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Grünbaum colorings extended to non-facial 3-cycles

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Abstract

We consider the question of when a triangulation with a Grünbaum coloring can be edge-colored with three colors such that the non-facial 3-cycles also receive all three colors; we will call this a *strong Grünbaum coloring*. It turns out that for the sphere, every triangulation has a strong Grünbaum coloring, and that the presence of a K_5 subgraph prohibits a strong Grünbaum coloring, but that K_5 is not the only such barrier. We investigate the ramifications of these facts. We also show that for every other topological surface there exist triangulations with a strong Grünbaum coloring and triangulations that have Grünbaum colorings but that cannot have a strong Grünbaum coloring. Finally, we reframe strong Grünbaum colorings as certain hypergraph edge colorings, and raise the question of how many colors are needed to achieve an edge coloring such that both facial and non-facial 3-cycles receive three colors.

Keywords: embedding, edge coloring, Grünbaum coloring, triangulation
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1. Introduction

We consider triangulations, which are embedded graphs in which every face has three edges. A triangulation has a *Grünbaum coloring* if there exists an edge 3-coloring in which each face is

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assigned all three colors. It is known that all planar triangulations and many classes of toroidal [1], projective-planar [4], and Klein-bottle [3] triangulations have Grünbaum colorings. One reason that Grünbaum colorings are of interest is because of their connection to the Four Color Theorem: Every planar graph has a Grünbaum coloring because the Four Color Theorem assures us of a face 4-coloring, and Tait's theorem implies that every face 4-colorable embedding has an edge 3-coloring (which is in turn dual to a Grünbaum coloring).

There are many ways to generalize Grünbaum colorings. For example, in [5] the authors consider graph embeddings that have all faces of size d and attempt to assign each face d edge colors; in [6] the authors consider simplicial complexes and attempt to assign every simplicial facet all colors. More widely studied is the situation of 3-coloring triangles of (ordinary, not embedded) graphs. The papers [7, 8] find conditions under which an edge-colored graph must have a rainbow triangle; [2] counts the maximum number of rainbow triangles in 3-edge-colored K_n.

Our focus is a different generalization of Grünbaum colorings: We say that a triangulation has a *strong Grünbaum coloring* when it can be edge 3-colored such that both facial and non-facial 3-cycles receive all three colors. Note that a triangulation is topologically dual to an embedded cubic graph. While a Grünbaum coloring of a triangulation corresponds under duality to a proper edge 3-coloring of the dual cubic graph, a strong Grünbaum coloring corresponds to a stronger edge coloring of the dual cubic graph—in addition to being a proper edge 3-coloring, selected triples of must receive three distinct colors.

For which Grünbaum-colorable triangulations do there exist strong Grünbaum colorings? We show in Section 2.1 that for the sphere, every triangulation has a strong Grünbaum coloring, and in Sections 2.2 and 2.3 that for every other topological surface there exist triangulations with a strong Grünbaum coloring *and* triangulations that have Grünbaum colorings but that cannot have a strong Grünbaum coloring. We explain in Section 2.4 why structural results are currently elusive. Finally, in Section 3 we raise, but do not resolve, the question of how many colors are needed to assign colors to edges of a triangulation of a surface such that every 3-cycle (both facial and non-facial) receives three distinct colors.

2. Results

2.1. Contractible triangles

As previously mentioned, every planar triangulation has at least one Grünbaum coloring. It turns out that not only does every planar triangulation have a strong Grünbaum coloring, but in fact every (regular) Grünbaum coloring on the plane will be a strong Grünbaum coloring. The proof is straightforward, and as Mark Ellingham has observed, this proof is essentially the dual to that of the parity lemma.

Theorem 2.1. *Every Grünbaum coloring of a planar triangulation is a strong Grünbaum coloring.*

Proof. Consider a non-facial triangle in a Grünbaum coloring of a planar triangulation. This triangle and its interior may be cut out of the triangulation of the sphere and placed on another sphere, where the non-facial triangle has become the exterior face.

