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Regular handicap graphs of order $n \equiv 4 \pmod 8$

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Abstract

A *handicap distance antimagic labeling* of a graph $G = (V, E)$ with n vertices is a bijection $\hat{f} : V \to \{1, 2, \dots, n\}$ with the property that $\hat{f}(x_i) = i$, the *weight* $w(x_i)$ is the sum of labels of all neighbors of x_i, and the sequence of the weights $w(x_1), w(x_2), \dots, w(x_n)$ forms an increasing arithmetic progression. A graph G is a *handicap distance antimagic graph* if it allows a handicap distance antimagic labeling. We construct r-regular handicap distance antimagic graphs of order $n \equiv 4 \pmod 8$ for all feasible values of r.

Keywords: graph labeling, handicap labeling, regular graphs, tournament scheduling
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1. Introduction

A *complete round robin tournament of n teams* is a tournament in which every team plays each of the remaining $n - 1$ teams. Complete round robin tournaments are generally considered to be fair. However, when we look at them more closely, we may realize that they in fact favor the strongest teams.

The reason is that when the teams are ranked $1, 2, \dots, n$ according to their standings, then the sum of rankings of all opponents of the i-th ranked team, called *weight* and denoted $w(i)$,

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is $w(i) = n(n+1)/2 - i$, and the sequence $w(1), w(2), \ldots, w(n)$ is a decreasing arithmetic progression with difference one. Therefore, the strongest team plays only lower ranked opponents and thus has the easiest schedule. On the other hand, the weakest team plays only stronger teams and thus has the most difficult schedule.

This property can be generalized for tournaments where each team plays the same number $r < n - 1$ games as follows. A tournament of n teams in which every team plays precisely r opponents, where $r < n - 1$ and the sequence $w(1), w(2), \ldots, w(n)$ is a decreasing arithmetic progression with difference one is called a *fair incomplete round robin tournament* and denoted $\mathrm{FIT}(n, r)$. Again, as in complete tournaments, strong teams play weaker teams, and weak teams play stronger teams.

This can be eliminated by scheduling tournaments where each team plays in aggregate opponents of the same total strength. Such tournaments are called *equalized incomplete round robin tournaments* (denoted $\mathrm{EIT}(n, r)$) and were introduced by the first author in [4] and [9].

We can take it one step further and try to design tournaments where weak teams have a better chance at winning than in equalized incomplete tournaments. Hence, we want to schedule tournaments where the sequence $w(1), w(2), \ldots, w(n)$ would be an increasing arithmetic progression. Such a tournament in which every team plays $r < n - 1$ games is called a *handicap incomplete round robin tournament*. The case of n teams where $n \equiv 0 \pmod 8$ was completely solved by the authors in [12]. A summary of results of handicap tournaments obtained by the authors and other researchers can be found in [10]. More detailed overview of these and other related results is provided in Section 3.

In this paper we provide a detailed construction for $n \equiv 4 \pmod 8$ for all feasible regularities.

Remark. Since this paper is a direct continuation of [12], this and the following section were adopted from that paper. Therefore, the reader may find here many similarities or even identical parts.

2. Basic Notions

By $G = (V, E)$ we mean a finite simple graph of order n. To simplify notation where possible, we identify vertex names with their labels, thus by stating i we refer to the vertex labeled i.

Our constructions are based on magic- and antimagic-type labelings defined below. Namely, *distance magic* and *distance antimagic labeling*. The distance magic labeling was originally coined as a sigma labeling by Vilfred [20] in 1994, and then by Miller et. al. [17] using the name 1-vertex magic vertex labeling.

Definition 2.1. A *distance magic labeling* of a graph G of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer μ such that

$$\sum_{y \in N(x)} f(y) = \mu \qquad \forall x \in V.$$

The constant μ is called the *magic constant* of the labeling f, and $N(x)$ denotes the set of all vertices adjacent to v. The sum $\sum_{y \in N(x)} f(y)$ is called the weight of vertex x and is denoted $w(x)$. A graph that admits a distance magic labeling is called a *distance magic graph*.

Definition 2.2. A *distance* d*-antimagic labeling* of a graph G with n vertices is a bijection $\bar{f} : V \to \{1, 2, \ldots, n\}$ with the property that there exists an ordering of the vertices of G such that the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an arithmetic progression with difference d. When $d = 1$, then $\bar{f}$ is called just distance antimagic labeling. A graph G is a *distance* d*-antimagic graph* if it allows a distance d-antimagic labeling, and a *distance antimagic graph* when $d = 1$.

A survey on distance magic graphs can be found in [1], while an often updated overview of results of all types of labelings can be found in [13].

The term *handicap labeling* is due to Kovářová [16]; it was previously referred to as *ordered distance antimagic labeling* by Froncek in [5]. We provide a more general version of the definition, as there was recently significant development in this direction.

Definition 2.3. A *handicap distance* d*-antimagic labeling* or simply d*-handicap labeling* of a graph G with n vertices is a bijection $\hat{f} : V \to \{1, 2, \ldots, n\}$ with the property that $\hat{f}(x_i) = i$ and the sequence of the weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an increasing arithmetic progression with difference d. A graph G is a *handicap distance* d*-antimagic graph* if it allows a distance d-antimagic labeling, and a *handicap distance antimagic graph* when $d = 1$.

Observe that in a handicap labeling a vertex with a lower label has a lower weight than a vertex with higher label. Thus, if we think of the vertices as teams and label them according to their strength, an r-regular handicap distance antimagic graph is in fact a representation of a handicap incomplete round robin tournament.

