



# Matching book thickness of generalized Petersen graphs

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## Abstract

The *matching book embedding* of a graph  $G$  is to place its vertices on the spine, and arrange its edges on the pages so that the edges in the same page do not intersect each other and the edges induced subgraphs of each page are 1-regular. The *matching book thickness* of  $G$  is the minimum number of pages required for any matching book embedding of  $G$ , denoted by  $mbt(G)$ . In this paper, the matching book thickness of generalized Petersen graphs is determined.

*Keywords:* matching book embedding, matching book thickness, generalized Petersen graph

Mathematics Subject Classification : 05C10; 68R10

DOI: 10.5614/ejgta.2022.10.1.11

## 1. Introduction

The book embedding of a graph was first introduced by Bernhart and Kainen [1]. The *book embedding* problem of a graph consists of arranging the vertices on the *spine* (a line) in order and placing the edges in the *pages* (the half planes with the spine as the common boundary), so that the edges in the same page do not intersect each other. The *book thickness* of a graph  $G$  is the least number of pages that  $G$  can be embedded under all vertex orders, denoted by  $bt(G)$ . On the basis of book embedding, an additional condition is added that the edge induced subgraphs on each page are 1-regular, which is called *matching book embedding*. The *matching book thickness* of  $G$  is the minimum number of pages that  $G$  can be matching embedded, denoted by  $mbt(G)$ . The graph  $G$  is

Received: 3 September 2021, Revised: 23 January 2022, Accepted: 13 February 2022.

said to be *dispersable* if it has a matching book embedding in  $\Delta(G)$  pages, i.e.  $mbt(G) = \Delta(G)$ , where  $\Delta(G)$  is the maximum degree of  $G$ .

The book embedding of graphs have numerous applications(see [2-3]). Reference [4-7] and its references study the book embedding of some graphs, and obtain the book thickness of graphs. But there are few conclusions about the matching book thickness of graphs, especially the exact value. Bernhart and Kainen [1] proposed that complete bipartite graphs  $K_{n,n}$  ( $n \geq 1$ ), even cycles  $C_{2n}$  ( $n \geq 2$ ), binary  $n$ -cube  $Q(n)$  ( $n \geq 0$ ) and trees are dispersable and Overbay [8] gave detailed proof. For cartesian product graphs, Kainen [9] and Shao, Liu and Li [10] obtained that the matching book thickness of  $C_m \times C_n$  ( $m$  is even) and  $K_m \times C_n$  ( $m, n \geq 3$ ), respectively. In 2021, Kainen and Overbay showed that cubic planar bipartite graphs are dispersable (see [11]). For a Halin graph  $H$ , Shao, Geng, and Li [12] proved that  $mbt(H) = 4$ , if  $\Delta(H) = 3$  and  $mbt(H) = \Delta(H)$ , if  $\Delta(H) \geq 4$ .

The generalized Petersen graphs are a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. These graphs were first defined by Watkins [13]. The generalized Petersen graph  $P(n, k)$ ,  $n \geq 3$  and  $1 \leq k \leq n - 1$ , has vertex-set  $\{i, i' \mid i = 1, 2, \dots, n\}$  and edge-set  $\{(i, i + k), (i', (i + 1)'), (i, i') \mid i = 1, 2, \dots, n\} \pmod n$ .

The generalized Petersen graph  $P(n, k)$  is always a 3-regular graph with  $2n$  vertices and  $3n$  edges, and  $P(5, 2)$  is the well known Petersen graph, it can be matching embedded in a 4-page book (see Figure 1). From the definition of  $P(n, k)$ ,  $P(n, k) \cong P(n, n - k)$  for  $k < \frac{n}{2}$ , thus we can assume  $1 \leq k < \frac{n}{2}$ .



Figure 1.  $P(5, 2)$  (left) and the matching book embedding of  $P(5, 2)$ (right).

