Modular irregularity strength of dense graphs

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Abstract

We solve the open problem posed in *Modular irregularity strength of graphs, Electron. J. Graph Theory and Appl. 8 (2020), 435–433*, asking about the modular irregularity strength of the complete graph K_n for all $n \geq 3$. Furthermore, we establish also the exact values of the modular irregularity strength of complete bipartite graphs $K_{n,n+t}$ for any positive integer n and $t = 0, 1, 2$.

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1. Introduction

Let G be a finite simple undirected graph. A mapping from a set of elements of G to a set of numbers is called a *labeling* for the graph G. The numbers (usually non-negative integers) which are in the range of the mapping are called *labels*. The name of labelings depend mainly on the domains of the mapping. In this paper we focus on *edge labelings*, labelings that have edge-set as their domain. Some specific type of edge labelings were introduced by Chartrand, Jacobson, Lehel, Oellerman, Ruiz and Saba in [6], where the label of every edge is a positive integer such that the sum of all edge labels incident to a vertex is different for all vertices. Such labelings are called

irregular assignments. The *irregularity strength* of a graph G, denoted by $\mathrm{s}(G)$, is the minimum k for which a graph admits an irregular assignment using the number k as the largest edge label. There are a lot of results regarding with irregularity strength of graphs. Among them can be seen in [1, 2, 5, 7, 8, 9, 10, 11].

Irregularity strength of graphs inspired Bača, Muthugurupackiam, Kathiresan, Ramya in [4] to introduce a new graph parameter, so called *modular irregularity strength* which is explained in the following way.

Consider a graph $G(V, E)$ of order n having no component of order 2. An edge labeling $\tau : E(G) \to \{1, 2, \ldots, k\}$, for some positive integer k, is called k-*labeling* of the graph G. Let Z_n be the set of integers modulo n and the function $\sigma : V(G) \to Z_n$, be defined by

$$\sigma(u) = \sum \tau(uv),$$

over all vertices v adjacent to u. The value of $\sigma(u)$ is called the *modular weight* of the vertex u. If σ is a bijective function, then the function τ is called *modular irregular k-labeling*. Furthermore, the minimum k for which the graph G admits a modular irregular k-labeling is named the *modular irregularity strength* and is denoted by $\mathrm{ms}(G)$. It is defined as $\mathrm{ms}(G) = \infty$ in the case when there is no modular irregular labeling of G. As it is noted in [4], we can easily observe that irregularity strength of graph is defined only for graphs containing at most one isolated vertex and no connected component of order 2. Modular irregularity strength of some classes of graph has been established (see e.g. [3, 4, 12]).

For positive integers n and d, $n > d$, let us denote by $Z_n^* := Z_n - \{0\}$ and let (n, d) be the greatest common divisor of n and d. From basic number theory we can immediately see that if $(n, d) = 1$, then $\{\underline{d}, \underline{2d}, \underline{3d}, \ldots, \underline{(n-1)d}\} = Z_n^*$, where $\underline{s}$ stands for the least residue of s modulo n. By using this equality, it is easy to see the following theorem.

Theorem 1.1. *Let $G = (V, E)$ be a graph of order n with $\mathrm{s}(G) = k$. Assume that there exists an irregular assignment of G with edge values at most k, where weights of vertices form the set $\{a, a + d, a + 2d, \ldots, a + (n-1)d\}$ for some positive integers a and $d < n$. If $(n, d) = 1$, then $\mathrm{s}(G) = \mathrm{ms}(G) = k$.*

If $d = 1$ in the above theorem, we obtain the following corollary which was proposed as Lemma 2.2 in [4].

Corollary 1.1. *Let $G = (V, E)$ be a graph with no component of order ≤ 2 and let $\mathrm{s}(G) = k$. If there exists an irregular assignment of G with edge values at most k, where weights of vertices constitute a set of consecutive integers, then $\mathrm{s}(G) = \mathrm{ms}(G) = k$.*