3. Preliminary and Related Results

An overview of results on fair and equalized tournaments $\mathrm{FIT}(n, r)$ and $\mathrm{EIT}(n, r)$ obtained in [4] and [9] can be found in [12].

Handicap tournaments of even order have been studied extensively and the following results with detailed constructions were published so far.

Theorem 3.1. [12] *There exists an* r*-regular* 1*-handicap graph of order* n *where* $n \equiv 0 \pmod 8$ *if and only if* r *is odd and* $3 \le r \le n - 5$.

Theorem 3.2. [11, 15] *There exists an* r*-regular* 1*-handicap graph of order* n *where* $n \equiv 2 \pmod 4$ *if and only if* $r \equiv 3 \pmod 4$ *and* $3 \le r \le n - 7$ *except when* $r = 3$ *and* $n \in \{10, 12, 14, 18, 22, 26\}$.

In an unpublished manuscript, Kovář, Kovářová, Krajc, Kravčenko, and Krbeček [15] independently proved a result that partially overlaps with our main result. However, their methods were different from those presented in the next section.

Theorem 3.3. [15] *There exists an* r*-regular* 1*-handicap graph of even order* n *for all* $n \ge 28$ *and* $3 \le r \le n - 11$ *if not both* $n \equiv 2 \pmod 4$ *and* $r \equiv 1 \pmod 4$.

For the small values $n < 28$, we only list their results relevant to our case.

Lemma 3.1. [15] *There exist* 3*- and* 5*-regular* 1*-handicap graphs on* 20 *vertices and a* 5*-regular* 1*-handicap graph on* 12 *vertices. On the other hand, a* 3*-regular* 1*-handicap graph on* 12 *vertices does not exist.*

The main result of this paper was published in [10] without proof. Therefore, we provide a detailed constructive proof in this paper. Since our construction presented in the next section only covers regularities greater than five, we state the case of $n \equiv 0 \pmod 4$ and $r = 3, 5$ that follows from Theorem 3.3 and Lemma 3.1 separately.

Corollary 3.1. [15] *There exists a* 3*-regular* 1*-handicap graph of order* n *where* $n \equiv 4 \pmod 8$ *if and only if* $n \geq 20$ *and a* 5*-regular such graph if and only if* $n \geq 12$*.*

For graphs of odd order, much less is known. The following result was proved by the first author.

Theorem 3.4. [8] *There exists an* r*-regular handicap graph of an odd order* n *for at least one value of* r *if and only if* $n = 9$ *or* $n \geq 13$*.*

Recently, some results on handicap distance d-antimagic graphs where $d = 2$ were obtained by the first author, including a full characterization for $n \equiv 0 \pmod{16}$.

Theorem 3.5. [6] *If G is an* r*-regular* 2*-handicap graph, then* r *is even.*

Theorem 3.6. [7] *There exists an* r*-regular* 2*-handicap graph of order* $n \equiv 0 \pmod{16}$ *if and only if* $n \geq 16$ *and* $4 \leq r \leq n - 6$*.*

Theorem 3.7. [6] *There exists an* r*-regular* 2*-handicap graph of order* n *for every positive* $n \equiv 8 \pmod{16}$*,* $n \geq 56$ *and every even* r *satisfying* $6 \leq r \leq n - 50$*.*

Theorem 3.6 also follows from a more general result by Freyberg [3].

Theorem 3.8. [3] *Let* $d > 1$ *be given and let* $n \equiv 0 \pmod{2^{d+2}}$*. Then an* r*-regular* d*-handicap graph of order* n *exists if and only if* $d + 2 \leq r \leq n - d - 4$ *and* $r \equiv d \pmod 2$*.*

Additional results have been obtained for more general d-handicap tournaments by Freyberg in [2]. These include a variety of results for even d, a partial characterization of order n that permits d odd, and multiple restrictions on the feasible regularities based on n and d.

For any graph with a given regularity r and order n a simple counting argument shows the weight of each vertex i is already known as in the following lemma (see [10]).

Lemma 3.2. *In an* r*-regular handicap graph with* n *vertices the weight of every vertex is* $w(i) = (r - 1)(n + 1)/2 + i$*.*

Each vertex weight is an integer value obtained as a sum of integers. The previous lemma is used in a number of non-existence results. The following can be found amongst other non-existence results, see, e.g., [10] or [18].

Lemma 3.3. *If* n *is even,* $n > 2$*, and* G *is an* r*-regular* 1*-handicap graph, then* r *is odd and* $3 \leq r \leq n - 5$*.*

More restrictive bounds may apply to some specific values of n, as can be seen in Theorem 3.2.

We now proceed to the primary focus of this paper. Since from now on we only deal with 1-handicap graphs, we will be using just the term *handicap graph* instead.

4. Construction for $n \equiv 4 \pmod 8$

We use the following convention. When i is joined to k by an edge, we will use the notation $[i \mid k]$. Further, $[a, b \mid c, d]$ will denote the complete bipartite graph $K_{2,2}$ where a and b are in one partite set and c and d in the other.

Our basic building block in the construction will be the complete bipartite graph $[a, b|c, d]$ in which $a + b = c + d$. The actual sum will differ in various parts of our construction. A graph in which every edge is in exactly one $K_{2,2}$ can be viewed as the *lexicographic product* or a *composition* of graphs H and $2K_1 \cong \overline{K}_2$, denoted $H[\overline{K}_2]$. This graph can be constructed from H by replacing each vertex u by two vertices u', u'' and each edge uv be the complete bipartite graph $[u', u'' \mid v', v'']$. A composition $H[\overline{K}_2]$ is sometimes called a *blown up graph* H.