In this paper, we obtain that the matching book thickness of a generalized Petersen graph  $P(n, k)$  as follows.

**Main Theorem:**

$$mbt(P(n, k)) = \begin{cases} 3, & n \text{ is even and } k \text{ is odd,} \\ 4, & \text{else.} \end{cases}$$

**2. The Proof of Main Theorem**

The proof will be completed by a sequence of lemmas.

**Lemma 2.1.** [8] *For any simple graph  $G$ ,  $\Delta(G) \leq \chi'(G) \leq mbt(G)$ , where  $\chi'(G)$  is the chromatic index of  $G$ .*

**Lemma 2.2.** [8] For a regular graph  $G$ ,  $G$  is dispersable only if  $G$  is bipartite.

Suppose  $X = \{x_1, \dots, x_i\}$  is an ordered vertex set with  $i$  vertices in  $P(n, k)$ , we define  $X' := \{x'_1, \dots, x'_i\}$  and  $X^{-1} := \{x_i, \dots, x_1\}$ , and call them paired ordering of  $X$  and reverse ordering of  $X$ , respectively.

**Lemma 2.3.** If  $P(n, k)$  is a generalized Petersen graph,  $n$  is even and  $1 \leq k < \frac{n}{2}$ , then

$$mbt(P(n, k)) = \begin{cases} 3, & k \text{ is odd,} \\ 4, & k \text{ is even.} \end{cases}$$

*Proof.* According to the parity of  $k$ , we need to consider two cases to prove the matching book thickness of  $P(n, k)$ .

**Case 1.**  $k$  is odd.

Since  $\Delta(P(n, k)) = 3$ ,  $mbt(P(n, k)) \geq 3$  by Lemma 2.1. Now we embed the vertices of  $P(n, k)$  along the spine according to the following ordering of  $1, n, 3, n - 2, \dots, n - 3, 4, n - 1, 2, 2', (n - 1)', 4', (n - 3)' \dots, (n - 2)', 3', n', 1'$ . All edges of  $P(n, k)$  can be placed on 3 pages in a matching manner as follows.

Page 1: edges  $\{(i, i + k), (i', (i + 1)') \mid 1 \leq i \leq n, i \text{ is odd} \} \pmod n$ .

Page 2: edges  $\{(i, i + k), (i', (i + 1)') \mid 1 \leq i \leq n, i \text{ is even} \} \pmod n$ .

Page 3: edges  $\{(i, i') \mid 1 \leq i \leq n\}$ .

Hence  $mbt(P(n, k)) \leq 3$ . (See Figure 2(a) for the case  $k = 3$  and  $n = 10$ ).

**Case 2.**  $k$  is even.

In this case,  $P(n, k)$  is 3-regular but not bipartite because it contains at least one odd cycle. According to Lemma 2.2,  $P(n, k)$  is not dispersable,  $mbt(P(n, k)) \geq \Delta(P(n, k)) + 1 = 4$ . Let  $d = \gcd(n, k)$  be the greatest common denominator of  $n$  and  $k$  (here  $n$  and  $k$  are even, so  $d$  is even),  $\sigma_i$  is an ordered vertex set and  $\sigma_i = \{i, i + k, i + 2k, \dots, i + (\frac{n}{d} - 1)k\} \pmod n$  ( $1 \leq i \leq d$ ), then  $V(P(n, k)) = \bigcup_{i=1}^d \sigma_i \sigma'_i$ . We put all vertices on the spine in the ordering  $\sigma_1, \sigma_2^{-1}, \dots, \sigma_{d-1}, \sigma_d^{-1}, (\sigma_d^{-1})', (\sigma_{d-1})', \dots, (\sigma_2^{-1})', (\sigma_1)'$  and assign the edges of  $P(n, k)$  to 4 pages as follows.

