odd sum labeling is not possible in G when $n = 3$.

When $n = 2$, an odd sum labeling is shown in Figure 2.4.

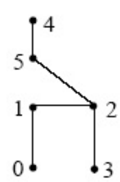


Figure 2.4. An odd sum labeling of G when $n = 2$ and $i = 1$.

Case 3. $e = u_iu_{i+1}$ for some $i, 2 \leqslant i \leqslant n-2$.

Let its duplication be $e' = u_i'u_{i+1}'$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 2n+4\}$ as follows:

For i is odd, $f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \end{cases}$ and $f(v_j) = \begin{cases} 1, & j = 1, \\ 2, & j = 2. \end{cases}$

For i is even, $f(u_j) = \begin{cases} 1, & j = 1 \\ 2, & j = 2 \end{cases}$ and $f(v_j) = \begin{cases} 0, & j = 1, \\ 3, & j = 2. \end{cases}$

$$f(u_j) = \begin{cases} f(u_{j-2}) + 4, & 3 \leqslant j \leqslant i \\ f(u_{j-2}) + 8, & j = i+1 \\ f(u_{j-2}) + 10, & j = i+2 \\ f(u_{j-2}) + 4, & i+3 \leqslant j \leqslant n, \end{cases}$$

$$f(v_j) = \begin{cases} f(v_{j-2}) + 4, & 3 \leqslant j \leqslant i \\ f(v_{j-2}) + 8, & j = i+1, i+2 \\ f(v_{j-2}) + 6, & j = i+3 \\ f(v_{j-2}) + 4, & i+4 \leqslant j \leqslant n, \end{cases}$$

$$f(u_i') = f(u_i) + 4 \text{ and } f(u_{i+1}') = f(u_{i+1}) - 2.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

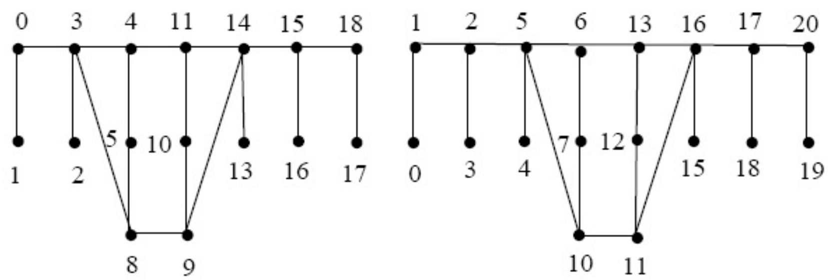


Figure 2.5. An odd sum labeling of G when $n = 7, i = 3$ and $n = 8, i = 4$.

Case 4. $e = u_1u_2$ (or $u_{n-1}u_n$).

Let its duplication be $e' = u_1'u_2'$.

For $n \geqslant 3$, we define $f : V(G) \to \{0, 1, 2, \ldots, 2n+3\}$ as follows:

$$f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \\ f(u_{j-2}) + 8, & j = 3, 4 \\ f(u_{j-2}) + 4, & 5 \leqslant j \leqslant n, \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 2, & j = 2 \\ f(v_{j-2}) + 8, & j = 3, 4 \\ f(v_{j-2}) + 4, & 5 \leqslant j \leqslant n, \end{cases}$$

$$f(u_1') = 6 \text{ and } f(u_2') = 7.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

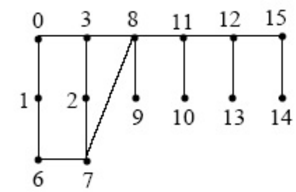


Figure 2.6. An odd sum labeling of G when $n = 6$ and $i = 1$.

□

Proposition 2.3. *Let G be the graph obtained by duplicating an edge of a cycle $C_n, n \geqslant 4$. Then, G is an odd sum graph if and only if n is even.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on the cycle C_n. Let G be the graph obtained by duplicating an edge of C_n. By Theorem 1.1 an odd sum labeling of G does not exist when n is odd. So assume that n is even.

Case 1. $n \equiv 2(mod\, 4)$ and $n \geqslant 14$.

Let $e' = v'_2v'_3$ be the duplicating edge of $e = v_2v_3$ in G.

