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On decompositions of complete graphs into unicyclic disconnected bipartite graphs on nine edges

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Abstract

We use Rosa-type labelings to decompose complete graphs into unicyclic, disconnected, bipartite graphs on nine edges – namely, those featuring cyclic component C_4, C_6, or C_8. For any such graph H, we prove there exists an H-design of K_{18k+1} and K_{18k} for all positive integers k.

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1. Introduction

All graphs considered within are without loops, parallel edges, and isolated vertices. A *decomposition* of a graph G is a set of pairwise edge-disjoint subgraphs H_i such that every edge of G belongs to exactly one H_i. More precisely, we say that a decomposition of G is a set $\xi = \{H_1, H_2, \dots, H_m\}$ where

$$E(G) = \bigcup_{i=1}^{m} E(H_i) \quad \text{and} \quad E(H_i) \cap E(H_j) = \emptyset, \ \text{for all } i \neq j.$$

When the complete graph of order p is decomposed into ξ with all H_i isomorphic to some graph H, we refer to ξ as an *H-design of K_p* or a *(K_p, H)-design*.

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For such a graph H on n edges with labeling $f : V(H) \to \mathbb{Z}_{2n+1}$, the term *clicking* refers to the application of the isomorphism $i \to i+1$ to the image of f. When a design is formed by clicking, it is said to be *cyclic*; a formal definition will be given in the following section.

If a graph contains exactly one cycle, it is said to be *unicyclic*. This study concerns graphs H that are unicyclic, bipartite, and disconnected with exactly nine edges. We then need only consider those H which are supergraphs of C_4, C_6, or C_8. Moreover, when decomposing K_p into copies of H, the number of edges in the complete graph is necessarily a multiple of nine. Thus,

$$\frac{p(p-1)}{2} = 9k, \text{ for some } k \in \mathbb{N}.$$

This narrows our search down to those p with $p \equiv 0, 1 \pmod 9$. It follows that only four values of p demand our attention: $p = 18k$, $p = 18k + 1$, $p = 18k + 9$, and $p = 18k + 10$. The tools in this paper allow us to address the first two values for $k \geq 1$, leaving the latter cases subject to further study.

For completeness, we note that the spectrum for graphs with at most eight edges has been determined almost completely; for a comprehensive overview, see [5] and [6]. In concurrence with our effort, another group of authors classified connected, unicyclic graphs on nine edges for which K_{18k} and K_{18k+1} admit H-designs, see [1]. We aim to settle the bipartite, disconnected case. To do this, we will use various tools developed by A. Rosa and many other researchers.

2. Tools and Methods

We begin by introducing some tools that will be used frequently in this paper. The first of which will be Rosa's ρ-labeling defined in [9].

Definition 2.1. *Let H be a graph on n edges. A ρ-labeling f_ρ is an injective (one-to-one) function $f : V(H) \to \{0, 1, 2, \ldots, 2n\}$ inducing the mapping $\ell : E(H) \to \{1, 2, \ldots, n\}$ defined as*

$$\ell(uv) = \min\{|f(u) - f(v)|,\ 2n + 1 - |f(u) - f(v)|\},$$

where the image satisfies

$$\{\ell(uv) : (uv) \in E(H)\} = \{1, 2, \ldots, n\}.$$

We assign the elements of the integer group $\mathbb{Z}_{2n+1}$ to the vertices of H such that we yield $\{1, 2, \ldots, n\}$ as the image of ℓ, often called the *length function*.

All Rosa-type labelings are special cases of Definition 2.1. One is the σ-labeling and it arises when the conditions of the ρ-labeling are satisfied using a restrictive length function ℓ'. This labeling was also presented by Rosa in [9].

Definition 2.2. *If f_ρ can instead be constructed using the restrictive length function $\ell'(uv) = |f(u) - f(v)|$, then it is called a σ-labeling (denoted f_σ).*

Most of the results in this paper are cyclic designs generated by clicking σ-labelings. While we have briefly discussed this idea, we will now provide a formal definition courtesy of Froncek and Kubesa in [6].

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Definition 2.3. *An H-design ξ of the complete graph K_p is cyclic if there exists an ordering $(x_0, x_1, \ldots, x_{p-1})$ of the vertices of K_p and a permutation $\varphi : V(K_p) \to V(K_p)$ defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \ldots, p-1$ inducing an automorphism on ξ, where the addition is performed modulo p.*

A slight modification of this idea allows us to decompose the odd-regular K_{18k}, but we first define another variant of the ρ-labeling, formed when the domain of f_ρ is altered to include ∞. Under certain conditions, such a labeling is said to be *1-rotational* and will be denoted $\tilde{f}_\rho$. It must be noted that this idea has been known for decades. While the following definition was given by Bunge in [2], its origins can be traced back to Huang and Rosa in [8].