Page 1: edges  $\{(n - i + ak, n - i + (a + 1)k) \mid 1 \leq i \leq d, 0 \leq a \leq \frac{n}{d} - 2, a \text{ is even} \} \pmod n$  and edges  $\{(i', (i + 1)') \mid 1 \leq i \leq n, i \text{ is odd} \}$ .

Page 2: edges  $\{(n - i + ak, n - i + (a + 1)k) \mid 1 \leq i \leq d, 0 \leq a \leq \frac{n}{d} - 2, a \text{ is odd} \} \pmod n$  and edges  $\{((d + jk)', (d + jk + 1)') \mid 0 \leq j \leq b\} \pmod n$ , where  $b = \min\{c \in \mathbb{Z}^+ \mid d + ck \equiv n - k \pmod n\}$ .

Page 3: edges  $\{(n - i - k, n - i) \mid 1 \leq i \leq d\}$ , edges  $\{((i + jk)', (i + jk + 1)') \mid 1 \leq i < d, i \text{ is even}, 0 \leq j \leq \frac{n}{d} - 1\} \pmod n$  and edges  $\{((d + jk)', (d + jk + 1)') \mid b + 1 \leq j \leq \frac{n}{d} - 1\} \pmod n$ , where  $b = \min\{c \in \mathbb{Z}^+ \mid d + ck \equiv n - k \pmod n\}$ .

Page 4: edges  $\{(i, i') \mid 1 \leq i \leq n\}$ .

Therefore we get a matching book embedding of  $P(n, k)$  in a 4-page book,  $mbt(P(n, k)) \leq 4$ . (See Figure 2(b) for the case  $k = 4$  and  $n = 10$ ).

The result is established. □

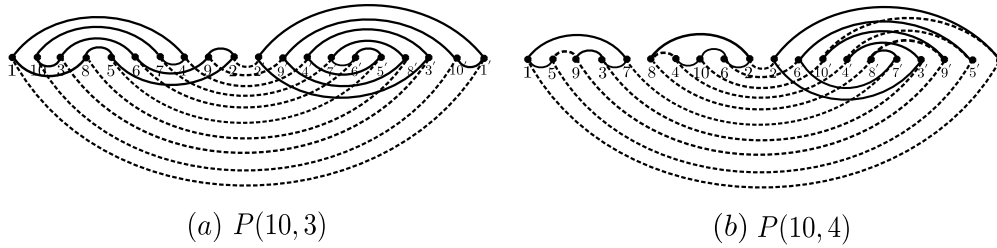


Figure 2. The matching book embedding of  $P(10, 3)$  and  $P(10, 4)$ .

**Lemma 2.4.** *If  $P(n, k)$  is a generalized Petersen graph,  $n$  is odd and  $1 \leq k < \frac{n}{2}$ , then  $mbt(P(n, k)) = \Delta(P(n, k)) + 1 = 4$ .*

*Proof.* Since  $n$  is odd,  $P(n, k)$  contains an odd cycle  $C_n$  according to its definition, it is not bipartite, and then  $P(n, k)$  is 3-regular, so it is not dispersible by Lemma 2.2,  $mbt(P(n, k)) \geq \Delta(P(n, k)) + 1 = 4$ .

Let  $n = ak + r$  ( $a \geq 2, 0 \leq r < k$ ),  $A_i, B_i, C_i$  and  $D_i$  be ordered vertex set,  $A_i = \{i, i + n - k\}$  ( $1 \leq i \leq k$ ),  $B_i = \{ik, ik - 1, \dots, ik - (k - 1)\}$  ( $2 \leq i \leq a$ ),  $C_i = \{ak - i, ak - i - k\}$  ( $0 \leq i \leq k - 1$ ),  $D_i = \{ik + 1, ik + 2, \dots, ik + k\}$  ( $1 \leq i \leq a$ ). According to the parity of  $a$ , we need to consider two cases to prove that  $mbt(P(ak + r, k)) \leq 4$  as follows.