The modular irregularity strength of some graphs was recently established in [4]. There we can find the following results: For a path P_n of $n \geq 3$ vertices, $\mathrm{ms}(P_n) = \lceil \frac{n}{2} \rceil$ if $n \not\equiv 2 \pmod 4$ and $\mathrm{ms}(P_n) = \infty$ otherwise; For a star $K_{1,n}$ of order $n + 1$ with $n \geq 2$, $\mathrm{ms}(K_{1,n}) = n$ if $n \equiv 0, 2$ (mod 4), $\mathrm{ms}(K_{1,n}) = n + 1$ if $n \equiv 3 \pmod 4$, and $\mathrm{ms}(K_{1,n}) = \infty$ if $n \equiv 1 \pmod 4$; For a cycle C_n of order $n \geq 3$, $\mathrm{ms}(C_n) = \lceil \frac{n}{2} \rceil$ if $n \equiv 1 \pmod 4$, $\mathrm{ms}(C_n) = \lceil \frac{n}{2} \rceil + 1$ if $n \equiv 0, 3 \pmod 4$, and $\mathrm{ms}(C_n) = \infty$ if $n \equiv 2 \pmod 4$; For a gear graph G_n of order $2n$ with $n \geq 3$, $\mathrm{ms}(G_n) = \frac{n+2}{2}$

if $n \equiv 0 \pmod 2$, and $\mathrm{ms}(G_n) = \infty$ if $n \equiv 1 \pmod 2$; and for a triangular graph T_n of order $2n - 1$ with $n \geq 2$, $\mathrm{ms}(T_n) = \frac{n+4}{2}$ if $n \equiv 0 \pmod 2$ and $\mathrm{ms}(T_n) = \frac{n+3}{2}$ if $n \equiv 1 \pmod 2$. For $n \geq 3$ the gear graph G_n is constructed from the cycle C_n by replacing each edge of the cycle with a triangle C_3. Whereas, the triangular graph T_n, $n \geq 2$, is constructed from the path on n vertices by replacing each edge of the path with a triangle C_3.

At the closing discussion in [4], an open research problem is proposed which is about the modular irregularity strength of complete graph $K_n, n \geq 3$. In this paper we solve this problem. Besides, we also derive the modular irregularity strength of three families of complete bipartite graphs: $K_{n,n}$, $K_{n,n+1}$ and $K_{n,n+2}$.

Before continuing to the discussion of our main results, we recall the following two theorems proved in [4]. The first of them gives a lower bound of the modular irregularity strength and the second of them gives a condition when no modular irregular labeling of a graph exists.

Theorem 1.2. [4] *Let G be a graph with no component of order ≤ 2. Then*

$$\mathrm{s}(G) \leq \mathrm{ms}(G).$$

Theorem 1.3. [4] *If G is a graph of order n, $n \equiv 2 (mod\ 4)$, then G has no modular irregular labeling i.e., $\mathrm{ms}(G) = \infty$.*

2. Main Results

A graph $G(V, E)$ is called *dense*, if $\frac{1}{2} < D(G) \leq 1$, with

$$D(G) = \frac{2|E|}{|V|(|V| - 1)}, \tag{1}$$

and it is called *sparse* graph if $0 \leq D(G) \leq \frac{1}{2}$.

For the complete graph K_n, $n \geq 2$, it is well known that $|E(K_n)| = \binom{n}{2}$. Therefore, we have $D(K_n) = 1$, and hence the graph K_n is dense.

Now, we observe the density of complete bipartite graphs. Let $K_{n,n+t}$ be the complete bipartite graph of $2n + t$ vertices, $0 \leq t < n$. It is clear that $|E(K_{n,n+t})| = n(n + t)$. Then, we have

$$
\begin{aligned}
D(K_{n,n+t}) &= \frac{2(n(n+t))}{(2n+t)(2n+t-1)} \\
&= \frac{n(2n+t)}{(2n+t)(2n+t-1)} + \frac{nt}{(2n+t)(2n+t-1)} \\
&= \frac{n}{(2n+t-1)} + \frac{nt}{(2n+t)(2n+t-1)} \\
&> \frac{n}{2n+t} + \frac{nt}{(2n+t)(2n+t-1)} \\
&= \frac{2n^2+2nt-n}{(2n+t)(2n+t-1)} \\
&= \frac{2n^2+2nt-n}{4n^2+4nt-2n-t} \\
&\geq \frac{2n^2+2nt-n}{4n^2+4nt-2n} = \frac{1}{2}.
\end{aligned}
$$

From the last inequality we conclude that the complete bipartite graph $K_{n,n+t}$, with $0 \leq t < n$, is a dense graph.