Definition 2.4. *Let H be a graph on n edges, one of which is incident with a vertex x of degree one (called a pendant vertex). A 1-rotational ρ-labeling $\tilde{f}_\rho$ of H is an injective (one-to-one) function $\tilde{f} : V(H) \to \{0, 1, 2, \ldots, 2n-2\} \cup \{\infty\}$ where $\tilde{f}(x) = \infty$ with the property that $\tilde{f}$ constitutes a ρ-labeling of $H - x$.*

The 1-rotational ρ-labeling is used to form 1-rotational designs by clicking. By assigning ∞ to a pendant vertex, this node is able to bypass the operation. As a consequence, the modulus on the clicking operator is reduced and the resulting design is instead called 1-rotational.

The tools presented above are not of much interest to us by themselves, but when their conditions are strengthened as in [4], they can prove quite useful.

Definition 2.5. *A labeling of a graph H with bipartition $V(H) = X \cup Y$ is said to be ordered if $f(x) < f(y)$ for all $xy \in E(H)$ where $x \in X$ and $y \in Y$. When a labeling is ordered, it is common to indicate this using a "+" superscript.*

We now introduce several theorems that will be used throughout this paper. The following result was proved by El-Zanati, Vanden Eynden, and Punnim in [3] and has been reconfigured to the context of our inquiry.

Theorem 2.6 (El-Zanati, Vanden Eynden, and Punnim, 2001)**.** *Let H be a graph on 9 edges. If H admits a ρ^+-labeling, then there exists a cyclic (K_{18k+1}, H)-design for all positive integers k.*

A stronger result was proved by Fahnenstiel and Froncek in [5]. We now take their general result and apply it to graphs on nine edges.

Theorem 2.7 (Fahnenstiel and Froncek, 2019)**.** *Let H be a graph with a σ^+-labeling on 9 edges such that the edge of length 9 is a pendant edge. Then there exists a 1-rotational (K_{18k}, H)-design for all positive integers k.*

The proof of Theorem 2.7 is based on the amalgamation of k edge-disjoint copies of H into one graph H^*. This graph can then be shown to admit a ρ-labeling. We then find that K_{18k} allows an H^*-decomposition, which can be further decomposed into copies of H. This result builds upon one in [4], which is loosely based on Definition 2.4 above. Since $\max\{\tilde{f}_\rho(x) \neq \infty : x \in V(H)\} = 16$, we add the additional restriction that if some nonempty subset of $\{17, 18\}$ is in the image of f^+_σ, it is a singleton and is necessarily the image of a pendant vertex on the edge of longest length. This change refines the proof in [5].

We end this section by noting that any such labeling is also capable of invoking Theorem 2.6; this fact will greatly reduce computational costs in the following sections. For the remainder of this paper, such a labeling will be called a σ^*- labeling and it will be the primary tool in our analysis.

3. Labeling Graphs Containing C_8

We will now begin labeling. It is natural to start with the simplest of the three families – namely, those H featuring cyclic component C_8. For unicyclic, disconnected graphs on nine edges, we can exhaust this family (up to isomorphism) by considering one graph.

Using the tools defined in the previous section, the process is simple – we find a σ^*-labeling then convert the result into one that is 1-rotational. This allows us to show that H decomposes K_{18k+1} and K_{18k} using one labeling. For illustrative purposes, we will now demonstrate this process, though it will be omitted from following sections.

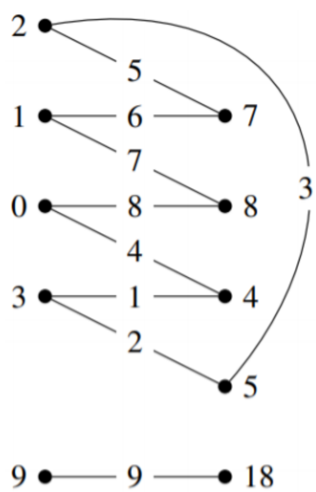

Figure 1: σ^+– labeling

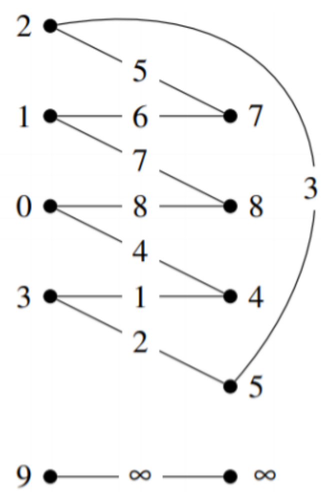

Figure 2: 1–rotational ρ^+– labeling

Figure 1 depicts an ordered ρ-labeling and thus can be used to invoke Theorem 2.6 directly. However, since it was formed using the restrictive length function ℓ' and features a pendant edge of length 9, it is in fact a σ^*-labeling. This allows us to use Theorem 2.7 by transforming the result into a 1-*rotational* ρ^+-*labeling* by mapping $18 \to \infty$.