We will also use the notion of a *bubble graph*. When one has a graph $H[\overline{K}_2]$, then the graph H can be viewed as an inverse image of it with respect to the composition with $\overline{K}_2$. The bubble graph is a slight generalization of such inverse. An illustration can be seen in Figures 5 and 6. For a detailed description of the construction and usage of the bubble graph, we refer the reader to [12].

Definition 4.1. Let J be a graph on $2m$ vertices with the vertex set $\{u_1', u_1'', u_2', u_2'', \dots, u_m', u_m''\}$ which contains no edge $u_i'u_i''$ for any i. Then the *bubble graph* of J, denoted $B(J)$ or just simply B, is a graph on m vertices with the vertex set $\{u_1, u_2, \dots, u_m\}$ in which u_iu_j is an edge if and only if at least one of the four edges $u_i'u_j', u_i'u_j'', u_i''u_j', u_i''u_j''$ is present in J.

In our construction, we will need a result on 1-factorization of certain regular graphs. Recall that a 1-*factor* of a graph H is a 1-regular spanning subgraph, also often called a *perfect matching*. Then a 1-*factorization* of H is a decomposition of the edge set of H into disjoint 1-factors.

If H has m vertices, then the *length* of an edge $[k \mid j]$ is the minimum of the set $\{|k - j|, m - |k - j|\}$. This can be visualized as follows. Place the vertices at uniform distance in a circle, starting with 1 at the top-center position, in a clock–wise fashion. Then the length of $[k \mid j]$ is the "circular distance" between the vertices k and j, i.e., the number of steps we need to take around the circle to get from k to j using the shorter of the two paths between them.

Definition 4.2. Let m be a positive integer and D a non-empty subset of $\{1, 2, \dots, \lfloor m/2 \rfloor\}$. The *circulant graph* $C_m(D)$ is the graph with vertex set $\{1, 2, 3 \dots, m\}$ and edge set consisting of all edges whose length is in D.

There is a very specific requirement for when a circulant graph admits a 1-factorization. The following lemma is due to Stern and Lenz [19].

Lemma 4.1. [19] *Let n be a positive integer and D a non-empty subset of $\{1, 2, \dots, \lfloor n/2 \rfloor\}$. If D contains an element d where $n/\gcd(d, n)$ is even, then the circulant graph $C_n(D)$ admits a* 1*-factorization.*

In other words, to see if $C_n(D)$ has a 1-factorization, all we need to do is find an edge length $d \in D$ so that $n/\gcd(d, n)$ is an even integer.

We want to prove the following.

Theorem 4.1. *For $n \equiv 4 \pmod 8$ and r odd, there exists an r-regular handicap graph G for* $7 \leq r \leq n - 5$.

Proof. By construction. To simplify our terminology and increase clarity, we split the edges up into three color classes. Suppose $r = 2s + 7$. We will have one red edge, six blue edges, and $2s$ black edges. We build our graph by first constructing the red subgraph, then adding blue edges, and finally the black ones. Since we have six blue edges we will break that stage up into three parts.

Step 1: Set $n = 8k + 4$ and start with the red edges as follows. First we draw edges between lower vertices as

$$[1 \mid 2k+2], [2 \mid 2k+3], [3 \mid 2k+4], \ldots, [2k+1 \mid 4k+2],$$

followed by edges between upper vertices,

$$[4k+3 \mid 6k+4], [4k+4 \mid 6k+5], \ldots, [6k+3 \mid 8k+4].$$

Let $w_r(i)$ denote the partial weight of vertex i obtained from the red edges. We have

$$w_r(i) = 2k + 1 + i \text{ for } i \in [1, 2k+1] \cup [4k+3, 6k+3]$$

and

$$w_r(i) = i - (2k+1) \text{ for } i \in [2k+2, 4k+2] \cup [6k+4, 8k+4]\,.$$

Step 2.1: In the first stage of adding blue edges, we construct multiple copies of $K_{2,2}$ that include exactly half of the vertices. Namely

$$[1, 6k{+}3 \mid 2, 6k{+}2], [2, 6k{+}2 \mid 3, 6k{+}1], \ldots, [2k, 4k{+}4 \mid 2k{+}1, 4k{+}3], [2k{+}1, 4k{+}3 \mid 1, 6k{+}3].$$

We will call the set of vertices used U, so $U = \{1, 2, \ldots, 2k+1, 4k+3, 4k+4, \ldots, 6k+3\}$.

Step 2.2: Similar to the first stage, we add $K_{2,2}$'s to the other half of the vertices, specifically

$$[2k{+}2, 8k{+}4 \mid 2k{+}3, 8k{+}3], \ldots, [4k{+}1, 6k{+}5 \mid 4k{+}2, 6k{+}4], [4k{+}2, 6k{+}4 \mid 2k{+}2, 8k{+}4]$$

and name the set of vertices used here W. Hence, $W = \{2k+2, 2k+3, \ldots, 4k+2, 6k+4, 6k+5, \ldots, 8k+4\}$.