**Case 1.**  $a$  is even.

It is easy to know that  $r$  is odd because  $n$  is odd and  $a$  is even. Assume  $\alpha_1 = \prod_{i=k-r+1}^k A_i^{(-1)^{i+1}} = A_{k-r+1}^{(-1)^{k-r+2}}, A_{k-r+2}^{(-1)^{k-r+3}}, \dots, A_k^{(-1)^{k+1}}$ ;  $\alpha_2 = \prod_{i=2}^a B_i^{(-1)^{i+1}} = B_2^{-1}, B_3, \dots, B_{a-1}, B_a^{-1}$ ;  $\varphi_1 = \{1, 2, \dots, k - r, \alpha_1, \alpha_2^{-1}\}$ . Now we can construct a matching book embedding of  $P(ak + r, k)$  in a 4-page book as follows. Put all  $2n$  vertices on the spine in the ordering  $\varphi_1(\varphi_1^{-1})'$  and assign all edges to 4 pages.

Page 1: edges  $\{(i + jk, i + jk + k) \mid 1 \leq i \leq k, 0 \leq j \leq a - 1, j \text{ is even}\}$  and edges  $\{(i', (i + 1)'), (j', (j + 1)'), \mid 1 \leq i \leq 2k, i \text{ is odd}, 2k \leq j \leq ak, j \text{ is even and } j \neq bk\}$ , where  $1 \leq b \leq a$  and  $b$  is even.

Page 2: edges  $\{(n - i - k, n - i) \mid 0 \leq i \leq r - 1\}$  and edges  $\{(i', (i + 1)'), (j', (j + 1)'), \mid 1 \leq i \leq 2k - 1, i \text{ is even}, 2k \leq j \leq n, j \text{ is odd}\} \pmod n$ .

Page 3: edges  $\{(k - i, n - i), (jk - i, jk - i + k) \mid 0 \leq i \leq k - 1, 1 \leq j \leq a - 1, j \text{ is even}\}$  and edges  $\{((ak + i)', (ak + i + 1)'), ((jk)', (jk + 1)'), \mid 1 \leq i \leq r, i \text{ is even}, 2 \leq j \leq a, j \text{ is even}\}$ .

Page 4: edges  $\{(i, i') \mid 1 \leq i \leq n\}$ .

Hence  $mbt(P(ak + r, k)) \leq 4$ . (See Figure.3(a) for the case  $a = 2, k = 4$  and  $r = 3$ ).

**Case 2.**  $a$  is odd.

In this case, we prove  $mbt(P(ak + r, k)) \leq 4$  from the following two situations.

- (1)  $k$  is even and  $r$  is odd.

We denote  $\beta_1 = \prod_{i=k-r+1}^k A_i^{(-1)^{i+1}} = A_{k-r+1}^{-1}, A_{k-r+2}, \dots, A_{k-1}, A_k^{-1}; \beta_2 = \prod_{i=0}^{k-1} C_i^{(-1)^{i+1}} = C_0^{-1}, C_1, \dots, C_{k-2}^{-1}, C_{k-1}; \beta_3 = \prod_{i=1}^{a-3} D_i^{(-1)^{i+1}} = D_1, D_2^{-1}, \dots, D_{a-4}, D_{a-3}^{-1}; \varphi_2 = \{1, 2, \dots, k-r, \beta_1, \beta_2, \beta_3^{-1}\}$ . All  $2n$  vertices of  $P(ak+r, k)$  are assigned on the spine by the ordering of  $\varphi_2(\varphi_2^{-1})'$  and all edges of  $P(ak+r, k)$  can be matching embedded in a 4-page book as follows.

Page 1: edges  $\{(i+jk, i+jk+k) \mid 1 \leq i \leq k, 0 \leq j \leq a-3, j \text{ is even}\}$  and edges  $\{(i', (i+1)'), (j', (j+1)'), (m', (m+1)'), \mid 1 \leq i \leq 3k, i \text{ is odd}, 3k+1 \leq j \leq ak-1, j \text{ is even and } j \neq bk\}$ , where  $3 < b < a, b \text{ is odd}$ .