We now turn our attention to Figure 2 for the results of this process. Notice there is no longer an edge of length 9; consequently, the omission of pendant vertex x yields a ρ^+-labeling of the induced subgraph. In terms of Definitions 2.1 and 2.2, we have a labeling of $H - x$ given by $f : V(H - x) \to \{0, 1, 2, 3, 4, 5, 7, 8\}$ inducing an image of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ under ℓ'.

Again, the transformation from Figure 1 to Figure 2 was demonstrated for illustrative purposes only. For the remainder of this paper, we will only be providing the σ^*-labelings of H without their 1-rotational ρ^+-counterparts. The argument above can be applied to any of the following results.

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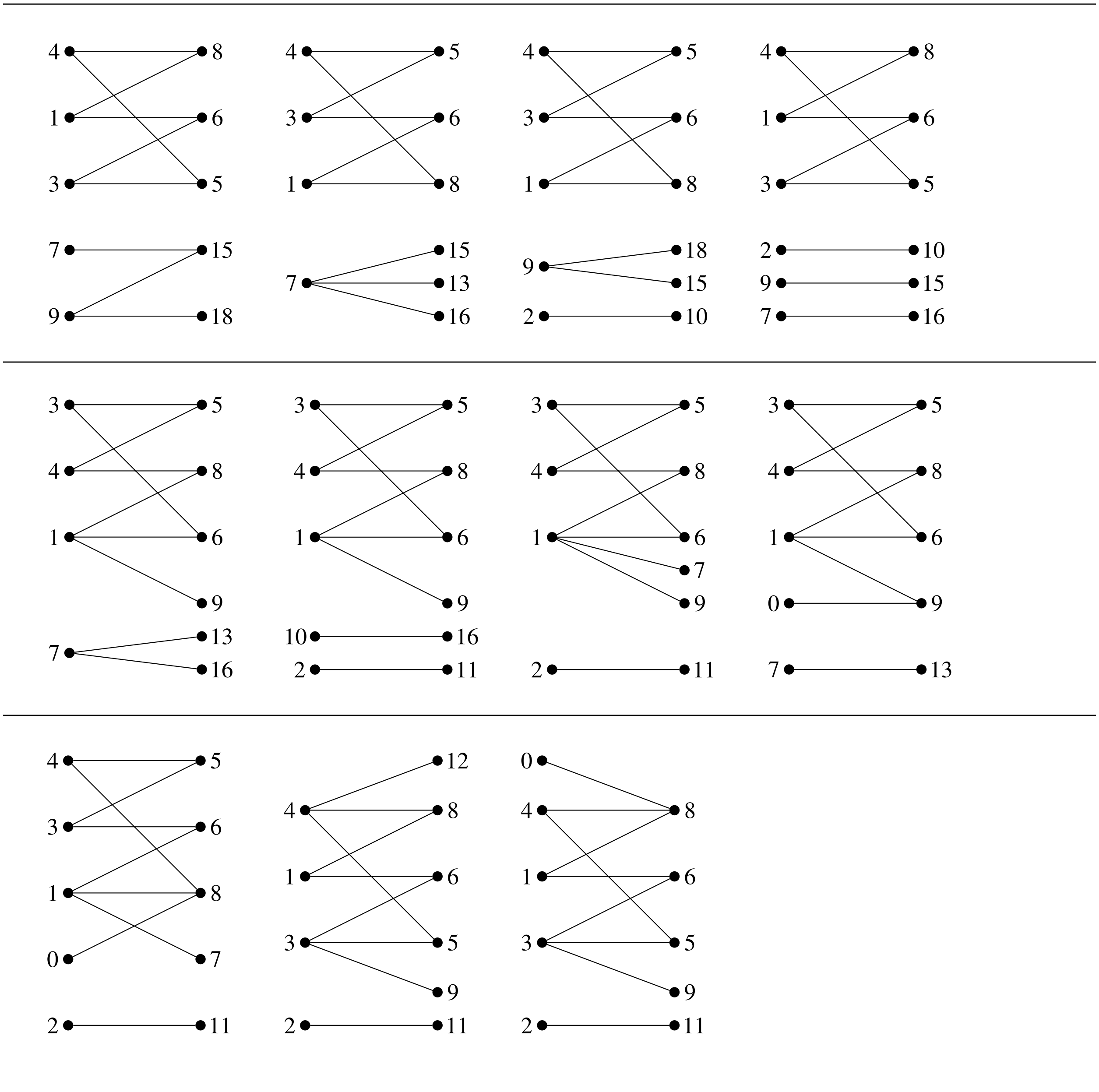
4. Labeling Graphs Containing C_6

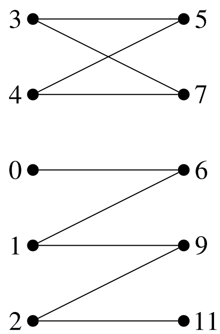
The class involving cycle C_6 is only slightly more involved. Once the cycle is established, there are three edges to be placed – one of which must be in a separate component. With our restriction of unicyclicity, we have eleven cases to consider.