Define $f : V(G) \rightarrow \{0, 1, 2, \ldots, n+3\}$ as follows:

$$f(v_i) = \begin{cases} i-1, & 1 \leqslant i \leqslant \frac{n+2}{4} \text{ and } i \text{ is odd} \\ i+1, & \frac{n+6}{4} \leqslant i \leqslant \frac{n-4}{2} \text{ and } i \text{ is odd} \\ i+3, & \frac{n}{2} \leqslant i \leqslant n-1 \text{ and } i \text{ is odd} \\ i+1, & 2 \leqslant i \leqslant \frac{n+2}{4} \text{ and } i \text{ is even} \\ i+3, & \frac{n+10}{4} \leqslant i \leqslant n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v'_2) = 1 \text{ and } f(v'_3) = \frac{n+2}{2}.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

When $n = 6$ and 10, the odd sum labeling of G is given in Figure 2.7.

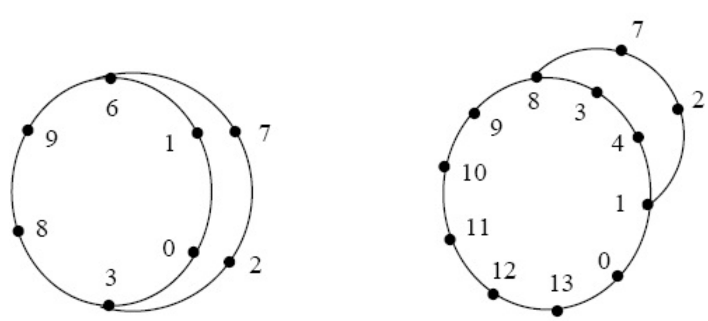


Figure 2.7. An odd sum labeling of G when $n = 6$ and 10.

Case 2. $n \equiv 0(mod\, 4)$ and $n \geqslant 16$.

Let $e' = v'_2v'_3$ be the duplicating edge of $e = v_2v_3$ in G.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+3\}$ as follows:

$$f(v_i) = \begin{cases} i-1, & i = 1, 3 \\ i+1, & 5 \leqslant i \leqslant \frac{n}{4} \text{ and } i \text{ is odd} \\ i+3, & \frac{n+4}{4} \leqslant i \leqslant n-1 \text{ and } i \text{ is odd} \\ i-1, & 2 \leqslant i \leqslant \frac{n}{4} \text{ and } i \text{ is even} \\ i+1, & \frac{n+4}{4} \leqslant i \leqslant \frac{n-4}{2} \text{ and } i \text{ is even} \\ i+3, & \frac{n}{2} \leqslant i \leqslant n \text{ and } i \text{ is even,} \end{cases}$$

$$f(v_2') = \frac{n+2}{2} \text{ and } f(v_3') = 4.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

When $n = 4, 8$ and 12, the odd sum labeling of G is given in Figure 2.8.

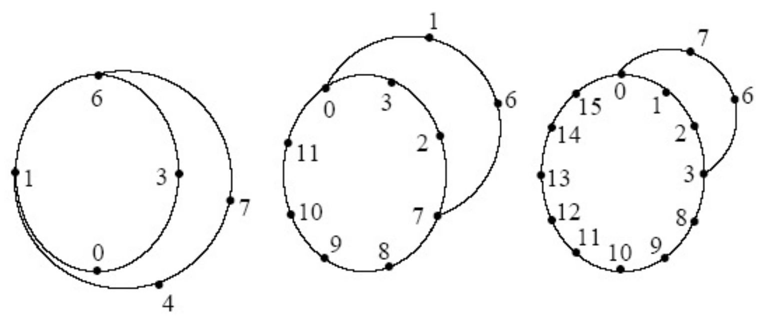


Figure 2.8. An odd sum labeling of G when $n = 4, 8$ and 12.

□

Proposition 2.4. *Let G be a graph obtained by duplicating an edge of $C_n \circ K_1, n \geqslant 4$. Then, G is an odd sum graph if and only if n is even.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices on the cycle C_n and $u_i, 1 \leqslant i \leqslant n$ be the pendant vertex attached at $v_i, 1 \leqslant i \leqslant n$ in $C_n \circ K_1$. Let G be a graph obtained by duplicating an edge e in $C_n \circ K_1$. By Theorem 1.1, an odd sum labeling of G does not exist when n is odd. So assume that n is even.

Case 1. Let $u'_{n-1}v'_{n-1}$ be the duplicating edge of $u_{n-1}v_{n-1}$ in G.