Page 2: edges  $\{(n-i-k, n-i) \mid 0 \leq i \leq k-1\}$  and edges  $\{(i', (i+1)'), (j', (j+1)'), \mid 1 \leq i \leq 3k-1, i \text{ is even}, 3k \leq j \leq n, j \text{ is odd}\} \pmod n$ .

Page 3: edges  $\{(k-i, n-i), (jk-i, jk-i+k) \mid 0 \leq i \leq k-1, 1 \leq j \leq a-2, j \text{ is even}\}$ , edges  $\{((a-2)k+i, (a-2)k+i+k) \mid 1 \leq i \leq r\}$  and edges  $\{((ik)', (ik+1)'), (j', (j+1)'), \mid 2 \leq i \leq a-1, i \text{ is odd}, ak \leq j \leq n, j \text{ is even}\}$ .

Page 4: edges  $\{(i, i') \mid 1 \leq i \leq n\}$ .

So  $mbt(P(ak+r, k)) \leq 4$ . (See Figure.3(b) for the case  $a = 5, k = 2$  and  $r = 1$ ).

(2)  $k$  is odd and  $r$  is even.

Let  $\gamma_1 = \prod_{i=k-r+1}^k A_i^{(-1)^{i+1}} = A_{k-r+1}^{-1}, A_{k-r+2}, \dots, A_{k-1}, A_k; \gamma_2 = \prod_{i=0}^{k-1} C_i^{(-1)^i} = C_0, C_1^{-1}, \dots, C_{k-2}^{-1}, C_{k-1}; \gamma_3 = \prod_{i=1}^{a-3} D_i^{(-1)^{i+1}} = D_1, D_2^{-1}, \dots, D_{a-4}, D_{a-3}^{-1}; \varphi_3 = \{1, 2, \dots, k-r, \gamma_1, \gamma_2, \gamma_3^{-1}\}$ .

(i)  $2 < r \leq k-1$

We can construct a matching book embedding of  $P(ak+r, k)$  in a 4-page book as follows. All the vertices are placed on the spine in the order  $\varphi_3(\varphi_3^{-1})'$ , and all the edges can be placed in 4 pages so that the edges of the same page do not intersect each other and the edges induced subgraphs of each page are 1-regular.

Page 1: edges  $\{(i+jk, i+jk+k) \mid 1 \leq i \leq k, 0 \leq j \leq a-3, j \text{ is even}\}$  and edges  $\{(i', (i+1)'), (j', (j+1)'), (m', (m+1)'), \mid 1 \leq i \leq (a-1)k, i \text{ is odd}, (a-1)k+1 \leq j \leq ak, j \text{ is even}, ak+1 \leq m \leq n-1, m \text{ is odd}\}$ .

Page 2: edges  $\{(n-i-k, n-i) \mid 0 \leq i \leq k-1\}$ , edges  $\{(i', (i+1)'), (j', (j+1)'), (m', (m+1)'), \mid 1 \leq i \leq (a-1)k-1, i \text{ is even}, (a-1)k \leq j < ak, j \text{ is odd}, ak+1 \leq m \leq n-2, m \text{ is even}\} \pmod n$  and edge  $(n', 1')$ .

Page 3: edges  $\{(k-i, n-i), (jk-i, jk-i+k) \mid 0 \leq i \leq k-1, 1 \leq j \leq a-2, j \text{ is even}\}$ , edges  $\{((a-2)k+i, (a-2)k+i+k) \mid 1 \leq i \leq r\}$ , edges  $\{((ik)', (ik+1)'), \mid a-1 \leq i \leq a\}$  and edge  $((n-1)', n')$ .

Page 4: edges  $\{(i, i') \mid 1 \leq i \leq n\}$ .