We now have a triangulation with f faces. Because a triangulation is dual to a cubic graph, it has an even number of faces; thus, $f-1$ is odd. For any color, the total number of edge-face incidences must be even, and therefore it must be that each color appears an odd number of times on the exterior face. The only way this can happen is if each color appears exactly once. □

The ideas in the proof of Theorem 2.1 extend in multiple ways to triangulations on general surfaces.

Corollary 2.2. *For a Grünbaum coloring of any triangulation on any orientable surface* $\mathbb{S}_g$ *or nonorientable surface* $\mathbb{N}_g$*, any contractible triangle must be Grünbaum colored.*

Corollary 2.3. *In a triangulation of edge-width at least* 4 *(so there are no noncontractible* 3*-cycles) on any surface* $\mathbb{S}_g$ *or* $\mathbb{N}_g$*, every Grünbaum coloring is a strong Grünbaum coloring.*

Corollary 2.4. *Every 4-vertex-colorable triangulation on any surface* $\mathbb{S}_g$ *or* $\mathbb{N}_g$ *is strong Grünbaum colorable.*

Corollary 2.4 follows directly from the proof in [1] that every 4-vertex-colorable triangulation is Grünbaum colorable.

Corollary 2.5. *A Grünbaum coloring of any triangulation with only contractible triangles is a strong Grünbaum coloring.*

2.2. Triangulations containing K_5

From Corollary 2.3 we see that it remains to consider triangulations with noncontractible 3-cycles. The most elementary nonplanar subgraph to consider is K_5.

Theorem 2.6. *Any triangulation containing* K_5 *is not strong Grünbaum colorable.*

Proof. We will show that six of the ten 3-cycles of K_5 cannot be simultaneously be Grünbaum colored, so for any embedding of K_5 there does not exist a strong Grünbaum coloring. For illustration, consider an embedding of K_5 on the torus where 4 of the 3-cycles are facial triangles and 6 of the 3-cycles are noncontractible. In Figure 1(left), we color the 1-2-5 facial triangle purple-

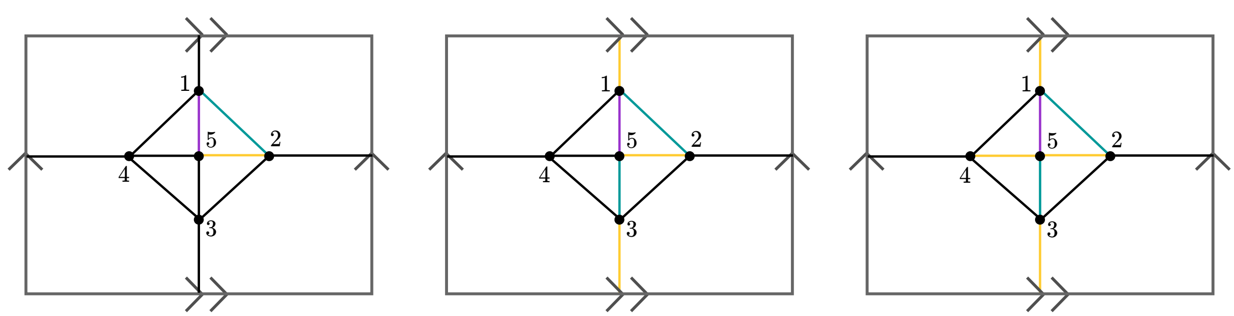

Figure 1. A K_5 on the torus with (left) one facial triangle colored (center) an additional noncontractible cycle colored (right) a contradiction to Grünbaum coloring a different noncontractible cycle.

teal-gold. Note that edge 5-3 cannot be gold if the 2-3-5 facial triangle is to have three colors; nor can edge 5-3 be purple if the 1-5-3 cycle is to have three colors. Thus, edge 5-3 must be teal (see

Figure 1(center)). Now, examine edge 4-5. This edge cannot be purple if the 1-4-5 facial triangle is to have three colors, nor can it be teal if the 3-4-5 facial triangle is to have three colors; thus, edge 4-5 must be gold. However, this prohibits the 4-5-2 noncontractible 3-cycle from receiving three colors, so it is not possible to create a strong Grünbaum coloring of K_5. □