Subcase (i). $n \equiv 0 (mod\, 4)$ and $n \geqslant 8$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 2n + 3\}$ as follows:

$$f(v_i) = \begin{cases} 2i - 2, & 1 \leqslant i \leqslant n - 1 \text{ and } i \text{ is odd} \\ 2i - 1, & 2 \leqslant i \leqslant \frac{n}{2} \text{ and } i \text{ is even} \\ 2i + 1, & \frac{n}{2} + 2 \leqslant i \leqslant n - 2 \text{ and } i \text{ is even} \\ 2i + 3, & i = n, \end{cases}$$

$$f(u_i) = \begin{cases} 2i - 1, & 1 \leqslant i \leqslant \frac{n}{2} + 1 \text{ and } i \text{ is odd} \\ 2i + 1, & \frac{n}{2} + 3 \leqslant i \leqslant n - 1 \text{ and } i \text{ is odd} \\ 2i - 2, & 2 \leqslant i \leqslant n - 2 \text{ and } i \text{ is even} \\ 2i + 2, & i = n, \end{cases}$$

$$f(u'_{n-1}) = 2n + 1 \text{ and } f(v'_{n-1}) = 2n.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

An odd sum labeling of G when $n = 4$ is given in Figure 2.9.

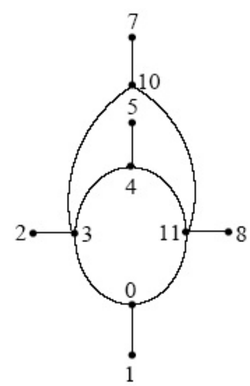


Figure 2.9. An odd sum labeling of G when $n = 4$.

Subcase (ii). $n \equiv 2(mod\ 4)$ and $n \geqslant 10$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 2n + 3\}$ as follows:

$$f(v_i) = \begin{cases} 2i - 2, & 1 \leqslant i \leqslant \frac{n}{2} \text{ and } i \text{ is odd} \\ 2i, & \frac{n}{2} + 2 \leqslant i \leqslant n - 3 \text{ and } i \text{ is odd} \\ 2i - 2, & i = n - 1 \\ 2i - 1, & 2 \leqslant i \leqslant n - 2 \text{ and } i \text{ is even} \\ 2i + 3, & i = n, \end{cases}$$

$$f(u_i) = \begin{cases} 2i - 1, & 1 \leqslant i \leqslant n - 1 \text{ and } i \text{ is odd} \\ 2i - 2, & 2 \leqslant i \leqslant \frac{n}{2} + 1 \text{ and } i \text{ is even} \\ 2i, & \frac{n}{2} + 3 \leqslant i \leqslant n - 4 \text{ and } i \text{ is even} \\ 2i + 4, & i = n - 2 \\ 2i - 2, & i = n, \end{cases}$$

$$f(u'_{n-1}) = 2n + 1 \text{ and } f(v'_{n-1}) = 2n + 2.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

When $n = 6$, an odd sum labeling of G is given in Figure 2.10.

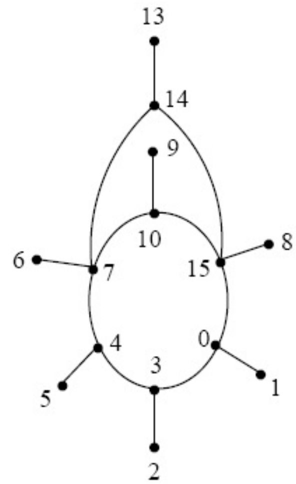


Figure 2.10. An odd sum labeling of G when $n = 6$.

Case 2. Let e' be the duplicating edge of an edge e on the cycle C_n in G.
Subcase (i). $n \equiv 0(mod\, 4)$ and $n \geqslant 4$.

Let $e' = v'_n v'_1$ be the duplicating edge of $v_n v_1$ in G.