Step 2.3: The graph induced by the blue edges is currently 4-regular. To add the last two edges we intertwine the copies of $K_{2,2}$ already created. For each new $K_{2,2}$ one partite set comes from U and one partite set comes from W. For example, we take the first partite set from Step 2.1, and connect it to the second partite set from Step 2.2. In general, connect

$$[1, 6k+3 | 2k+3, 8k+3], [2, 6k+2 | 2k+5, 8k+1], \ldots, [2k+1, 4k+3 | 2k+2, 8k+4].$$

Let $w_b(i)$ denote the partial weight of vertex i obtained from the blue edges. We have that

$$w_b(i) = 22k + 14 \text{ for } i \in [1, 2k+1] \cup [4k+3, 6k+3]$$

and

$$w_b(i) = 26k + 16 \text{ for } i \in [2k+2, 4k+2] \cup [6k+4, 8k+4]\,.$$

Then for $i \in [1, 2k+1] \cup [4k+3, 6k+3]$ we obtain

$$w_b(i) + w_r(i) = 22k + 14 + 2k + i = 24k + 15 + i$$

and for $i \in [2k+2, 4k+2] \cup [6k+4, 8k+4]$ we have

$$w_b(i) + w_r(i) = 26k + 16 + i - (2k+1) = 24k + 15 + i\,.$$

So we have a 7-regular handicap graph. If we can have the black edges contribute the same weight μ to each vertex, we will not be affecting the arithmetic progression of our weights, and therefore, still have a handicap graph with higher regularities.

Step 3: Recall that $r = 2s + 7$. Our goal now is to show that we can add $2s$ black edges such that the graph induced by the black edges is distance magic. Pair the vertices 1 with $8k+4$, 2 with $8k+3, \ldots,$ and $4k+2$ with $4k+3$, so that sum of these pairs is $8k+5$. Each pair can be thought of as a graph $\overline{K}_2$ with no edges and becomes a vertex in our bubble graph B. In B, there will be an edge between two bubbles $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ if and only if there would be a red or blue edge (or both) between either x_1 or x_2 and y_1 or y_2.

To more easily understand the structure of the bubble graph, we look at the edge lengths. We refer to each bubble by the minimum of the two labels it contains. Place the bubbles at uniform distance in a circle, starting with 1 at the top-center position, in a clock–wise fashion.

In Step 1, we define our red edges, all of which are of length $2k+1$. In Step 2.1, we see blue edges come in a couple different lengths, namely 1 and $2k$. In Step 2.2, we see blue edges also come in length 1 and $2k$. In Step 2.3, we have blue edges of lengths 1 and $2k$ as well. Thus, in B, the edges are all of length 1, $2k$, and $2k+1$. Since $n = 8k+4$ we have exactly $n' = 4k+2$ bubbles. For any given bubble, there are 2 bubbles at length 1 away (one clockwise and one counter-clockwise), 2 bubbles at length $2k$ away, and exactly one bubble at length $2k+1$ away. If all edges of lengths 1, $2k$, and $2k+1$ are used in B, B is 5-regular. Thus, $\overline{B}$ will have all edges of the lengths that are not present in B, so $\overline{B}$ is isomorphic to a circulant graph $C_{n'}(\{3, \ldots, 2k-1\})$.

Recall Lemma 4.1, which says that $\overline{B}$ can be 1-factored if there exists an edge length d of $\overline{B}$ so that $n'/\gcd(d, n')$ is an even integer. This can be done as follows. Let d be an odd edge length in the edge set of $\overline{B}$. Such a d exists since 3 will always be an edge length used in $\overline{B}$. Recall that $n' = 4k+2$. Let $n'' = n'/2 = 2k+1$, thus n'' is odd. Now, since d is odd and n' is even we have that

$$\gcd(d, n') = \gcd(d, n'/2) = \gcd(d, n'')$$

is an odd integer since both d and n'' are odd. Now, since the $\gcd(d, n'')$ divides both n'' and d, $n''/\gcd(d, n'')$ is an integer. Thus,

$$\frac{n''}{\gcd(d, n'')} = \frac{n''}{\gcd(d, n')} \Rightarrow \frac{n'}{\gcd(d, 2n'')} = \frac{n'}{\gcd(d, n')}$$

is an even integer. And so, by Lemma 4.1, $\overline{B}$ can be 1-factored.

Each black edge in $\overline{B}$ equates to a $K_{2,2}$ in the blown up graph $\overline{B}[\overline{K}_2]$. Therefore, each 1-factor in $\overline{B}$ contributes a 2-regular distance magic graph to the red and blue edges. We can add $2(\frac{n}{2}-6) = n-12$ black edges to increase regularity, if desired, for a max of $n - 12 + 1 + 6 = n - 5$. □

The reader my find it useful to see an example of the construction for Theorem 4.1, so we present one here.

5. Example Construction of 7-regular Handicap Graph on $n = 28$ Vertices

In this example, $n = 28 = 8(3) + 4$, so $k = 3$. The resulting graph is just 7-regular, but with 28 vertices it is somewhat dense for the human eye to digest. Thus at the end of the example we offer an alternative view of the graph by seperating red and blue edges. This more clearly indicates what the structure of these graphs look like.