Assume that $K_{n,n+t}$ is the complete bipartite graph of $2n+t$ vertices, $0 \le t < n$, with maximum degree $\Delta(K_{n,n+t}) = n + t$ and φ is an irregular assignments. Let $s(K_{n,n+t}) = k$. Consider the weights of vertices in $K_{n,n+t}$. The smallest among these weights is at least n. The largest weight must be at least $n + |V(K_{n,n+t})| - 1 = 3n + t - 1$ and at most $\Delta(K_{n,n+t})k = (n + t)k$. Thus, $3n + t - 1 \le (n + t)k$ and

$$k = s(K_{n,n+t}) \ge \left\lceil \frac{3n + t - 1}{n + t} \right\rceil.$$

Evidently, $s(K_{n,n+t}) \ge 3$ for $t = 0, 1, 2$.

2.1. The modular irregularity strength of complete graph K_n

Next theorem gives the precise value of the modular irregularity strength of complete graph K_n, $n \ge 3$, and presents a solution of the Problem 1 in [4].

Theorem 2.1. *Let K_n be a complete graph on n vertices with $n \ge 3$. Then*

$$ms(K_n) = \begin{cases} 3, & \text{if } n \not\equiv 2 \pmod 4, \\ \infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. The exact value of the irregularity strength of complete graphs is determined in [6] and it is $s(K_n) = 3$, $n \ge 3$. According to Theorem 1.2 we have that $ms(K_n) \ge 3$ for $n \ge 3$.

Name the vertices of the complete graph K_n with $v_1, v_2, \ldots, v_n$. Consider an edge labeling $\phi : E(K_n) \to \{1, 2, 3\}$. Moreover, let matrix $A = [a_{ij}]$ be the $(n \times n)$-matrix whose entry $a_{ij} = \phi(v_i v_j)$, and A_i be the ith row of the matrix A. Let us consider the following four cases, according to the order of the graph K_n.

Case 1. $n \equiv 0 \pmod 4$. In this case, we set a function ϕ as follows.

$$\begin{aligned} \phi(v_i v_i) &= 0, & \text{if } i = 1, 2, \ldots, n, \\ \phi(v_i v_j) &= 1, & \text{if } i + j \le n + 1, i \ne j, \\ \phi(v_i v_n) = \phi(v_n v_i) &= 3, & \text{if } i = \frac{n+2}{2} \\ \phi(v_i v_{i+1}) = \phi(v_{i+1} v_i) &= 3, & \text{if } \frac{n+4}{2} \le i \le n - 2; i \text{ even}, n \ge 8, \\ \phi(v_i v_j) &= 2, & \text{otherwise.} \end{aligned}$$

Using the mapping ϕ we can see the following properties of the matrix A:

(i) For every $i, 1 \le i \le \frac{n}{2}$, A_i contains no integer 3, $i - 1$ integers 2, and $n - i$ integers 1. This implies that the weight of vertex v_i, $w(v_i)$, is equal to

$$w(v_i) = 3(0) + 2(i - 1) + 1(n - i) = n + i - 2.$$

(ii) For each $i, \frac{n}{2} + 1 \le i \le n$, A_i contains exactly one integer 3, $i - 3$ integers 2, and $n - i + 1$ integers 1. Therefore the weight of vertex v_i, $w(v_i)$, is equal to

$$w(v_i) = 3(1) + 2(i - 3) + 1(n - i + 1) = n + i - 2.$$

From (i) and (ii) it follows that the weights of vertices v_i, $i = 1, 2, \ldots, n$, constitute the set of consecutive integers $W = \{n - 1, n, n + 1, \ldots, \frac{3n}{2} - 2, \frac{3n}{2} - 1, \ldots, 2n - 2\}$.

Case 2. $n \equiv 1 \pmod 4$. Now, we define a function ϕ in the following way.

$$
\begin{aligned}
\phi(v_i v_i) &= 0, \quad \text{if } i = 1, 2, \ldots, n, \\
\phi(v_i v_j) &= 1, \quad \text{if } i + j \leq n + 1, i \neq j, \\
\phi(v_i v_n) = \phi(v_n v_i) &= 3, \quad \text{if } i = \frac{n+3}{2}, \\
\phi(v_i v_{i+1}) = \phi(v_{i+1} v_i) &= 3, \quad \text{if } \frac{n+5}{2} \leq i \leq n - 2, i \text{ odd}, n \geq 9, \\
\phi(v_i v_j) &= 2, \quad \text{otherwise.}
\end{aligned}
$$

Observe that under the previous labeling ϕ from the matrix A follows:

(i) For every $i, 1 \leq i \leq \frac{n+3}{2} - 1$, A_i has no integer 3, has $i - 1$ integers 2 and $n - i$ integers 1. Based on this condition, we obtain that the weight of vertex v_i, $w(v_i)$, is equal to

$$w(v_i) = 3(0) + 2(i - 1) + 1(n - i) = n + i - 2.$$

(ii) For each $i, \frac{n+3}{2} \leq i \leq n$, A_i consists of exactly one integer 3, $i - 3$ integers 2, and $n - i + 1$ integers 1. Hence, the weight of vertex v_i, $w(v_i)$, is equal to

$$w(v_i) = 3(1) + 2(i - 3) + 1(n - i + 1) = n + i - 2.$$

Again, from (i) and (ii), we can conclude that the set of weights of vertices of K_n, namely $W = \{n - 1, n, n + 1, \ldots, \frac{3n-1}{2} - 1, \frac{3n-1}{2}, \ldots, 2n - 2\}$ contains n consecutive integers.

Case 3. $n \equiv 3 \pmod 4$. We construct a function ϕ such that

$$
\begin{aligned}
\phi(v_i v_i) &= 0, \quad \text{if } i = 1, 2, \ldots, n, \\
\phi(v_i v_j) &= 1, \quad \text{if } i + j \leq n, i \neq j, \\
\phi(v_i v_j) = \phi(v_j v_i) &= 3, \quad \text{if } \frac{n+1}{2} \leq i \leq \frac{3n-1}{4}; j = \frac{3(n+1)}{4}, \\
\phi(v_i v_j) &= 2, \quad \text{otherwise.}
\end{aligned}
$$

In this case we describe weights of the vertices v_i as follows:

(i) For every $i, 1 \leq i \leq \frac{n+1}{2} - 1$, A_i has no integer 3. But it has i integers 2 and $n - i - 1$ integers 1. This implies that for $i, 1 \leq i \leq \frac{n+1}{2} - 1$, the weight of vertex v_i, $w(v_i)$, is equal to

$$w(v_i) = 2(i) + 1(n - i - 1) = n + i - 1.$$

(ii) For all i in the set $\{\frac{n+1}{2}, \frac{n+3}{2}, \ldots, \frac{3n-1}{4}\}$, row A_i consists of exactly one integer 3, $i - 2$ integers 2 and $n - i$ integers 1. Therefore the weight of vertex v_i is equal to

$$w(v_i) = 3(1) + 2(i - 2) + 1(n - i) = n + i - 1.$$

(iii) If i runs over the set $\{\frac{3n+7}{4}, \frac{3n+11}{4}, \ldots, n\}$, row A_i consists only of $n - i$ integers 1 and $i - 1$ integers 2. This leads to the weight of vertex v_i as

$$w(v_i) = 2(i - 1) + 1(n - i) = n + i - 2.$$

(iv) For $i = \frac{3n-1}{4} + 1 = \frac{3(n+1)}{4}$, A_i contains $n - i = \frac{n-3}{4}$ integers 1, $\frac{n-1}{2}$ integers 2, and $\frac{n+1}{4}$ integers 3. This gives the following weight of vertex v_i

$$w(v_i) = 3(\frac{n+1}{4}) + 2(\frac{n-1}{2}) + 1(\frac{n-3}{4}) = 2n - 1.$$

By combining the weight sets expressed in (i) - (iv), we get the set of weights of vertices of K_n

$$W = \{n, \ldots, \frac{3(n-1)}{2}, \frac{3n-1}{2}, \ldots, \frac{7n-5}{4}, \frac{7n-1}{4}, 2n-2, 2n-1\},$$

which contains consecutive integers.

In any of the previous three cases we get a set of consecutive integers as the set of vertex weights of K_n and by Corollary 1.1 it proves that $s(K_n) = ms(K_n) = 3$ for $n \not\equiv 2 \pmod{4}$.

Case 4. $n \equiv 2 \pmod{4}$. The result follows immediately from Theorem 1.3. □

2.2. The modular Irregularity strength of $K_{n,n}$

In this section we derive the modular irregularity strength of $K_{n,n}$ for every positive integer $n \geq 2$.

Theorem 2.2. *Let $K_{n,n}$ be a complete bipartite graph on $2n$ vertices with $n \geq 2$. Then*

$$ms(K_{n,n}) = \begin{cases} 3, & \text{if } n \text{ is even,} \\ \infty, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let U and V be the partitions of vertices of $K_{n,n}$, say $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. We discuss the following three cases.

