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Multi-bridge graphs are anti-magic

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Abstract

An anti-magic graph is a graph whose $|E|$ edges can be labeled with the first $|E|$ natural numbers such that each edge receives a distinct number and each vertex receives a distinct vertex sum which is obtained by taking the sum of the labels of all the edges incident to it. We prove that the multi-bridge graph is anti-magic.

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1. Introduction

Let $G = (V, E)$ be a graph with neither loop nor multiple edges. An *anti-magic labeling* of G is a bijection φ from E to $\{1, 2, \dots, |E|\}$ such that the sum of the labels on the edges incident to a vertex, called the *vertex sum*, is distinct for each vertex. A graph is *anti-magic* if it admits an anti-magic labeling.

The concept of anti-magic graphs has its origin from the book [7] where Hartsfield and Ringel conjectured that all connected graphs but the single edge K_2 are anti-magic. Since then, the problem of deciding which graphs are anti-magic has attracted much attention. Nevertheless the conjecture remains unsettled despite concerted efforts by mathematicians in graph theory.

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In the same book, Hartsfield and Ringel remarked that even when the conjecture is restricted to trees, no complete affirmative answer has been offered. Some results concerning the anti-magicness of trees are given in [8] and [9].

On the other hand, by confining the attention on regular graphs, the situation turns out to be a lot more delightful. In [4], Cranston showed that every regular bipartite graph with degree at least 2 is anti-magic. In [5], Cranston et al. proved that Hartsfield and Ringel's conjecture is true for all odd regular graphs. Shortly afterwards, in [3], Chang et al. proved that all even regular graphs are anti-magic. By modifying the argument used in [5], Bérczi et al. in [2] also proved that even regular graphs are anti-magic. For more details on anti-magic graphs, we refer the reader to [6]. For some recent results on anti-magic graphs, we refer the reader to [10].

In view of this, we turn our attention to graphs which are close to being regular.

Consider a graph with only two vertices and having r multiple edges joining them, $r \geq 3$. Subdivide the edges of this graph arbitrarily so that at most one edge is not subdivided. Call the result graph an *r-bridge graph* and denote it by $\theta(m_1, m_2, \ldots, m_r)$ if the lengths of the paths are $m_1, m_2, \ldots, m_r$ respectively.

The purpose of this paper is to prove the following result.

Theorem 1.1. *Every r-bridge graph is anti-magic.*

In a forth-coming paper, we shall make use of the above result to prove the anti-magicness of a class of not quite regular graphs. Hence it is an appetizer result for a more general result which is to appear later.

We note in passing that in [1], Alon et al. proved that all dense graphs are anti-magic while in [11], Wang initiated the investigation on the anti-magicness of sparse graphs. Incidentally, the graphs in this paper and those in our forth-coming papers are sparse graphs.

2. The proof of Theorem 1.1

Throughout this section, we shall assume that in the graph $\theta(m_1, m_2, \ldots, m_r)$, the path lengths satisfy the condition $m_1 \geq m_2 \geq \cdots \geq m_r$. Also, we shall call the paths in $\theta(m_1, m_2, \ldots, m_r)$ the m_i-path, $i = 1, 2, \ldots, r$.

Let x and y denote the two vertices of degree r in $\theta(m_1, m_2, \ldots, m_r)$ and let $w(x), w(y)$ denote the vertex sums of x, y respectively.

The proof is divided into three cases.

Case 1. $r = 3k$.

Suppose $k = 1$.

The labelings depicted in Figure 1 show that if $m_1 \leq 2$, the 3-bridge graph is anti-magic. Hence we assume that $m_1 \geq 3$.

Subcase 1.1. $m_1 + m_2 + m_3$ is odd.

Let φ_0 denote the following edge labeling on the 3-bridge graph.

(i) Label the edges of the m_1-path with $1, 2, \ldots, m_1$ successively starting from the vertex x.

(ii) Label the edges of the m_3-path with $m_1 + 1, m_1 + 2, \ldots, m_1 + m_3$ successively starting from the vertex y.

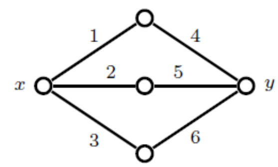

Figure 1. Anti-magic labelings where $m_1 = 2$.