Step 1: We start with the red edges by connecting $[1 \mid 2k+2], [2 \mid 2k+3], [3 \mid 2k+4], \ldots,$ and $[2k+1 \mid 4k+2]$, followed by $[4k+3 \mid 6k+4], [4k+4 \mid 6k+5], \ldots,$ and $[6k+3 \mid 8k+4]$. So for the lower vertices we have $[1 \mid 8], [2 \mid 9], [3 \mid 10], \ldots,$ and $[7 \mid 14]$. For the upper vertices we have $[15 \mid 22], [16 \mid 23], \ldots,$ and $[21 \mid 28]$. This is shown in Figures 1 and 7. Let $w_r(i)$ denote the weight of vertex i obtained from the red edges. We have that

$$w_r(i) = 7 + i \text{ for } i \in [1, 7] \cup [15, 21]$$

and

$$w_r(i) = i - 7 \text{ for } i \in [8, 14] \cup [22, 28] .$$

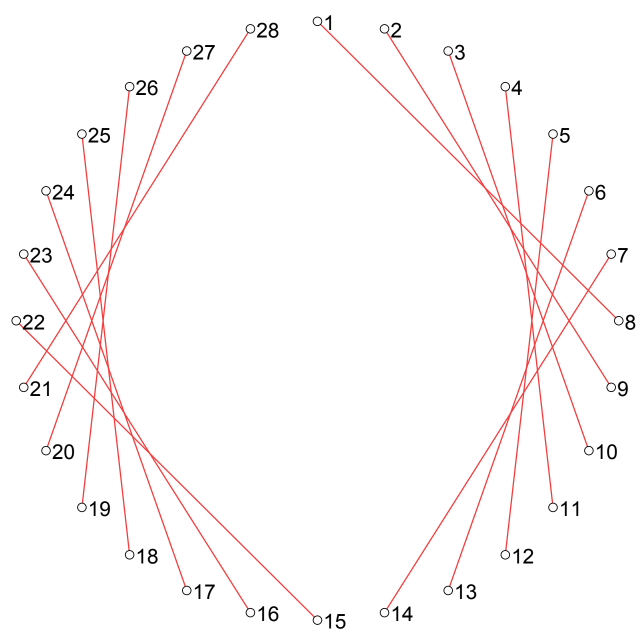

Figure 1. Step 1 on 28 vertices

Step 2.1: We now add the first set of blue copies of $K_{2,2}$. Namely $[1, 21 \mid 2, 20]$, $[2, 20 \mid 3, 19], \ldots, [6, 16 \mid 7, 15], [7, 15 \mid 1, 21]$. See Figure 2. Thus, in this example $U = \{1, 2, \ldots, 7, 15, 16, \ldots, 21\}$.

Step 2.2: We now add the second set of blue copies of $K_{2,2}$, $[8, 28 \mid 9, 27], \ldots, [13, 23 \mid 14, 22], [14, 22 \mid 8, 28]$. See Figure 3. In this example $W = \{8, 9, \ldots, 14, 22, 23, \ldots, 28\}$.

Steps 2.1 and 2.2 are shown in the alternative view in Figure 8.

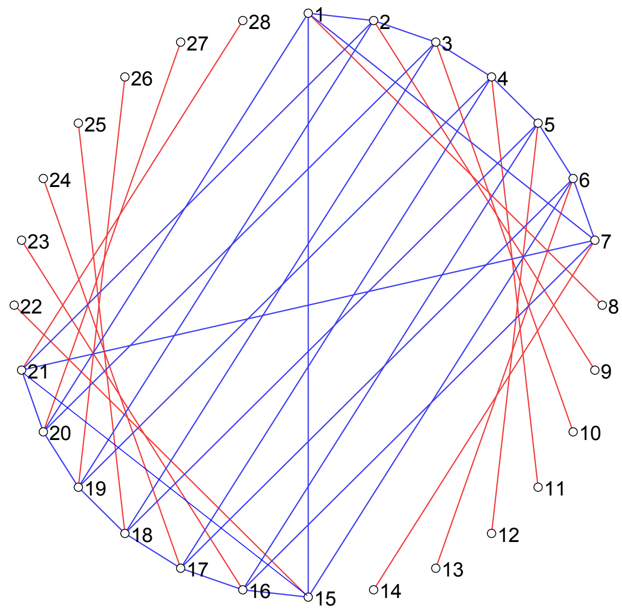

Figure 2. Step 2.1 on 28 vertices

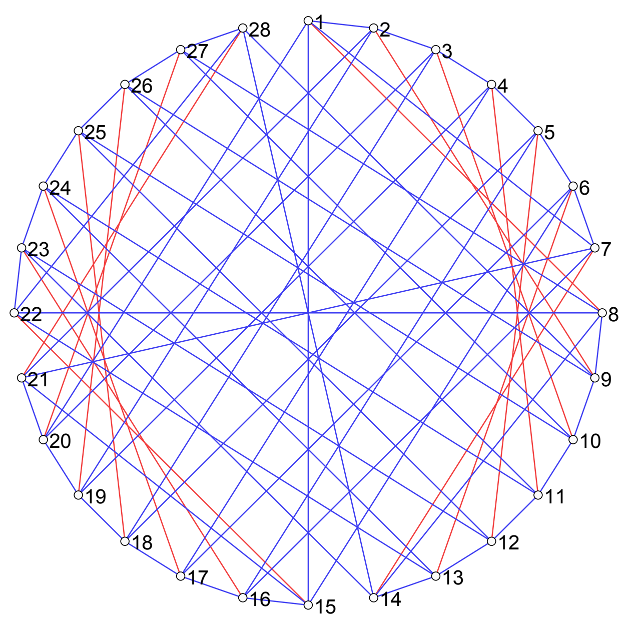

Figure 3. Step 2.2 on 28 vertices

Step 2.3: In this step, each new $K_{2,2}$ has one partite set that comes from U and one partite set from W. For example, the first will be $[1, 21 \mid 9, 27]$. This can be seen in Figure 9. In a similar fashion we complete the process, adding $[2, 20 \mid 14, 22], \ldots, [7, 15 \mid 8, 28]$. The completion of this process can be seen in Figures 4 and 10. Let $w_b(i)$ denote the weight obtained from the blue edges for vertex i. Then