Define $f : V(G) \to \{0, 1, 2, \ldots, 2n + 5\}$ as follows:

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 2i, & 3 \leqslant i \leqslant \frac{n}{2} - 1 \text{ and } i \text{ is odd} \\ 2i + 4, & \frac{n}{2} + 1 \leqslant i \leqslant n - 1 \text{ and } i \text{ is odd} \\ 2i + 1, & 2 \leqslant i \leqslant n - 2 \text{ and } i \text{ is even} \\ 2i + 5, & i = n, \end{cases}$$

$$f(u_i) = \begin{cases} 1, & i = 1 \\ 2i + 1, & 3 \leqslant i \leqslant n - 1 \text{ and } i \text{ is odd} \\ 2i, & 2 \leqslant i \leqslant \frac{n}{2} \text{ and } i \text{ is even} \\ 2i + 4, & \frac{n}{2} + 2 \leqslant i \leqslant n \text{ and } i \text{ is even} \end{cases}$$

$$\text{and } f(v'_i) = \begin{cases} 2i, & i = 1 \\ 2i + 1, & i = n. \end{cases}$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

Subcase (ii). $n \equiv 2(mod\, 4)$ and $n \geqslant 10$.

Let $e' = v'_2 v'_3$ be the duplicating edge of $v_2 v_3$ in G.

Define $f : V(G) \rightarrow \{0, 1, 2, \ldots, 2n+5\}$ as follows:

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 2i+2, & 3 \leqslant i \leqslant \frac{n}{2} - 2 \text{ and } i \text{ is odd} \\ 2i+4, & \frac{n}{2} \leqslant i \leqslant n-1 \text{ and } i \text{ is odd} \\ 3, & i = 2 \\ 2i+5, & 4 \leqslant i \leqslant n \text{ and } i \text{ is even,} \end{cases}$$

$$f(u_i) = \begin{cases} 1, & i = 1 \\ 2i+3, & i = 3 \\ 2i+5, & 5 \leqslant i \leqslant n-1 \text{ and } i \text{ is odd} \\ 2, & i = 2 \\ 2i+2, & 4 \leqslant i \leqslant \frac{n}{2} - 1 \text{ and } i \text{ is even} \\ 2i+4, & \frac{n}{2} + 1 \leqslant i \leqslant n \text{ and } i \text{ is even} \end{cases}$$

$f(v_2') = 7$ and $f(v_3') = 6$.

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

When $n = 6$, an odd sum labeling of G is given in Figure 2.11.

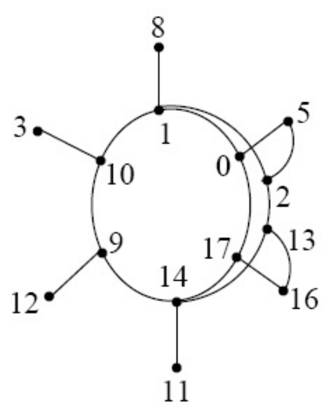


Figure 2.11. An odd sum labeling of G when $n = 6$.

□

Proposition 2.5. *Let G be the graph obtained by duplicating an edge e of the ladder $L_n, n \geqslant 2$. Then, G is not an odd sum graph only when $n = 3$ and e is the edge between the pendant vertices of the paths of length $n-1$ in L_n.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the paths of length $n-1$ in the ladder L_n. Let G be the graph obtained by duplicating any one of the edge e of L_n.

Case 1. e' is the duplicating edge of an pendant edge e of the paths of length $n-1$.

Let $e' = u_1'u_2'$ be the duplicating edge of $e = u_1u_2$. Assume that $n \geqslant 4$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 3n + 2\}$ as follows:

$$f(u_i) = \begin{cases} 1, & i = 1 \\ 3i + 4, & 2 \leqslant i \leqslant n - 1 \\ 3i + 2, & i = n, \end{cases}$$

$$f(v_i) = \begin{cases} 0, & i = 1 \\ 3i - 1, & 2 \leqslant i \leqslant n \end{cases}$$

$$\text{and } f(u'_i) = i + 2, i = 1, 2.$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

When $n = 2$ and 3, an odd sum labeling of G is given in Figure 2.12.

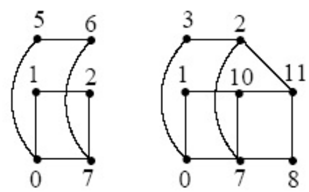


Figure 2.12. An odd sum labeling of G when $n = 2$ and 3.

Case 2. e' is the duplicating edge of an edge $e = u_iu_{i+1}$(or v_iv_{i+1}), for some $i, 2 \leqslant i \leqslant n - 2$.