Note that this does not extend to subdivisions of K_5, as the example in Figure 2 shows. However, Theorem 2.6 allows us to produce Grünbaum-colored triangulations that are not strong

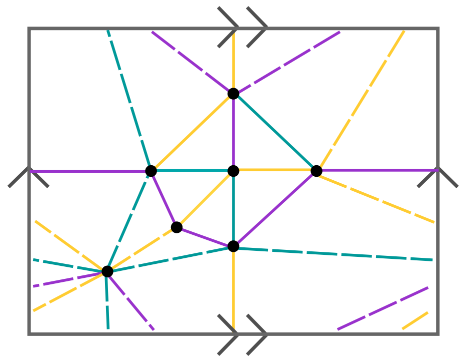

Figure 2. This embedded subdivision of K_5 easily extends to a strong Grünbaum colored toroidal triangulation.

Grünbaum colorable on every nonspherical surface.

Corollary 2.7. *For $g \geq 1$, there exists a triangulation T_g embedded on $\mathbb{S}_g$ such that T_g has a 3-cycle corresponding to every homotopy generator and has a Grünbaum coloring but not a strong Grünbaum coloring. Moreover, T_g has at least g non-3-colored noncontractible 3-cycles (one on each torus in the connected sum).*

Proof. Consider the triangulation N of the torus given in Figure 3; it has a Grünbaum coloring but, because it contains K_5, not a strong Grünbaum coloring.

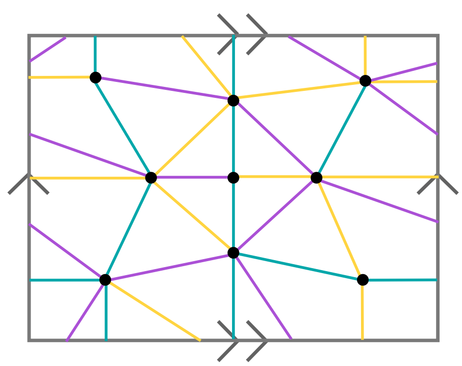

Figure 3. A Grünbaum colored toroidal triangulation containing K_5.

Now consider the Grünbaum-colored triangulation of the cylinder C shown in Figure 4. First

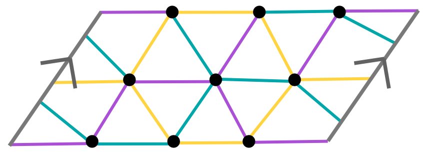

Figure 4. An even triangulated cylinder with a strong Grünbaum coloring.

we construct a Grünbaum-colored triangulation on the two-holed torus. Begin with two copies of N, say N_1 and N_2. Remove the interior of one (Grünbaum-colored) face of N_1, and identify its edges with the edges on one boundary of C; then, remove the interior of a face of N_2 and identify its edges with the other cylinder boundary. This produces the desired triangulation.

We can use this same process to construct a Grünbaum-colored triangulation on $\mathbb{S}_g$ as follows: Using copies $N_1, \ldots N_g$ of N and copies $C_1, \ldots C_{g-1}$ of C, remove the interior of one face of N_1, N_g and remove the interior of two faces of each of $N_2, \ldots N_{g-1}$. For each $1 \leq i \leq g-1$, identify the edges bounding a removed face of N_i with one boundary of C_i and identify the edges bounding a (different) removed face of N_{i+1} with the other boundary of C_i. This produces a triangulation T_g embedded on $\mathbb{S}_g$, for $g \geq 1$, such that T_g has a K_5 contained in every torus of the connected sum, and therefore a 3-cycle corresponding to every homotopy generator. T_g therefore also has a Grünbaum coloring but not a strong Grünbaum coloring. □

The toroidal triangulation containing K_5 shown in Figure 3 has chromatic number 6. A natural question is whether the situation is different for toroidal triangulations with chromatic number 5, or for even triangulations. This is not the case. Figure 5 shows a toroidal triangulation that has a Grünbaum coloring but cannot have a strong Grünbaum coloring, but that is even and has chromatic number 5.