(iii) Label the edges of the m_2-path with $m_1 + m_3 + 1, m_1 + m_3 + 2, \ldots, m_1 + m_3 + m_2$ successively starting from the vertex x.

Figure 2(i) illustrates the case $(m_1, m_2, m_3) = (5, 4, 2)$.

Note that the vertex sums of the degree-2 vertices consist of distinct odd natural numbers and that the vertex sums of x and y are both even and are given by $w(x) = 2(m_1 + m_3 + 1)$ and $w(y) = 2m_1 + m_1 + m_2 + m_3 + 1$ respectively.

This shows that φ_0 is an anti-magic labeling of the 3-bridge graph.

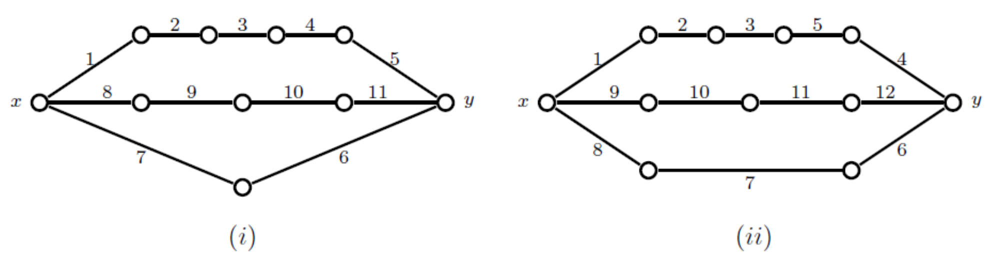

Figure 2. Two anti-magic labelings on 3-bridges.

Subcase 1.2. $m_1 + m_2 + m_3$ is even.

In this case, an anti-magic labeling is obtained by swapping the labels $m_1 - 1, m_1$ (on the last two edges of the m_1-path) from the anti-magic labeling φ_0 given in Subcase 1.1. Note that there are only three vertices whose vertex-sums are even, namely x, y and the second last vertex on the m_1-path. Since the vertex-sums are $2(m_1+m_3+1), 2m_1+m_1+m_2+m_3$ and $2m_1-2$ respectively, they are distinct natural numbers.

The vertex-sums of the rest of the vertices are distinct odd natural numbers.

Figure 2(ii) illustrates the case $(m_1, m_2, m_3) = (5, 4, 3)$.

Now suppose $k \geq 2$.

For each $i = 1, 2, \ldots, k$, let H_i denote the 3-bridge subgraph induced by the m_{3i-2}-path, m_{3i-1}-path and the m_{3i}-path.

Define $p_0 = 0$ and $p_i = p_{i-1} + m_{3i-2} + m_{3i-1} + m_{3i}$ for $i \geq 1$.

For each $i = 1, 2, \ldots, k$, label the edges of H_i so that

(i) the edges of the m_{3i-2}-path receive the labels $p_{i-1}+1, p_{i-1}+2, \ldots, p_{i-1}+m_{3i-2}$ successively starting from the vertex x,

(ii) and then label the edges of the m_{3i}-path with $p_{i-1} + m_{3i-2} + 1, p_{i-1} + m_{3i-2} + 2, \ldots, p_{i-1} + m_{3i-2} + m_{3i}$ successively starting from the vertex y.

(iii) Finally, label the edges of the m_{3i-1}-path with $p_{i-1}+m_{3i-2}+m_{3i}+1, p_{i-1}+m_{3i-2}+m_{3i}+2, \ldots, p_{i-1}+m_{3i-2}+m_{3i}+m_{3i-1}$ starting from the vertex x.

Figure 3 illustrates the cases $(m_1, m_2, \ldots, m_6) = (6,6,5,4,3,2)$ and $(m_1, m_2, \ldots, m_6) = (2,2,\ldots,2)$.