Case 1. Let n be odd, $n \geq 3$. It is clear that in this case $|V(K_{n,n})| \equiv 2 \pmod{4}$ and according to Theorem 1.3 the modular irregularity strength of $K_{n,n}$ is infinity.

Case 2. Let $n = 2$. For $j = 1, 2$ define a function $\tau : E(K_{2,2}) \to \{1, 2, 3\}$ as follows.

$$\tau(u_1 v_j) = 1 \text{ and } \tau(u_2 v_j) = j + 1.$$

The weights of the vertices u_1, u_2, v_1 and v_2 of the complete bipartite graph $K_{2,2}$ under the labeling τ admit the values

$$\begin{aligned} w(u_1) &= \tau(u_1 v_1) + \tau(u_1 v_2) = 2, \\ w(u_2) &= \tau(u_2 v_1) + \tau(u_2 v_2) = 5, \\ w(v_1) &= \tau(u_1 v_1) + \tau(u_2 v_1) = 3, \\ w(v_2) &= \tau(u_1 v_2) + \tau(u_2 v_2) = 4. \end{aligned}$$

We can see that all edge labels are at most 3 and the vertex weights are consecutive.

Case 3. Let n be even, $n \geq 4$. For $1 \leq i, j \leq n$, we set an edge labeling $\tau : E(K_{n,n}) \to \{1, 2, 3\}$ as follows.

$$
\begin{aligned}
\tau(u_2 v_n) &= 2, \\
\tau(u_{n-1} v_1) &= 2, \\
\tau(u_i v_j) &= 2, \quad \text{if } i + j = n + 1, i = \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \ldots, n \\
\tau(u_i v_j) &= 3, \quad \text{if } i + j > n + 1, i = 3, 4, \ldots, n \\
\tau(u_i v_j) &= 1, \quad \text{otherwise.}
\end{aligned}
$$

We split the vertex set of $K_{n,n}$ into four mutually disjoint subsets
$A = \{u_1, u_2, v_1, v_2\}$,
$B = \{u_i : i = 3, 4, \ldots, \tfrac{n}{2}\} \cup \{v_j : j = 3, 4, \ldots, \tfrac{n}{2}\}$,
$C = \{v_j : j = \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \ldots, n - 2\} \cup \{u_i : i = \tfrac{n}{2} + 1, \tfrac{n}{2} + 2, \ldots, n - 2\}$,
$D = \{v_{n-1}, v_n, u_{n-1}, u_n\}$.
Clearly, $A \cup B \cup C \cup D = V(K_{n,n})$.
Observe that under the edge labeling τ the weights of the vertices

(i) from the set A are equal to

$$
w(u_1) = \sum_{j=1}^{n} \tau(u_1 v_j) = \sum_{j=1}^{n} 1 = n,
$$

$$
w(u_2) = \sum_{j=1}^{n-1} \tau(u_2 v_j) + \tau(u_2 v_n) = \sum_{j=1}^{n-1} 1 + 2 = n + 1,
$$

$$
w(v_1) = \sum_{i=1}^{n-2} \tau(u_i v_1) + \sum_{i=n-1}^{n} \tau(u_i v_1) = \sum_{i=1}^{n-2} 1 + \sum_{i=n-1}^{n} 2 = n + 2,
$$

$$
w(v_2) = \sum_{i=1}^{n-2} \tau(u_i v_2) + \tau(u_{n-1} v_2) + \tau(u_n v_2) = \sum_{i=1}^{n-2} 1 + 2 + 3 = n + 3,
$$

and they create the corresponding set of weights $W_A = \{n, n+1, n+2, n+3\}$.

(ii) from the set B admit the integers

$$
w(u_i) = \sum_{j=1}^{n-i+1} \tau(u_i v_j) + \sum_{j=n-i+2}^{n} \tau(u_i v_j) = \sum_{j=1}^{n-i+1} 1 + \sum_{j=n-i+2}^{n} 3 = n + 2i - 2,
$$

$$
w(v_j) = \sum_{i=1}^{n-j} \tau(u_i v_j) + \tau(u_{n-j+1} v_j) + \sum_{i=n-j+2}^{n} \tau(u_i v_j)
$$

$$
= \sum_{i=1}^{n-j} 1 + 2 + \sum_{i=n-j+2}^{n} 3 = n + 2j - 1,
$$

and they form the common set of consecutive integers $W_B = \{n + 4, n + 5, n + 6, n + 7, \ldots, 2n - 4, 2n - 3, 2n - 2, 2n - 1\}$.