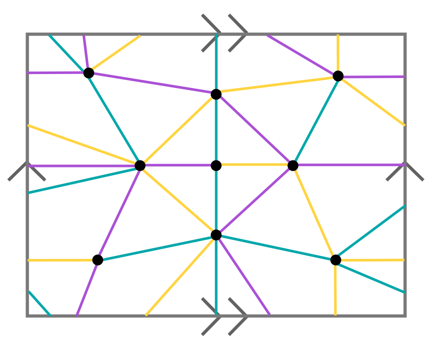

Figure 5. A 5-chromatic even toroidal triangulation that has a Grünbaum coloring but cannot have a strong Grünbaum coloring.

We next treat the case of nonorientable surfaces.

Corollary 2.8. *For $g \geq 1$, there exists a triangulation T'_g embedded on $\mathbb{N}_g$ such that T'_g has a 3-cycle corresponding to every homotopy generator and has a Grünbaum coloring but not a strong*

Grünbaum coloring. Moreover, T'_g has at least g non-3-colored noncontractible 3-cycles (one on each projective plane in the connected sum).

Proof. Consider the triangulation of the projective plane given in Figure 6; it has a Grünbaum coloring but, because it contains K_5, not a strong Grünbaum coloring.

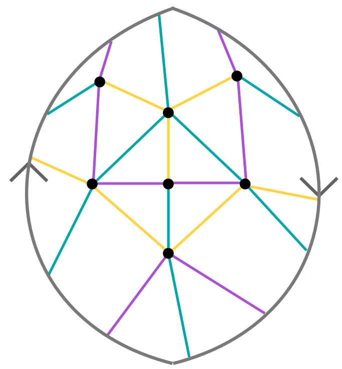

Figure 6. A Grünbaum-colored projective-planar triangulation containing K_5.

Using the same construction given in the proof of Proposition 2.7 but with the triangulation from Figure 6, we obtain a triangulation T'_g embedded on $\mathbb{N}_g$, for $g \geq 1$, such that T'_g has a 3-cycle corresponding to every homotopy generator and has a Grünbaum coloring but not a strong Grünbaum coloring. □

2.3. Structural Observations

After finding that any triangulation containing K_5 cannot be strong Grünbaum colorable, the next natural question to ask is whether a Grünbaum colorable triangulation without K_5 must have a strong Grünbaum coloring. As Proposition 2.9 attests, the answer is 'no.' However, at the same time we can find many strong Grünbaum colorings of K_5-free nonplanar triangulations. In Proposition 2.10 we construct a family of such triangulations with at least one 3-cycle corresponding to each homotopy generator. We close the section with an example of a Grünbaum colored triangulation with a single 3-cycle and no strong Grünbaum coloring.

Proposition 2.9. *There exists a K_5-free triangulation T_g embedded on $\mathbb{S}_g$, for $g \geq 1$, and there exists a K_5-free triangulation T'_{2g} embedded on $\mathbb{N}_{2g}$, for $2g \geq 2$, such that T_g, T'_{2g} has a 3-cycle corresponding to every homotopy generator and has a Grünbaum coloring but not a strong Grünbaum coloring.*

Proof. By examination of Figure 7 we can see that these triangulations, of the torus and Klein bottle respectively, have Grünbaum colorings. Using the construction given in the proof of Corollary 2.7 analogous triangulations can be constructed for any $\mathbb{S}_g$, $g > 1$ and for any $\mathbb{N}_{2g}$, $2g > 2$. To show that the triangulations in Figure 7 cannot be strong Grünbaum colored, we work in the dual with 3-regular graphs. Here, any triple of edges dual to a noncontractible 3-cycle in the triangulation—a triple of edges crossing a noncontractible loop in the surface—must get 3 different colors.