Let $e' = u'_iu'_{i+1}$ be the duplicating edge of the edge $e = u_iu_{i+1}$ for some $i, 2 \leqslant i \leqslant n - 2$. Assume that $n \geqslant 5$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 3(n + 1)\}$ as follows:

$$f(u_j) = \begin{cases} 3j - 2, & 1 \leqslant j \leqslant i \\ 3j + 2, & j = i + 1 \\ 3j + 4, & j = i + 2 \\ 3j + 2, & i + 3 \leqslant j \leqslant n, \end{cases}$$

$$f(v_j) = \begin{cases} 3j - 3, & 1 \leqslant j \leqslant i \\ 3j + 1, & j = i + 1, i + 2 \\ 3j + 3, & i + 3 \leqslant j \leqslant n \end{cases}$$

$$\text{and } f(u'_j) = \begin{cases} 3j + 2, & j = i \\ 3j, & j = i + 1. \end{cases}$$

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

When $n = 4$, an odd sum labeling of G is given in Figure 2.13.

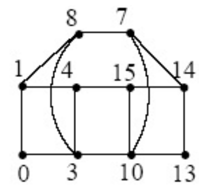


Figure 2.13. An odd sum labeling of G when $n = 4$ and $i = 2$.

Case 3. Let $e' = u_1'v_1'$ be the duplicating edge of the edge $e = u_1v_1$.
Subcase (i). $n \equiv 0(mod\,2)$ and $n \geqslant 4$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 3n+1\}$ as follows:

$$f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \\ 3j+3, & 3 \leqslant j \leqslant n-1 \text{ and } j \equiv 3(mod\,4) \\ 3j+1, & 4 \leqslant j \leqslant n \text{ and } j \equiv 0(mod\,4) \\ 3j-1, & 5 \leqslant j \leqslant n \text{ and } j \equiv 1,2(mod\,4), \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 8, & j = 2 \\ 3j, & 3 \leqslant j \leqslant n \text{ and } j \equiv 2,3(mod\,4) \\ 3j-2, & 4 \leqslant j \leqslant n \text{ and } j \equiv 0(mod\,4) \\ 3j+4, & 5 \leqslant j \leqslant n-1 \text{ and } j \equiv 1(mod\,4), \end{cases}$$

$f(u_1') = 2$ and $f(v_1') = 5$.

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

For $n = 2$, the graph G is isomorphic to the graph on Figure 2.14.

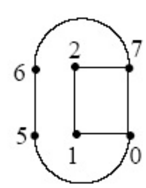


Figure 2.14. An odd sum labeling of G when $n = 2$.

Subcase (ii). $n \equiv 1(mod\,2)$ and $n \geqslant 5$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 3n + 1\}$ as follows:

$$f(u_j) = \begin{cases} 0, & j = 1 \\ 3, & j = 2 \\ 3j + 3, & j = 3, 4 \leqslant j \leqslant n - 1 \text{ and } j \equiv 0(mod\, 4) \\ 3j + 1, & 5 \leqslant j \leqslant n \text{ and } j \equiv 1(mod\, 4) \\ 3j - 1, & 6 \leqslant j \leqslant n \text{ and } j \equiv 2, 3(mod\, 4), \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 8, & j = 2 \\ 3j, & j = 3, 7 \leqslant j \leqslant n \text{ and } j \equiv 0, 3(mod\, 4) \\ 3j - 2, & j = 4, 5 \leqslant j \leqslant n \text{ and } j \equiv 1(mod\, 4) \\ 3j + 4, & 6 \leqslant j \leqslant n - 1 \text{ and } j \equiv 2(mod\, 4), \end{cases}$$

$f(u'_1) = 2$ and $f(v'_1) = 5$.

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

When $n = 3$, it is observed that there does not exist an odd sum labeling of G.

Case 4. Let $e' = u'_i v'_i$ be the duplicating edge of the edge $e = u_i v_i, 2 \leqslant i \leqslant \frac{n+1}{2}$.