(iii) from the set C receive the integers

$$
w(v_j) = \sum_{i=1}^{n-j+1} \tau(u_i v_j) + \sum_{i=n-j+2}^{n} \tau(u_i v_j)
$$

$$
= \sum_{i=1}^{n-j+1} 1 + \sum_{i=n-j+2}^{n} 3 = n + 2j - 2,
$$

$$
w(u_i) = \sum_{j=1}^{n-i} \tau(u_i v_j) + \tau(u_i v_{n-i+1}) + \sum_{j=n-i+2}^{n} \tau(u_i v_j)
$$

$$
= \sum_{j=1}^{n-i} 1 + 2 + \sum_{j=n-i+2}^{n} 3 = n + 2i - 1,
$$

and they are consecutive elements of the set $W_C = \{2n, 2n+1, 2n+2, 2n+3, \ldots, 3n - 8, 3n - 7, 3n - 6, 3n - 5\}$.

(iv) from the set D obtain the integers

$$w(v_{n-1}) = \sum_{i=1}^{2} \tau(u_i v_{n-1}) + \sum_{i=3}^{n} \tau(u_i v_{n-1}) = \sum_{i=1}^{2} 1 + \sum_{i=3}^{n} 3 = 3n - 4,$$

$$w(v_n) = \tau(u_1 v_n) + \tau(u_2 v_n) + \sum_{i=3}^{n} \tau(u_i v_n) = 1 + 2 + \sum_{i=3}^{n} 3 = 3n - 3,$$

$$w(u_{n-1}) = \sum_{j=1}^{2} \tau(u_{n-1} v_j) + \sum_{j=3}^{n} \tau(u_{n-1} v_j) = \sum_{j=1}^{2} 2 + \sum_{j=3}^{n} 3 = 3n - 2,$$

$$w(u_n) = \tau(u_n v_1) + \sum_{j=2}^{n} \tau(u_n v_j) = 2 + \sum_{j=2}^{n} 3 = 3(n-1) + 2 = 3n - 1,$$

and they create the associated set of weights $W_D = \{3n - 4, 3n - 3, 3n - 2, 3n - 1\}$.

Evidently, all edge labels are at most 3 and the weights of the vertices form the resulting set $W = W_A \cup W_B \cup W_C \cup W_D$ of consecutive integers. Thus the function τ is the desired modular irregular 3-labeling.

$\square$

2.3. The modular irregularity strength of $K_{n,n+1}$

We partition the vertex set of $K_{n,n+1}$ into $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$. Here we have $E(K_{n,n+1}) = \{u_i v_j : 1 \le i \le n, 1 \le j \le n+1\}$.

It is clear that for all positive integers $n \ge 1$ there is $|V(K_{n,n+1})| = 2n + 1 \not\equiv 2 \pmod 4$.

Theorem 2.3. *Let $K_{n,n+1}$ be a complete bipartite graph on $2n + 1$ vertices with $n \ge 1$. Then*

$$\mathrm{ms}(K_{n,n+1}) = \begin{cases} 2, & \text{if } n = 1, \\ 3, & \text{if } n \ge 2. \end{cases}$$

Proof. It is evident that $\mathrm{ms}(K_{1,2}) = 2$.

For $n \ge 2$ define a mapping $f : E(K_{n,n+1}) \to \{1, 2, 3\}$ as follows.

$$\begin{array}{ll} f(u_i v_j) = f(v_j u_i) = 1 & \text{if } i + j \le n + 1, \\ f(u_i v_j) = f(v_j u_i) = 2 & \text{if } i + j = n + 2, \\ f(u_i v_j) = f(v_j u_i) = 3 & \text{otherwise.} \end{array}$$

Since

$$w(u_i) = \sum_{j=1}^{n-i+1} f(u_i v_j) + f(u_i v_{n-i+2}) + \sum_{j=n-i+3}^{n+1} f(u_i v_j) = \sum_{j=1}^{n-i+1} 1 + 2 + \sum_{j=n-i+3}^{n+1} 3 = n + 2i,$$

for $1 \le i \le n$, so the vertex weights form the arithmetic sequence of difference 2 with the initial weight $n + 2$ and the last weight $3n$.