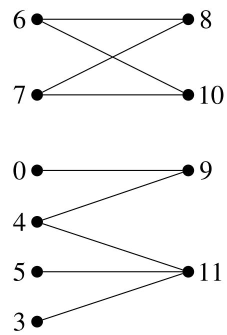
We now show σ^*-labelings for all graphs H, up to isomorphism. We sort them in increasing order of the number of trees attached to the cycle. Within each category, we prioritize those with fewer components. Partite sets are vertically oriented and edge labels are omitted.

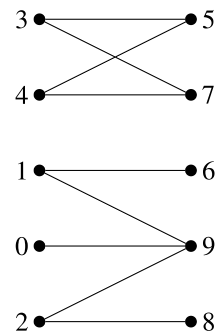


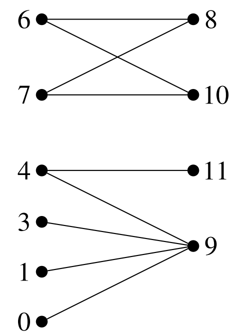
5. Labeling Graphs Containing C_4

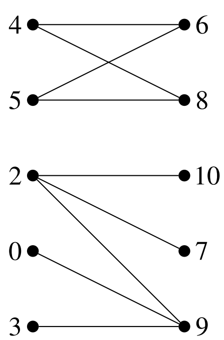
Our final class consists of graphs H with cyclic component C_4. This section will contain σ^*-labelings for all such graphs, up to isomorphism. This catalog is rather extensive – we will again sort them in increasing order of the number of trees attached to the cycle, though those of similar structure are grouped together. We prioritize those H with fewer components.