Subcase (i). $n - i \equiv 1(mod\, 2)$ and $n - i \geqslant 3$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 3(n + 1)\}$ as follows:

$$f(u_j) = \begin{cases} 3j - 2, & 1 \leqslant j \leqslant i - 1 \\ 3j - 4, & j = i \\ 3j + 4, & j = i + 1 \\ 3j + 2, & i + 2 \leqslant j \leqslant n \text{ and } j - i \equiv 1, 2(mod\, 4) \\ 3j, & i + 2 \leqslant j \leqslant n \text{ and } j - i \equiv 3(mod\, 4) \\ 3j + 6, & i + 2 \leqslant j \leqslant n \text{ and } j - i \equiv 0(mod\, 4), \end{cases}$$

$$f(v_j) = \begin{cases} 3j - 3, & 1 \leqslant j \leqslant i - 1 \\ 3j - 1, & j = i, i + 1 \\ 3j + 5, & i + 2 \leqslant j \leqslant n \text{ and } j - i \equiv 2(mod\, 4) \\ 3j + 3, & i + 2 \leqslant j \leqslant n \text{ and } j - i \equiv 3(mod\, 4) \\ 3j + 1, & i + 2 \leqslant j \leqslant n \text{ and } j - i \equiv 0, 1(mod\, 4), \end{cases}$$

$f(u'_i) = 3i + 4$ and $f(v'_i) = 3i + 3$.

From this vertex labeling, the required induced edge labeling for G will be attained.

Thus f is an odd sum labeling of G.

Subcase (ii). $n - i \equiv 0(mod\, 2)$ and $n - i \geqslant 4$.

Define $f : V(G) \to \{0, 1, 2, \ldots, 3(n+1)\}$ as follows:

$$f(u_j) = \begin{cases} 3j-2, & 1 \leqslant j \leqslant i-1 \\ 3j-4, & j = i \\ 3j+4, & j = i+1 \\ 3j+2, & j = i+2 \\ 3j, & j = i+3 \\ 3j, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 0 (mod\, 4) \\ 3j+6, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 1 (mod\, 4) \\ 3j+2, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 2, 3 (mod\, 4), \end{cases}$$

$$f(v_j) = \begin{cases} 3j-3, & 1 \leqslant j \leqslant i-1 \\ 3j-1, & j = i, i+1 \\ 3j+5, & j = i+2, i+3 \\ 3j+3, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 0 (mod\, 4) \\ 3j+1, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 1, 2 (mod\, 4) \\ 3j+5, & i+4 \leqslant j \leqslant n \text{ and } j-i \equiv 3 (mod\, 4), \end{cases}$$

$f(u'_i) = 3i + 4$ and $f(v'_i) = 3i + 3$.

From this vertex labeling, the required induced edge labeling for G will be attained. Thus f is an odd sum labeling of G.

An odd sum labelings of the graphs in Case 4 which do not fall on subcase (i) and (ii) are given in Figure 2.15.

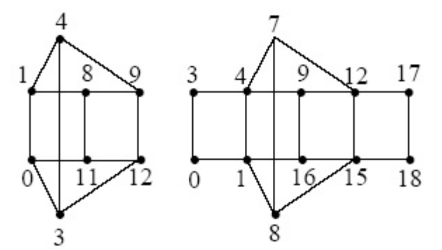


Figure 2.15. An odd sum labeling of G when $n = 3, i = 2$ and $n = 5, i = 3$.

□

References

[1] S. Arockiaraj and P. Mahalakshmi, On odd sum graphs, *International Journal of Mathematical Combinatorics*, **4** (2013), 58–77.

[2] S. Arockiaraj, P. Mahalakshmi and P. Namasivayam, Odd sum labeling of some subdivision graphs, *Kragujevac Journal of Mathematics*, (To appear)

[3] F. Buckley and F. Harary, *Distance in graphs*, Addison-Wesley, Reading, 1990.

[4] R. Balakrishnan, A. Selvam and V. Yegnanarayanan, On felicitous labeling of graphs, *Graph theory and its applications*, (1996), 47–61.

[5] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, **17** (2014), # DS6.

[6] K. Manickam and M. Marudai, Odd mean labelings of graphs, *Bulletin of Pure and Applied Sciences*, **25E**(1) (2006), 149–153.

[7] S. Somasundaram and R. Ponraj, Mean labelings of graphs, *National Academy Science Letter*, **26** (2003), 210–213.

[8] Selvam Avadyappan and R. Vasuki, Some results on mean graphs, *Ultra Scientist of Physical Sciences*, **21M**(1) (2009), 273–284.

[9] S. K. Vaidya and C. M. Barasara, Harmonic mean labeling in the context of duplication of graph elements, *Elixir Discrete Mathematics*, **48** (2012), 9482–9485.