$$w_b(i) = 22(3) + 14 \text{ for } i \in [1, 7] \cup [15, 21]$$

and

$$w_b(i) = 26(3) + 16 \text{ for } i \in [8, 14] \cup [22, 28]$$

so we have that

$$\text{for } i \in [1, 7] \cup [15, 21]$$

$$w_b(i) + w_r(i) = 22(3) + 14 + 2(3) + i = 24(3) + 15 + i$$

and

$$\text{for } i \in [8, 14] \cup [22, 28]$$

$$w_b(i) + w_r(i) = 26(3) + 16 + i - (2(3) + 1) = 24(3) + 15 + i\,.$$

Thus we have a 7-regular handicap graph. For completeness, we will illustrate the process of Step 3 even though we are not going to add any black edges to this example.

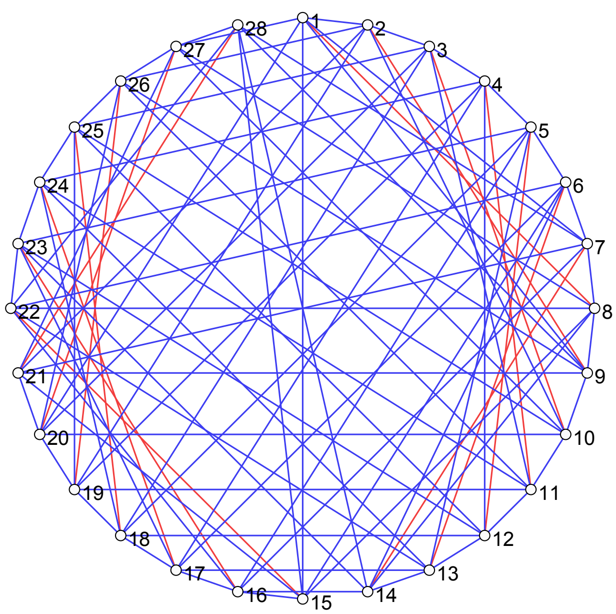

Figure 4. Step 2.3 on 28 vertices

Step 3: Now we take a look at which black edges are available to use. First we construct the bubble graph B by pairing the vertices to form bubbles so that the sum of each pair is 29. Then we draw red or blue edges between bubbles for edges already used in Step 1 or 2. The beautiful structure of this graph is shown in Figure 5.

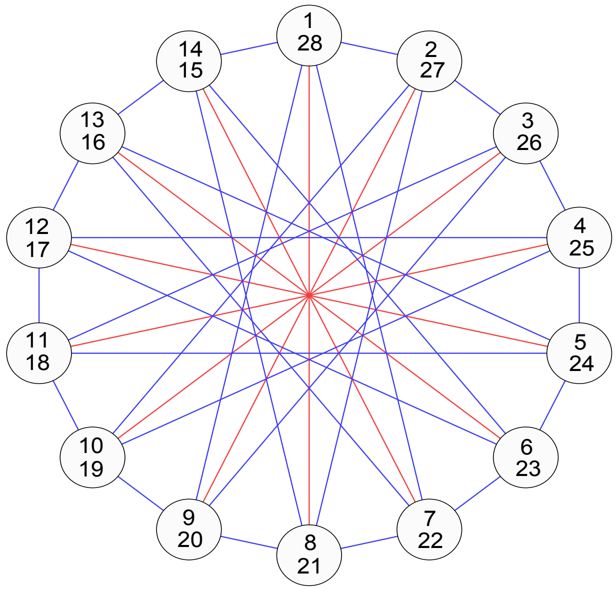

Figure 5. Bubble graph on 28 vertices

We then would take the complement of this to get $\overline{B}$, shown in Figure 6. This is where we would pull black edges from to increase regularity. $\overline{B}$ is 8-regular, and since each black edge equates to a $K_{2,2}$ in the blown up graph, we can have a handicap graph that has maximum regularity equal to $8(2) + 1 + 6 = 23$, i.e. $n - 5 = 28 - 5 = 23$, if desired.