Hence  $mbt(P(ak+r, k)) \leq 4$ . (See Figure.3(c) for the case  $a = 3, k = 5$  and  $r = 4$ ).

(ii)  $0 \leq r \leq 2$

When  $r = 0$ , let  $\varphi_4 = \{A_1, A_2^{-1}, \dots, A_{k-1}^{-1}, A_k, D_{a-2}^{-1}, D_{a-3}, \dots, D_1^{-1}\}$ . We place all  $2n$  vertices on the spine in the ordering  $\varphi_4(\varphi_4^{-1})'$  and assign the edges of  $P(ak+r, k)$  to 4 pages as

follows.

Page 1: edges  $\{(i + jk, i + jk + k) \mid 1 \leq i \leq k, 0 \leq j \leq a - 2, j \text{ is even}\}$ , edge  $\{(n - i + 1, n - i + 1 - k) \mid 0 \leq i \leq \frac{r}{2}, i \text{ is odd}\}$  and edges  $\{(i', (i + 1)') \mid 1 \leq i \leq n - 1, i \text{ is odd}\}$ .

Page 2: edges  $\{(n - i - k, n - i) \mid \frac{r}{2} \leq i \leq k - 1\}$ , edge  $(n', 1')$  and edges  $\{(i', (i + 1)') \mid 1 \leq i \leq (a - 1)k + r, i \text{ is even and } i \neq bk + r\}$ , where  $2 \leq b \leq a - 1$  and  $b$  is even.

Page 3: edges  $\{(k - i, n - i), (i + jk, i + jk + k) \mid 0 \leq i \leq k - 1, 0 \leq j \leq a - 3, j \text{ is odd}\}$  and edges  $\{((ik + r)', (ik + r + 1)'), (j', (j + 1)') \mid 2 \leq i \leq a - 1, i \text{ is even}, (a - 1)k + r + 1 \leq j \leq n, j \text{ is even}\}$ .

Page 4: edges  $\{(i, i') \mid 1 \leq i \leq n\}$ .

So  $P(ak + r, k)$  can be matching embedded into 4 pages.

When  $r = 2$ , let  $\varphi_5 = \{A_1, A_2^{-1}, \dots, A_{k-1}^{-1}, A_k, (a-1)k+1, (a-1)k+2, D_{a-2}^{-1}, D_{a-3}, \dots, D_1^{-1}\}$ . Let the order of  $2n$  vertices of  $P(ak + r, k)$  on the spine be as  $\varphi_5(\varphi_5^{-1})'$ , the edges of page 1, 2 and 4 are embedded in the same matching manner as when  $r = 0$ , but the edges  $\{((a - 2)k + i, (a - 1)k + i) \mid 1 \leq i \leq 2\}$  are added in page 3 based on  $r = 0$ .

In summary, we can get a matching book embedding of  $P(ak + r, k)$  in in a 4-page book. Hence  $mbt(P(ak + r, k)) \leq 4$ . (See Figure.3(d) for the case  $a = 3, k = 5$  and  $r = 2$ )

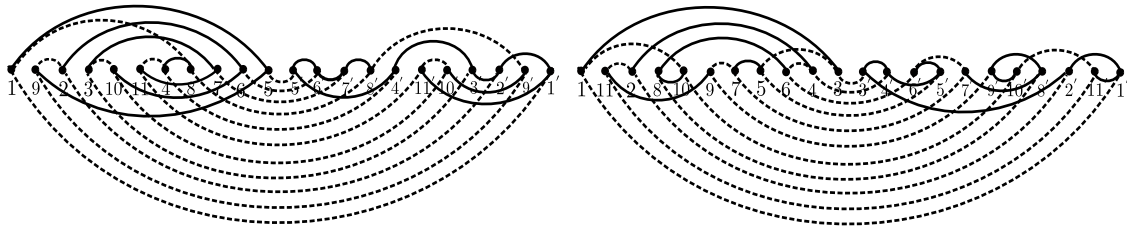
The result is obtained. □

### 3. Conclusions

In this paper, we study the matching book embedding of the generalized Petersen graph  $P(n, k)$ , and give different matching book embedding manners for the different parity of  $n$  and  $k$ . Finally, we determine the matching book thickness of  $P(n, k)$ , that is,  $mbt(P(n, k)) = 3$  when  $n$  is even and  $k$  is odd, and  $mbt(P(n, k)) = 4$  in other cases.