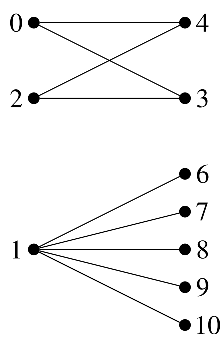


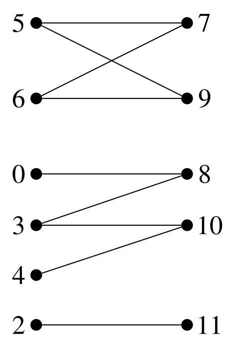


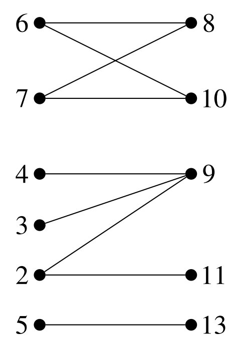


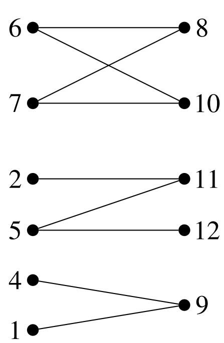


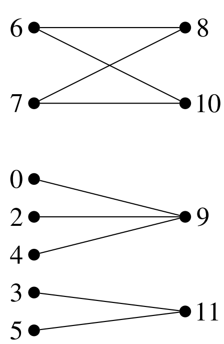


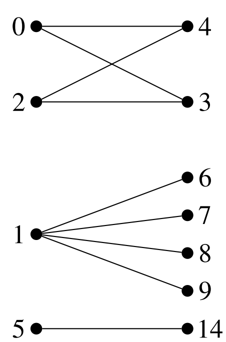