For the weight of the vertex v_1 we get

$$w(v_1) = \sum_{i=1}^{n} f(u_i v_1) = \sum_{i=1}^{n} 1 = n$$

and for weights of the vertices v_j, $j = 2, 3, \ldots, n+1$, we have

$$w(v_j) = \sum_{i=1}^{n-j+1} f(u_i v_j) + f(u_{n-j+2} v_j) + \sum_{i=n-j+3}^{n} f(u_i v_j) = \sum_{i=1}^{n-j+1} 1 + 2 + \sum_{i=n-j+3}^{n} 3 = n + 2j - 3.$$

We can see that these vertex weights create the arithmetic sequence of difference 2 with the initial weight $n + 1$ and the last weight $3n - 1$.

By combining both arithmetic sequences of vertex weights of difference 2 with $w(v_1) = n$ we obtain the resulting set of consecutive integers from n to $3n$ and according to Corollary 1.1 the function f is a suitable modular irregular 3-labeling and $\mathrm{ms}(K_{n,n+1}) = 3$ for $n \geq 2$.

$\square$

2.4. The modular irregularity strength of $K_{n,n+2}$

Here, we partition the vertex set of $K_{n,n+2}$ into $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}\}$. Then for the edge set we have $E(K_{n,n+2}) = \{u_i v_j : 1 \leq i \leq n, 1 \leq j \leq n+2\}$.

Theorem 2.4. *Let $K_{n,n+2}$ be a complete bipartite graph on $2n + 2$ vertices with $n \geq 1$. Then*

$$\mathrm{ms}(K_{n,n+2}) = \begin{cases} 4, & \text{if } n = 1, \\ 3, & \text{if } n \geq 3 \text{ is odd}, \\ \infty, & \text{if } n \text{ is even}. \end{cases}$$

Proof. In [4] is stated that for the graph $K_{1,3}$ as a star on 4 vertices and with edge labels 1,2,3, the irregularity strength is 3. But this irregular 3-labeling is not modular. The modular irregularity strength of $K_{1,3}$ is 4 with edge values 1,2 and 4.

On can see that for even positive integer n, $|V(K_{n,n+2})| \equiv 2 \pmod 4$ and according to Theorem 1.3 the modular irregularity strength of $K_{n,n+2}$ is infinity.

For n odd, $n \geq 3$, we define a function $g : E(K_{n,n+2}) \to \{1, 2, 3\}$ in the following way

$$\begin{aligned}
g(u_i v_j) = g(v_j u_i) &= 1 \quad \text{if } i + j \leq n + 1, \\
g(u_i v_j) = g(v_j u_i) &= 1 \quad \text{if } i + j = n + 2, i \leq n - 3, \text{ and } i \text{ even}, \\
g(u_i v_j) = g(v_j u_i) &= 2 \quad \text{if } n + 2 \leq i + j \leq n + 4, \text{ and } i \text{ odd}, \\
g(u_{n-1} v_j) = g(v_j u_{n-1}) &= 2 \quad \text{if } j = 3, 4, \\
g(u_i v_j) = g(v_j u_i) &= 3 \quad \text{otherwise}.
\end{aligned}$$

Let us split the vertex set of $K_{n,n+2}$ into five mutually disjoint subsets
$A = \{u_1, u_{n-1}, v_1, v_2, v_3, v_4, v_{n+2}\}$,
$B = \{u_i : i = 3, 5, 7, \ldots, n-4, n-2, n\}$,
$C = \{u_i : i = 2, 4, 6, \ldots, n-5, n-3\}$,

$D = \{v_j : j = 5, 7, 9, \ldots, n - 4, n - 2, n\}$,
$E = \{v_j : j = 6, 8, 10, \ldots, n - 3, n - 1, n + 1\}$.
Clearly, $A \cup B \cup C \cup D \cup E = V(K_{n,n+2})$.
Observe that under the edge labeling g the weights of the vertices