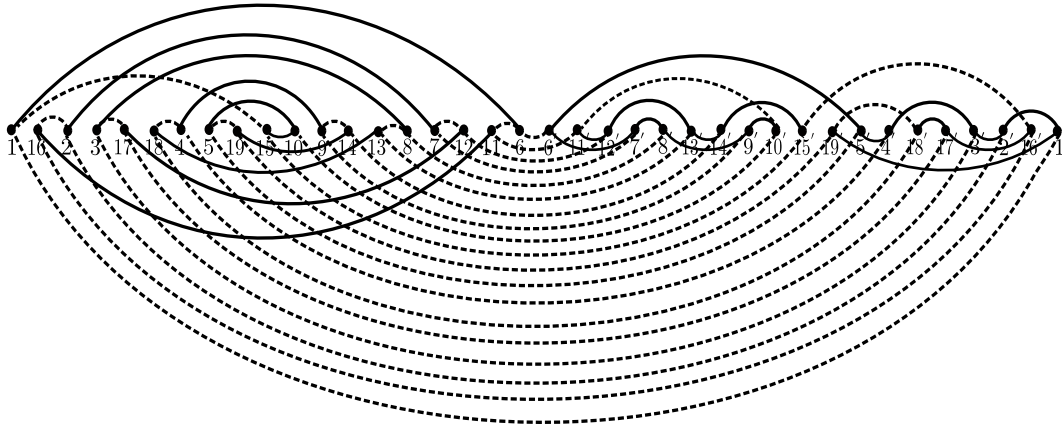
### Acknowledgement

We sincerely thank the anonymous referee for useful comments to improve the paper. This work was supported in part by Key Projects of Natural Science Research in College and universities of Hebei Province, China (No.ZD2020130) and the Natural Science Foundation of Hebei Province, China (No. A2021202013).

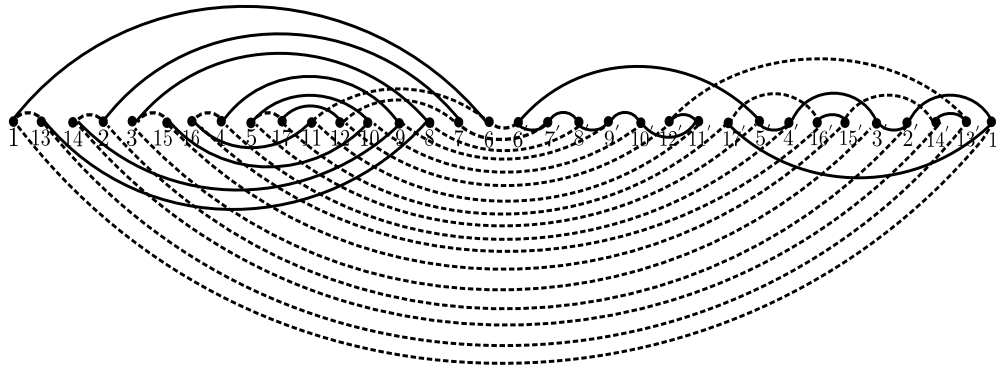


(a)  $P(11,4)$

(b)  $P(11,2)$



(c)  $P(19,5)$



(d)  $P(17,5)$

Figure 3. The matching book embedding of  $P(11,4)$ ,  $P(11,2)$ ,  $P(19,5)$  and  $P(17,5)$ .

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