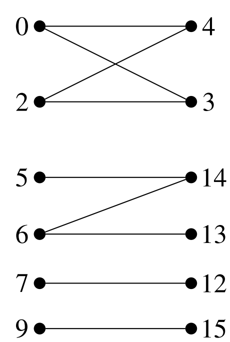


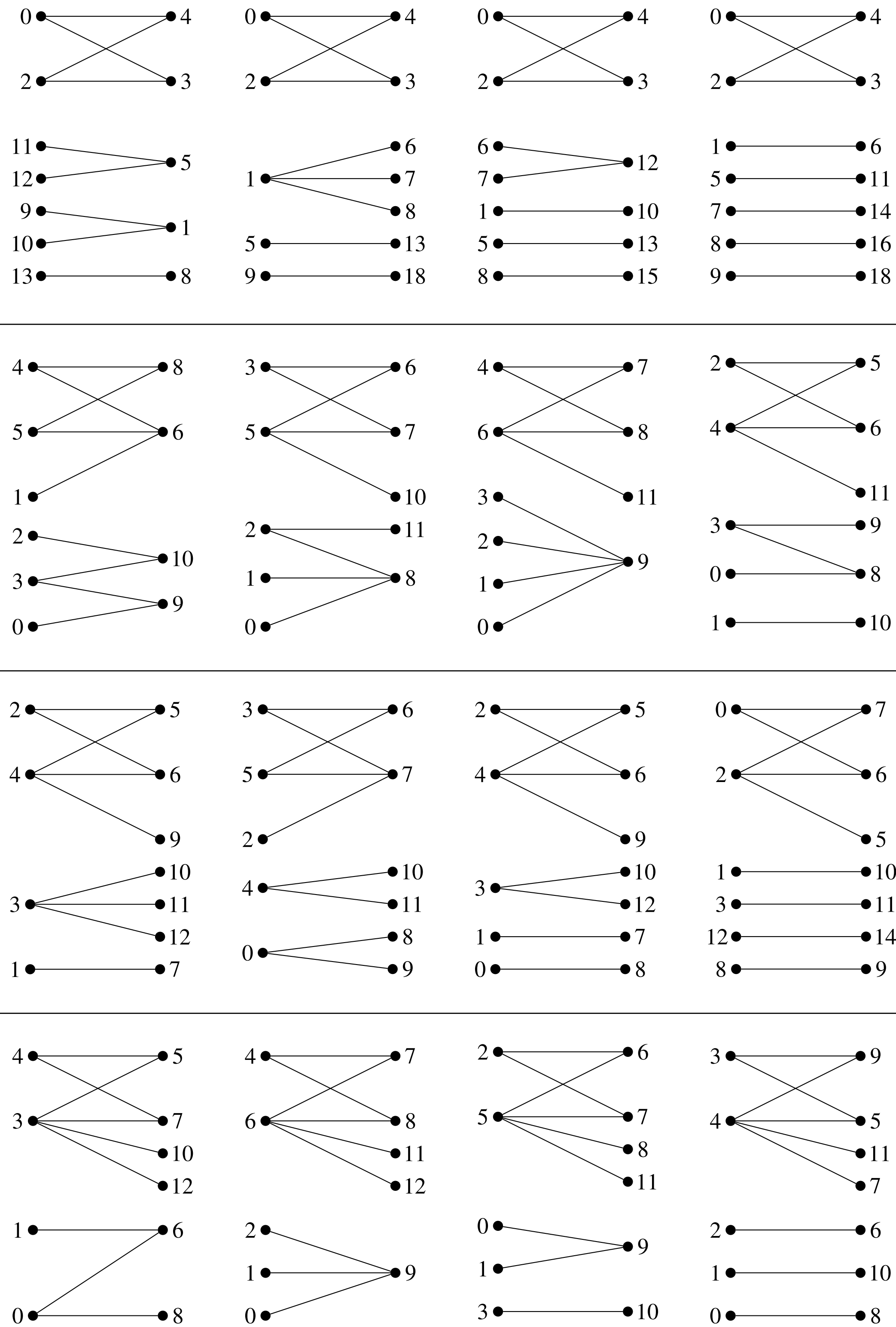












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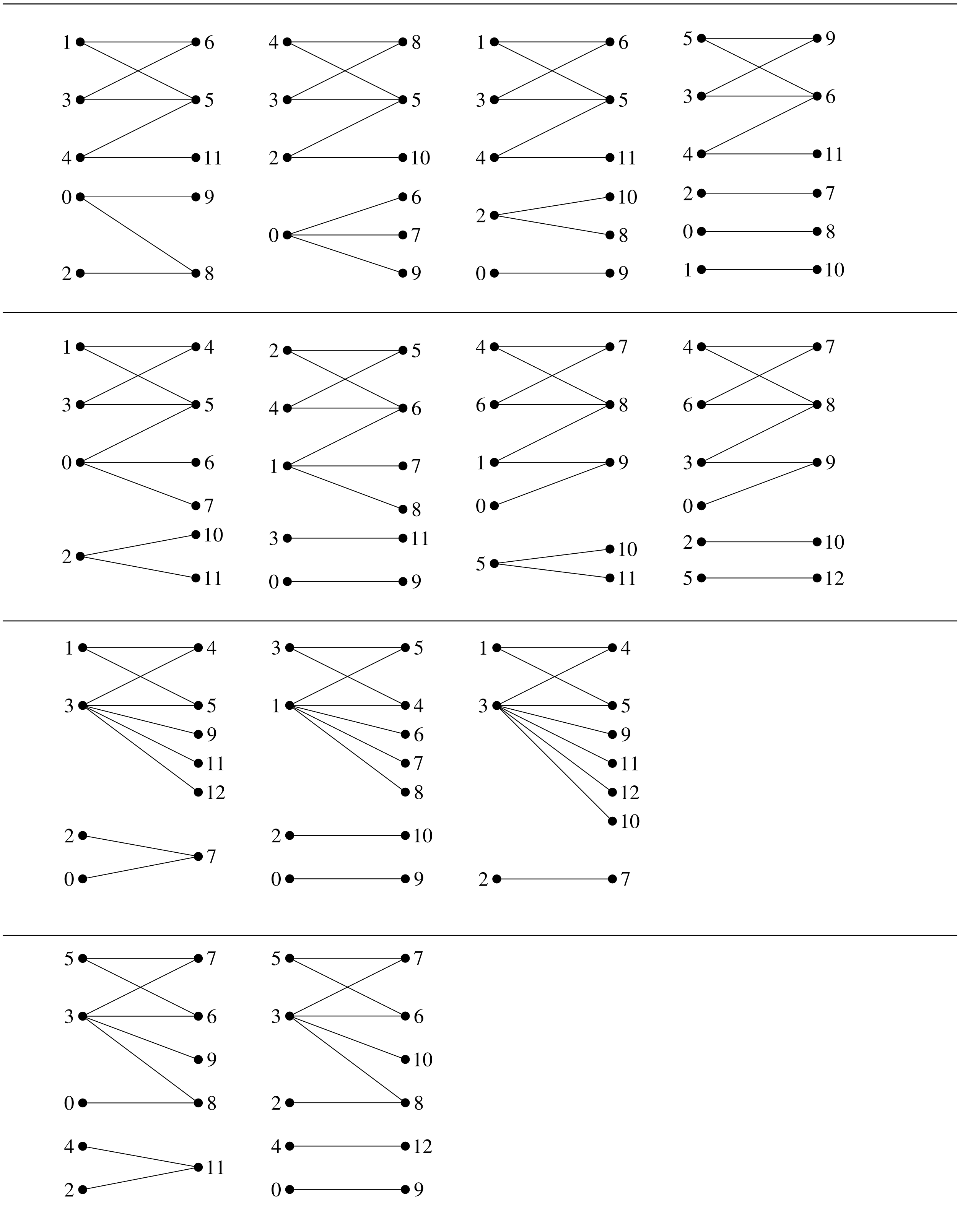