(i) from the set A are equal to

$$w(u_1) = \sum_{j=1}^{n} g(u_1 v_j) + \sum_{j=n+1}^{n+2} g(u_1 v_j) = n + 4,$$

$$w(u_{n-1}) = \sum_{j=1}^{2} g(u_{n-1}v_j) + \sum_{j=3}^{4} g(u_{n-1}v_j) + \sum_{j=5}^{n+2} g(u_{n-1}v_j)$$
$$= 1 \cdot (2) + 2 \cdot (2) + 3 \cdot (n - 2) = 3n,$$

$$w(v_1) = \sum_{i=1}^{n} g(u_i v_1) = 1 \cdot (n) = n,$$

$$w(v_2) = \sum_{i=1}^{n-1} g(u_i v_2) + g(u_n v_2) = 1 \cdot (n - 1) + 2 = n + 1,$$

$$w(v_3) = \sum_{i=1}^{n-2} g(u_i v_3) + \sum_{i=n-1}^{n} g(u_i v_3) = 1 \cdot (n - 2) + 2 \cdot (2) = n + 2,$$

$$w(v_4) = \sum_{i=1}^{n-3} g(u_i v_4) + \sum_{i=n-2}^{n} g(u_i v_4) = 1 \cdot (n - 3) + 2 \cdot (3) = n + 3,$$

$$w(v_{n+2}) = g(u_1 v_{n+2}) + \sum_{i=2}^{n} g(u_i v_{n+2}) = 2 + 3 \cdot (n - 1) = 3n - 1,$$

and they create the corresponding set of weights $W_A = \{n, n + 1, n + 2, n + 3, n + 4, 3n - 1, 3n\}$.

(ii) from the set B admit the integers

$$w(u_i) = \sum_{j=1}^{n-i+1} g(u_i v_j) + \sum_{j=n-i+2}^{n-i+4} g(u_i v_j) + \sum_{j=n-i+5}^{n+2} g(u_i v_j)$$
$$= 1 \cdot (n - i + 1) + 2 \cdot (3) + 3 \cdot (i - 2) = n + 2i + 1,$$

and they form the set of members of arithmetic sequence of difference 4, $W_B = \{n + 7, n + 11, n + 15, \ldots, 3n - 7, 3n - 3, 3n + 1\}$.

(iii) from the set C receive the integers

$$w(u_i) = \sum_{j=1}^{n-i+1} g(u_i v_j) + g(u_i v_{n-i+2}) + \sum_{j=n-i+3}^{n+2} g(u_i v_j)$$
$$= 1 \cdot (n - i + 1) + 1 + 3 \cdot i = n + 2i + 2,$$

and they create the associated set of members of arithmetic sequence of difference 4, $W_C = \{n + 6, n + 10, n + 14, \ldots, 3n - 12, 3n - 8, 3n - 4\}$.

(iv) from the set D obtain the integers

$$w(v_j) = \sum_{i=1}^{n-j+1} g(u_i v_j) + g(u_{n-j+2} v_j) + g(u_{n-j+3} v_j) + \sum_{i=n-j+4}^{n} g(u_i v_j)$$
$$= 1 \cdot (n - j + 1) + 1 + 2 + 3 \cdot (j - 3) = n + 2j - 5,$$

and they form the set of members of arithmetic sequence of difference 4, $W_D = \{n + 5, n + 9, n + 13, \ldots, 3n - 13, 3n - 9, 3n - 5\}$.

(v) from the set E receive the integers

$$
\begin{aligned}
w(v_j) &= \textstyle\sum_{i=1}^{n-j+1} g(u_i v_j) + g(u_{n-j+2} v_j) + g(u_{n-j+3} v_j) + g(u_{n-j+4} v_j) + \sum_{i=n-j+5}^{n} g(u_i v_j) \\
&= 1 \cdot (n-j+1) + 2 + 3 + 2 + 3 \cdot (j-4) = n + 2j - 4,
\end{aligned}
$$

and they create the associated set of members of arithmetic sequence of difference 4, $W_E = \{n+8, n+12, n+16, \ldots, 3n-10, 3n-6, 3n-2\}$.

Indeed, all edge labels are at most 3 and combining the previous sets of weights we obtain the resulting set $W = W_A \cup W_B \cup W_C \cup W_D \cup W_E$ of consecutive integers. Thus the function g is the desired modular irregular 3-labeling and we are done.

□

3. Discussion

We solved a very small portion of the modular irregularity strength of complete bipartite graphs. The rest is still unsolved therefore for further investigation we propose the following problem.

Problem 1. *For the complete bipartite graphs $K_{n,n+t}$, $n \geq 1$, $3 \leq t < n$, determine the exact value of the modular irregularity strength.*

It seems to be interesting to investigate the modular irregularity strength of the complete tripartite graphs $K_{m,n,s}$ for $m, n, s \geq 1$. We conclude the paper with the following open problem.

Problem 2. *For the complete tripartite graphs $K_{m,n,s}$, $m, n, s \geq 1$, determine the exact value of the modular irregularity strength.*

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