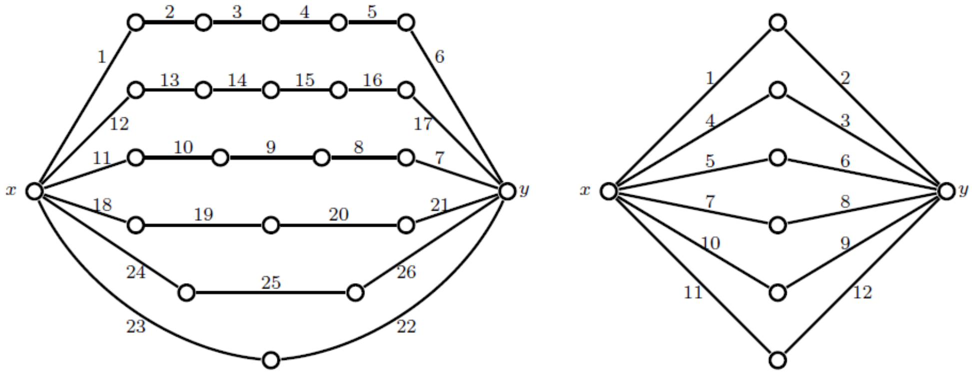

Figure 3. Two anti-magic labelings on 6-bridges.

It is routine to check that the vertex sums of x and y are given by

$w(x) = 2k + 2p_k - 2\sum_{i=1}^{k} m_{3i-1} + 3\sum_{i=1}^{k-1} p_i$

and

$w(y) = k + p_k + 2\sum_{i=1}^{k} m_{3i-2} + 3\sum_{i=1}^{k-1} p_i.$

respectively.

Also, note that the vertex sums of the degree-2 vertices consist of odd distinct natural numbers and are less than either of $w(x)$ and $w(y)$.

This completes the proof for Case 1.

Case 2. $r = 3k+1$.

Suppose $k = 1$.

Subcase 2.1. Not all paths have the same length.

Let φ_1 denote the following edge labeling on the 4-bridge graph.

(i) Label the edges of the m_1-path with $1, 2, \ldots, m_1$ successively starting from the vertex x.

(ii) Label the edges of the m_2-path with $m_1+1, m_1+2, \ldots, m_1+m_2$ successively starting from the vertex x.

(iii) Label the edges of the m_3-path with $m_1+m_2+1, m_1+m_2+2, \ldots, m_1+m_2+m_3$ successively starting from the vertex y.

(iv) Label the edges of the m_4-path with $m_1+m_2+m_3+1, m_1+m_2+m_3+2, \ldots, m_1+m_2+m_3+m_4$ successively starting from the vertex y.

Figure 4(i) illustrates the case $(m_1, m_2, m_3, m_4) = (5,4,3,2)$.

Note that the vertex sums $w(x)$ and $w(y)$ of x and y are given by $3m_1+2m_2+2m_3+m_4+2$ and $4m_1+3m_2+m_3+2$ respectively. Note that the vertex sums of the degree-2 vertices consist of distinct natural odd numbers and they are all less than either of $w(x)$ and $w(y)$.

This means that φ_1 is an anti-magic labeling of the 4-bridge.

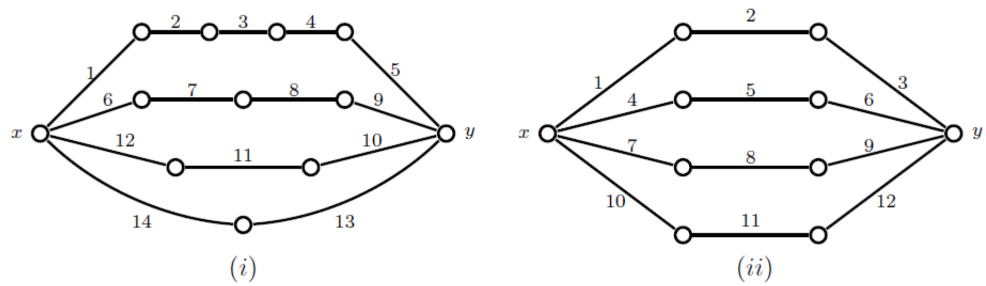

Figure 4. Two anti-magic labelings on 4-bridges.

Subcase 2.2. All paths have the same length m.

In this case, an anti-magic labeling is obtained by labeling the edges of the i-th path with the labels $(i-1)m+1, (i-1)m+2, \dots, im$ successively all starting from x to y. In this case $w(x) = 6m+4$ and $w(y) = 10m$. The rest of the vertex sums consist of distinct odd natural numbers.

Figure 4(ii) illustrates the case $m = 3$.