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4 6 3 0 2 5 7 8 9 10 12
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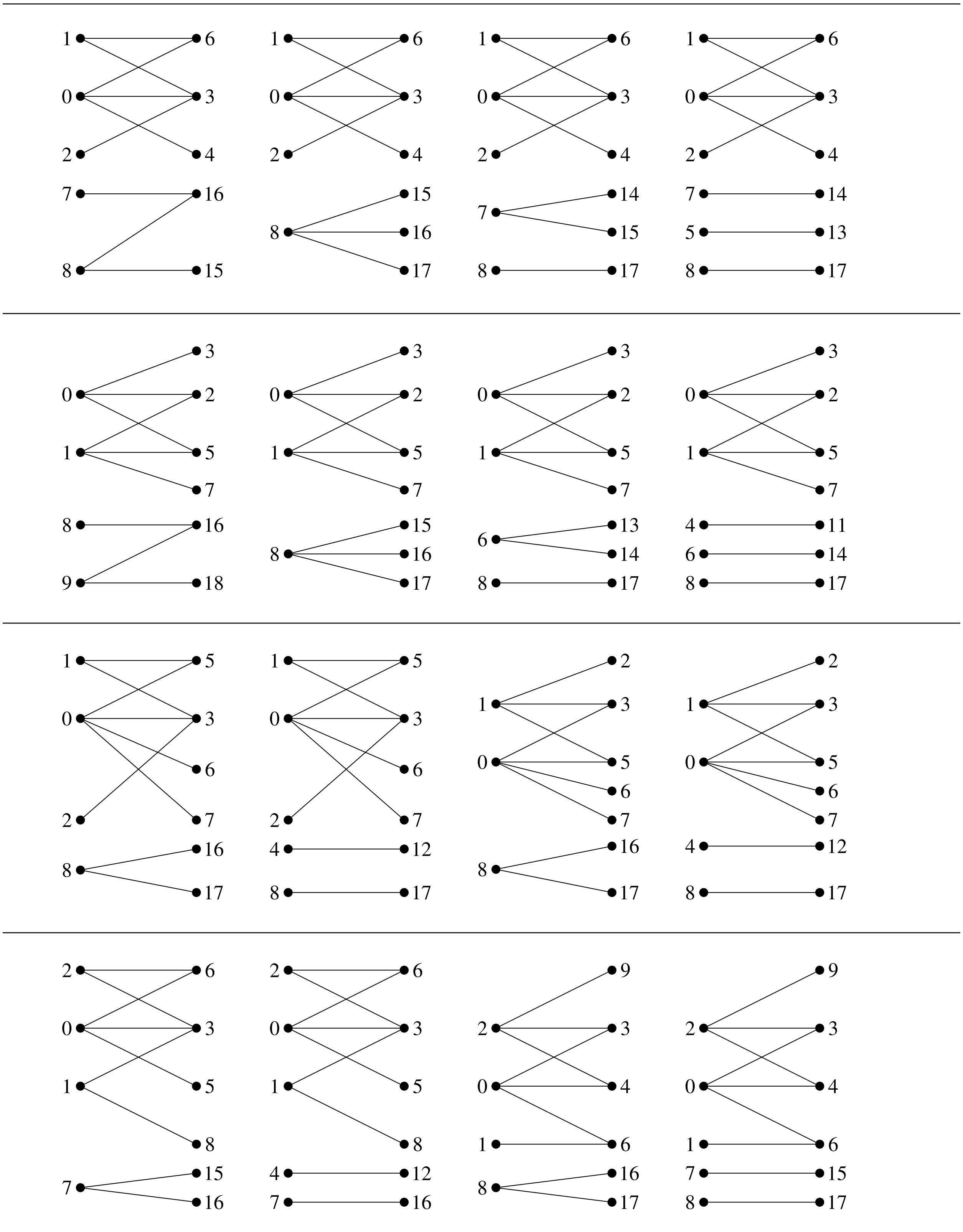
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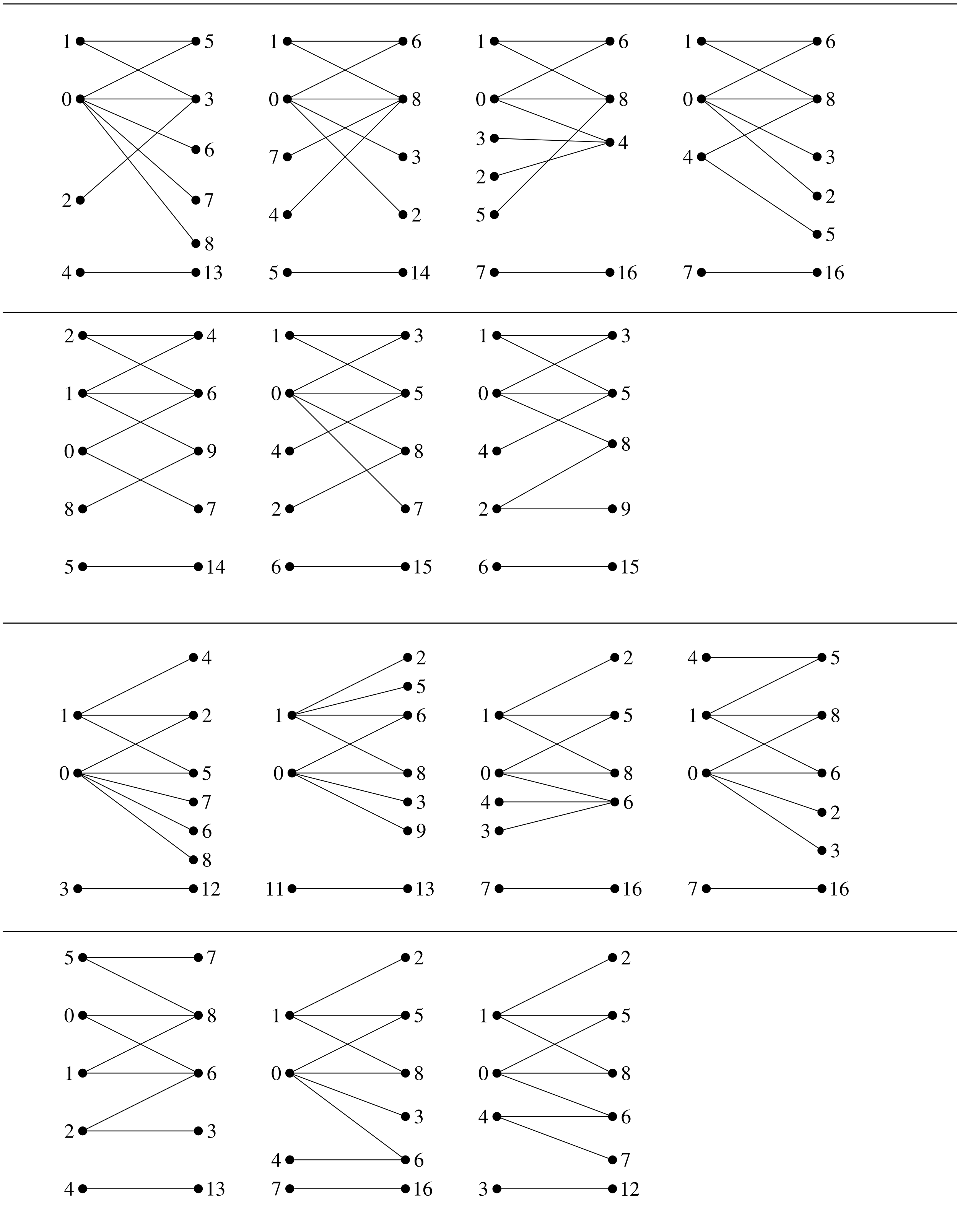