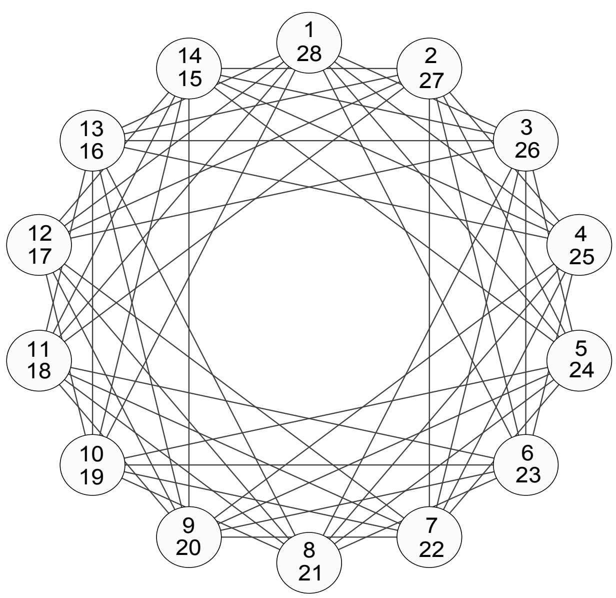

Figure 6. Complement of bubble graph on 28 vertices

Figure 7. Construction of Step 1 on 28 vertices

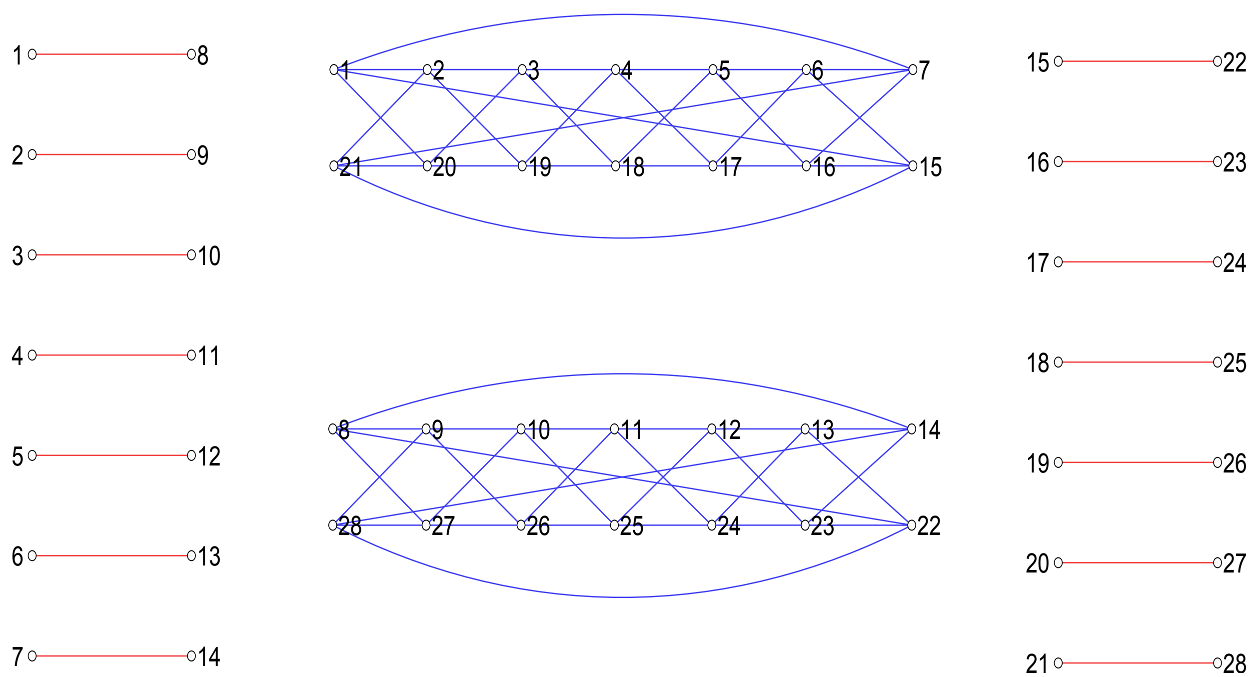

Figure 8. Construction of Step 2.1 and 2.2 on 28 vertices

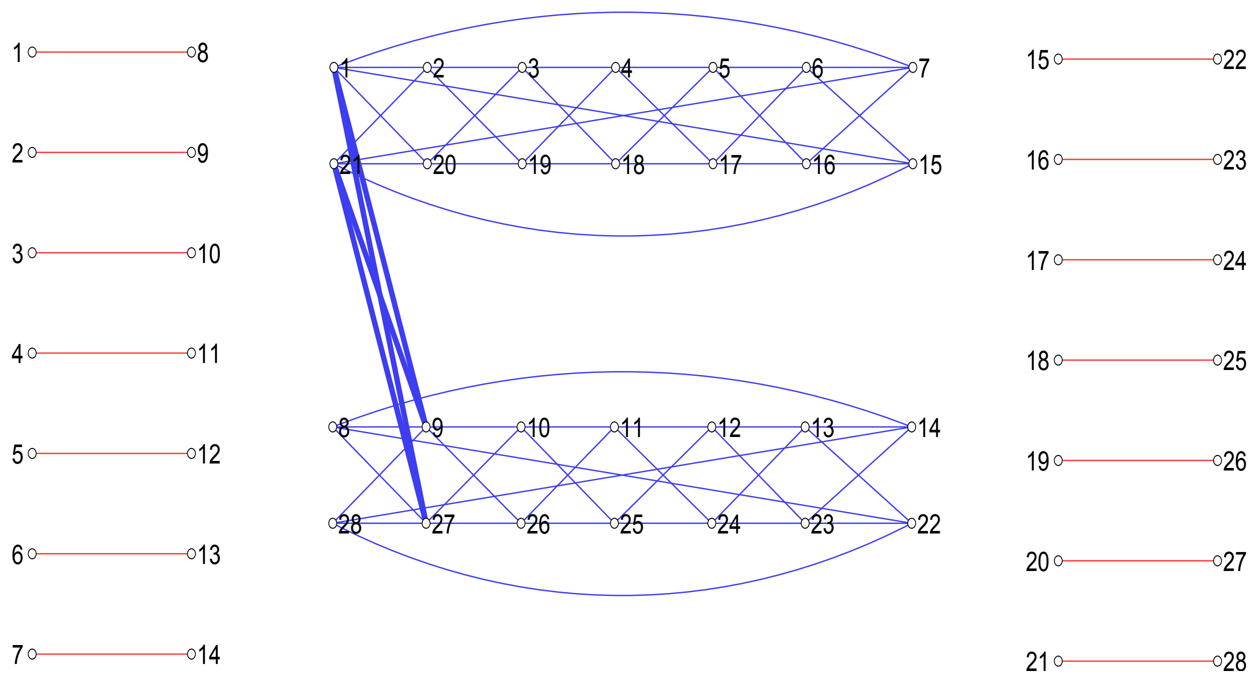

Figure 9. Construction of Step 2.3 on 28 vertices, adding first $K_{2,2}$

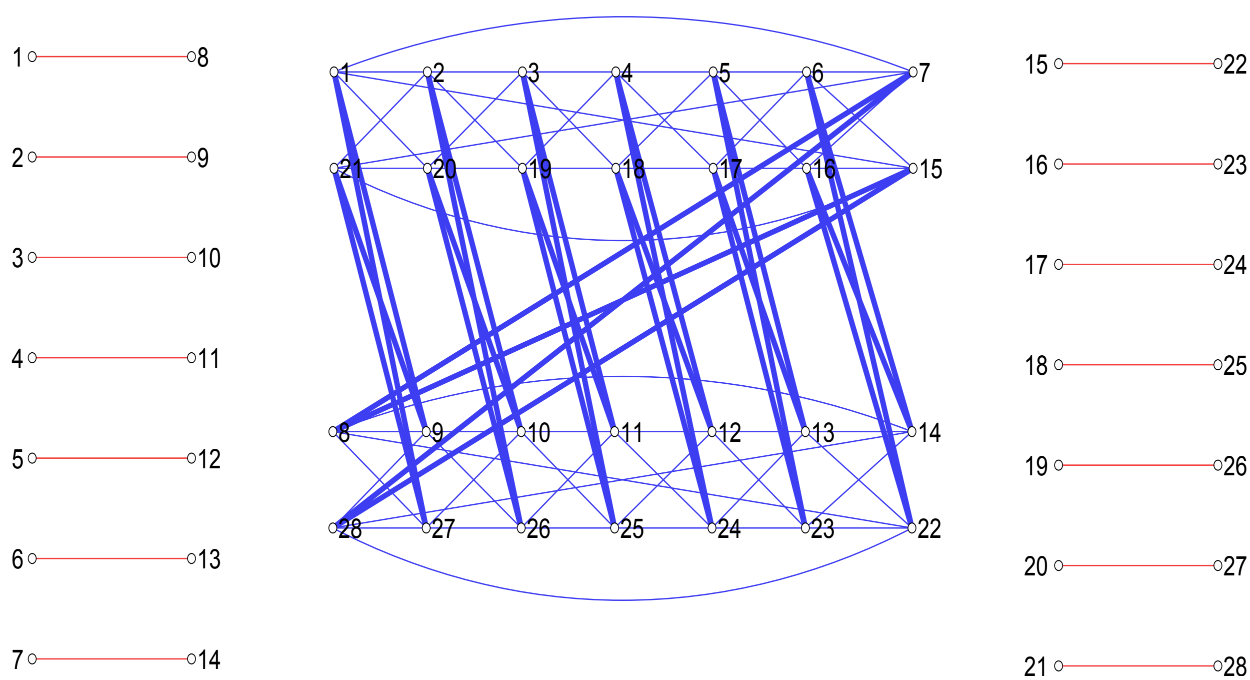

Figure 10. Construction of Step 2.3 on 28 vertices

6. Conclusion

We can now summarize our result and other related results on handicap graphs with n vertices where $n \equiv 4 \pmod 8$ as follows.

Theorem 6.1. *Let n be a positive integer, $n \equiv 4 \pmod 8$. Then there exists an r-regular 1-handicap graph G if and only if r is odd, $n \geq 12$ and $3 \leq r \leq n-5$ except when $r = 3$ and $n = 12$.*

Proof. Follows directly from Theorem 3.3, Lemmas 3.1 and 3.3 and Theorem 4.1.

From Lemma 3.3 we have the necessary conditions that r is odd and $3 \leq r \leq n-5$. The only odd-regular graph with $n = 4$, r odd, and $r \geq 3$ is K_4, which is obviously not a 1-handicap graph.

For $n < 28$ and $r = 3, 5$ the existence (including the exception for $r = 3$ and $n = 12$) follows from Lemma 3.1. The lower part of the spectrum follows from Theorem 3.3, stating that such graphs exist whenever $n \geq 28, n \equiv 0 \pmod 4$ and $3 \leq r \leq n-11$. Finally, the upper part of the spectrum for $7 \leq r \leq n-5$ follows from our result in Theorem 4.1 proved in Section 4. □

For the sake of completeness, below we also state the complete characterization of 1-handicap graphs with even number of vertices, which was published with references to the original sources ([11, 12, 15]) in [10].

Theorem 6.2. *Let n be an even positive integer. Then an r-regular handicap graph G on n vertices exists if and only if $n \geq 8$ and either*

- $n \equiv 0 \pmod 4$, $3 \leq r \leq n-5$ *and r is odd, or*
- $n \equiv 2 \pmod 4$, $3 \leq r \leq n-7$ *and* $r \equiv 3 \pmod 4$,

except when $r = 3$ and $n \in \{10, 12, 14, 18, 22, 26\}$.

The question of existence of 1-handicap graphs on odd numbers of vertices has been barely touched in [8] and remains widely open.

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