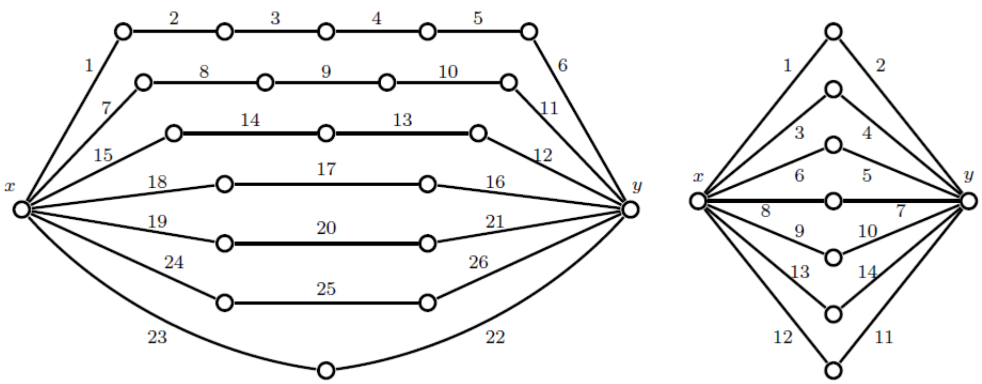

Figure 5. Two anti-magic labelings on 7-bridges.

Now suppose $k \geq 2$.

Let H_1 denote the 4-bridge subgraph induced by the m_j-path, $j = 1, 2, 3, 4$. Also, for each $i = 2, \dots, k$, let H_i denote the 3-bridge subgraph induced by the m_{3i-1}-path, m_{3i}-path and the m_{3i+1}-path.

Define $p_0 = 0$, $p_1 = m_1 + m_2 + m_3 + m_4$ and $p_i = p_{i-1} + m_{3i-1} + m_{3i} + m_{3i+1}$ for $i \geq 2$.

Label H_1 using φ_1 first. Then for each $i = 2, \dots, k$, label the edges of H_i so that

(i) the edges of the m_{3i-1}-path receive the labels $p_{i-1}+1, p_{i-1}+2, \dots, p_{i-1}+m_{3i-1}$ successively starting from the vertex x, and

(ii) label the edges of the m_{3i+1}-path with $p_{i-1} + m_{3i-1} + 1, p_{i-1} + m_{3i-1} + 2, \dots, p_{i-1} + m_{3i-1} + m_{3i+1}$ successively starting from the vertex y.

(iii) Finally, label the edges of the m_{3i}-path with $p_{i-1} + m_{3i-1} + m_{3i+1} + 1, p_{i-1} + m_{3i-1} + m_{3i+1} + 2, \dots, p_{i-1} + m_{3i-1} + m_{3i+1} + m_{3i}$ starting from the vertex x.

Figure 5 illustrates the cases $(m_1, m_2, \ldots, m_7) = (6, 5, 4, 3, 3, 3, 2)$ and $(m_1, m_2, \ldots, m_7) = (2, 2, \ldots, 2)$.

It is routine to check that the vertex sums of x and y are given by

$w(x) = 2p_k + 2k + m_1 - m_4 + \sum_{i=2}^{k}(3p_{i-1} - 2m_{3i})$

and

$w(y) = k + 1 + 4m_1 + 3m_2 + m_3 + 2(p_1 - p_k) + \sum_{i=2}^{k}(3p_i + 2m_{3i-1})$.

respectively.

Also, note that the vertex sums of the degree-2 vertices consist of distinct odd natural numbers each of which is less than either of $w(x)$ and $w(y)$.

This completes the proof for Case 2.

Case 3. $r = 3k + 2$.

Suppose $k = 1$.

Let φ_2 denote the following edge labeling on the 5-bridge graph.

(i) Label the edges of the m_1-path with $1, 2, \ldots, m_1$ successively starting from the vertex x.

(ii) Label the edges of the m_2-path with $m_1 + 1, m_1 + 2, \ldots, m_1 + m_2$ successively starting from the vertex y.

(iii) For each $i \in \{3, 4, 5\}$, label the edges of the m_i-path with $q_i + 1, q_i + 2, \ldots, q_i + m_i$ successively all starting from x to y. Here $q_3 = m_1 + m_2$ and $q_j = q_{j-1} + m_{j-1}$ for $j \in \{4, 5\}$.

Figure 6 illustrates the case $(m_1, m_2, m_3, m_4, m_5) = (6, 5, 4, 3, 2)$.