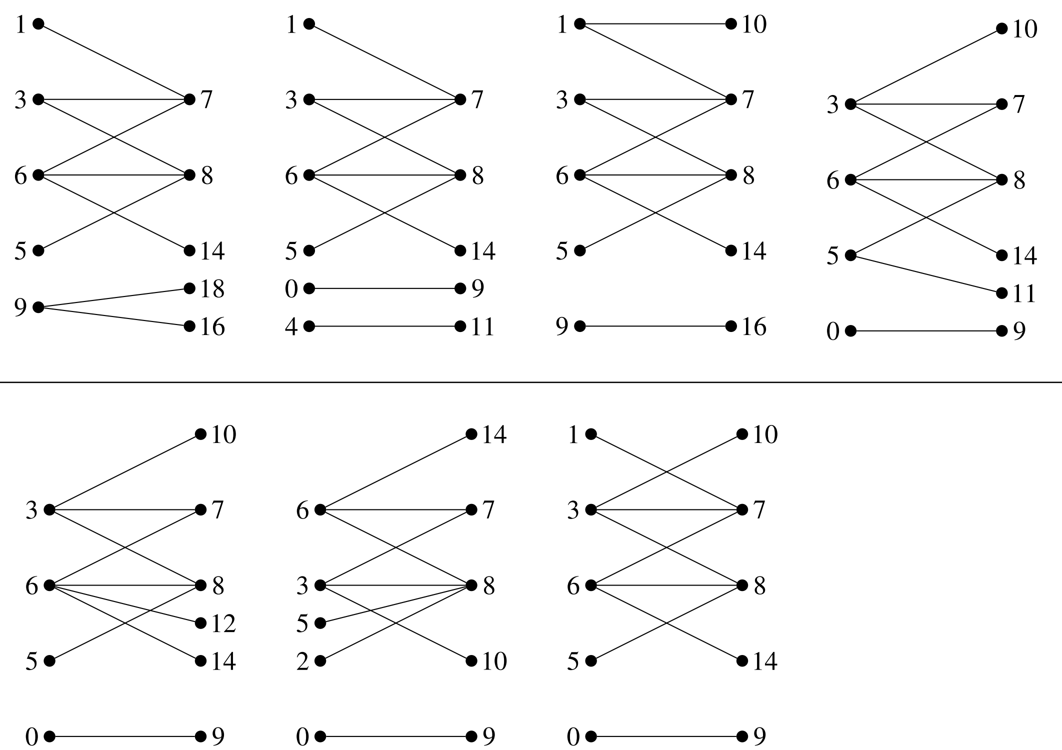

6. Results

With these labelings, we have exhausted the 90 nonisomorphic unicyclic, bipartite, disconnected graphs on nine edges. We now discuss the significance of these labelings in our decomposition of K_{18k+1} and K_{18k}.

Each labeling is capable of invoking Theorem 2.6 and can then be used directly to form a (K_{18k+1}, H)-design. This is done by clicking the labeling until the edge set is "complete." In decomposing K_{18k+1}, the modulus on the clicking operator is taken to be $18k + 1$. Moreover, the utility of this operation is that it does not alter the edge lengths – that is,

$$|i - j| = |(i + 1) - (j + 1)|.$$

The repeated application of this idea will yield $18k^2 + k$ edge-disjoint copies of H and embed them onto K_{18k+1}, eventually exhausting the edge set. The result is the desired cyclic H-design of K_{18k+1}, for any non-negative integer k.

The process is a bit more complex when we decompose the odd-regular K_{18k}. After transforming our σ^*-labeling into one that is 1-rotational, we repeatedly click the result (mod $18k - 1$) to form a 1-rotational decomposition of K_{18k}, for any $k \in \mathbb{N}$. For more information on 1-rotational decompositions, see [2].

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In this paper, we have shown σ^*-labelings for all unicyclic, bipartite, disconnected graphs on 9 edges – each with the special property that there exists a pendant vertex on the edge of longest length. Combining these with Theorems 2.6 and 2.7, we are left with the following result.

Theorem 6.1. *Let H be a unicyclic, bipartite, disconnected graph on 9 edges. Then there exists H-designs of K_{18k+1} and K_{18k} for all positive integers k.*

Proof. For all nonisomorphic unicyclic, disconnected graphs on nine edges featuring C_4, C_6, or C_8, we have presented σ^*-labelings. It follows from Theorems 2.6 and 2.7 that H decomposes K_{18k+1} and K_{18k} for all positive integers k. This completes the proof. □

7. Future Research

We note there are many classes of graphs that have not been cataloged. This paper focuses on bipartite graphs, though decompositions of K_p into non-bipartite H surely exist. Further study may be conducted for those H featuring cycle C_3, C_5, or C_7, though the former class may be large enough to warrant its own paper. We also remind the reader that we omitted from consideration K_p for $p = 18k + 9$ and $p = 18k + 10$, as our tools are not suitable for these values.

For graphs with more than nine edges, the prospects of thorough research are slim. The combinatorial explosion involved makes it a rather unrealistic catalog, though perhaps some lemmata could be introduced to expedite the process.

Furthermore, we note that surprisingly few papers have been published regarding the decomposition of complete graphs into non-spanning, acyclic subgraphs (i.e. forests). Most of what is known concerns the decomposition of complete graphs into isomorphic trees and can generally be traced back to Huang and Rosa in [8], though we note some progress has been made in cataloging forests F on n edges for which K_{2n} and K_{2n+1} admit F-designs – namely, caterpillars in [7] and linear forests in [10]. It may be worth further exploring this territory and expanding our catalog by adding those on nine edges, should decompositions of K_{18k+1} and K_{18k} be allowed.

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