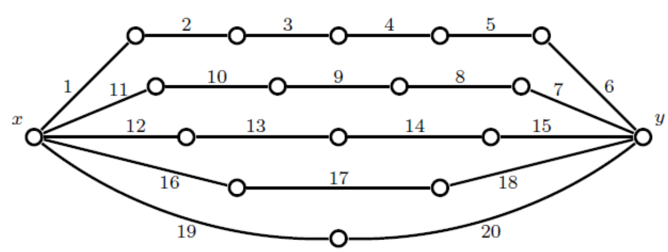

Figure 6. Anti-magic labeling of a 5-bridge.

Note that the vertex sums of x and y are given by $w(x) = 4(m_1 + m_2) + 2m_3 + m_4 + 4$ and $w(y) = 5m_1 + 3(m_2 + m_3) + 2m_4 + m_5 + 1$ respectively.

Clearly the vertex sums of the degree-2 vertices in φ_2 consist of odd distinct natural numbers and each is less than either of $w(x)$ and $w(y)$.

Hence φ_2 is an anti-magic labeling of the 5-bridge.

Now suppose $k \geq 2$.

Let H_1 denote the 5-bridge induced by the m_j-path, $j = 1, 2, \ldots, 5$. Also, for each $i = 2, \ldots, k$, let H_i denote the 3-bridge subgraph induced by the m_{3i}-path, m_{3i+1}-path and the m_{3i+2}-path.

Define $p_0 = 0$, $p_1 = m_1 + m_2 + \cdots + m_5$ and $p_i = p_{i-1} + m_{3i} + m_{3i+1} + m_{3i+2}$ for $i \geq 2$.

Label H_1 using φ_2 first. Then for each $i = 2, \ldots, k$, label the edges of H_i so that

(i) the edges of the m_{3i}-path receive the labels $p_{i-1} + 1, p_{i-1} + 2, \ldots, p_{i-1} + m_{3i}$ successively starting from the vertex x, and

(ii) label the edges of the m_{3i+2}-path with $p_{i-1}+m_{3i}+1, p_{i-1}+m_{3i}+2, \ldots, p_{i-1}+m_{3i}+m_{3i+2}$ successively starting from the vertex y.

(iii) Finally, label the edges of the m_{3i+1}-path with $p_{i-1}+m_{3i}+m_{3i+2}+1, p_{i-1}+m_{3i}+m_{3i+2}+2, \ldots, p_{i-1}+m_{3i}+m_{3i+2}+m_{3i+1}$ starting from the vertex x.

Figure 7 illustrates the case $(m_1, m_2, \ldots, m_8) = (6, 5, 4, 3, 3, 3, 2, 2)$.

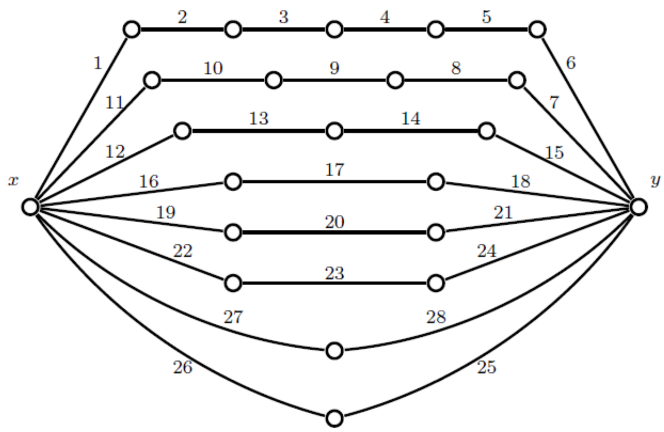

Figure 7. Anti-magic labeling of an 8-bridge.

It is routine to check that the vertex sums of x and y are given by

$w(x) = 2(p_k + k + 1 + m_1 + m_2 - m_5) - m_4 + \sum_{i=2}^{k}(3p_{i-1} - 2m_{3i+1})$

and

$w(y) = 2(2m_1 + m_2 + m_3) + m_4 + k + p_k + \sum_{i=2}^{k}(3p_{i-1} + 2m_{3i})$

respectively.

Also, note that the vertex sums of the degree-2 vertices consist of distinct odd natural numbers each of which is less than either of $w(x)$ and $w(y)$.

This completes the proof for Case 3 and so is the proof for Theorem 1.1. □

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