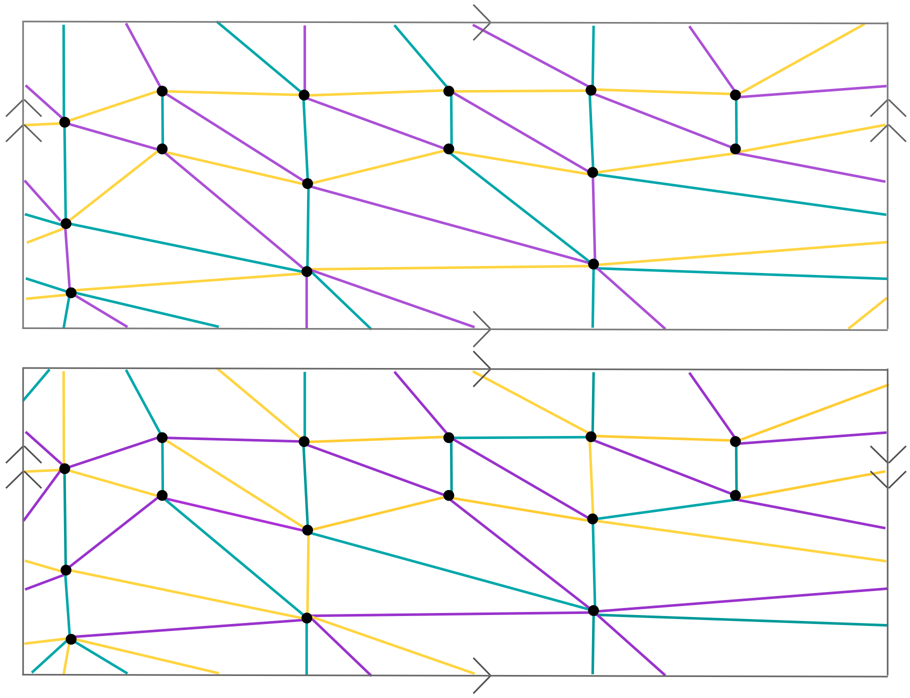

Figure 7. A Grünbaum-colored toroidal triangulation and a Grünbaum-colored triangulation of the Klein bottle, neither containing K_5.

Consider the embedded gadget shown at left in Figure 8. If it is part of the dual to a strong Grünbaum coloring, then it must have three different colors on the left-hand triple of edges. Check-

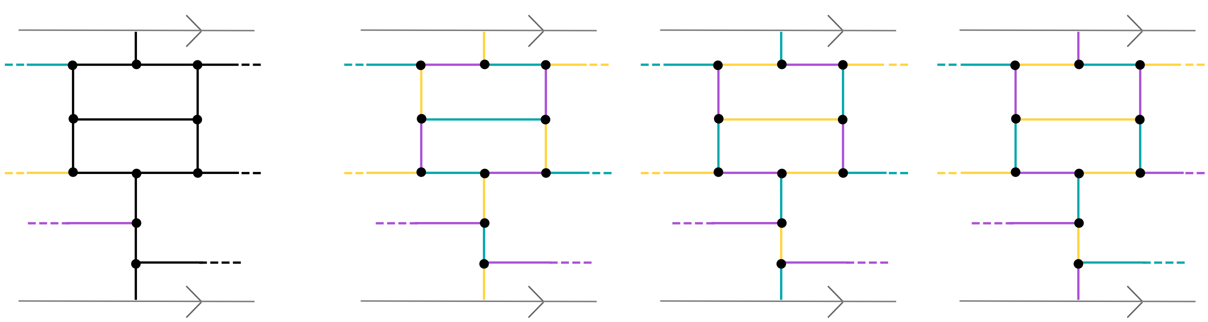

Figure 8. A gadget and its possible "strong" proper 3-colorings.

ing possible edge colorings by brute force produces only the three proper edge colorings shown at right. Note that for each of these three colorings, the middle-left partial edge is always the same color as the upper-right partial edge.

Now examine the toroidal 3-regular graph shown in Figure 9, composed of three copies of the gadget from Figure 8. It is dual to the triangulation in Figure 7. Suppose that edge e is assigned color a in a "strong" proper edge 3-coloring (so that it will be dual to a strong Grünbaum coloring),

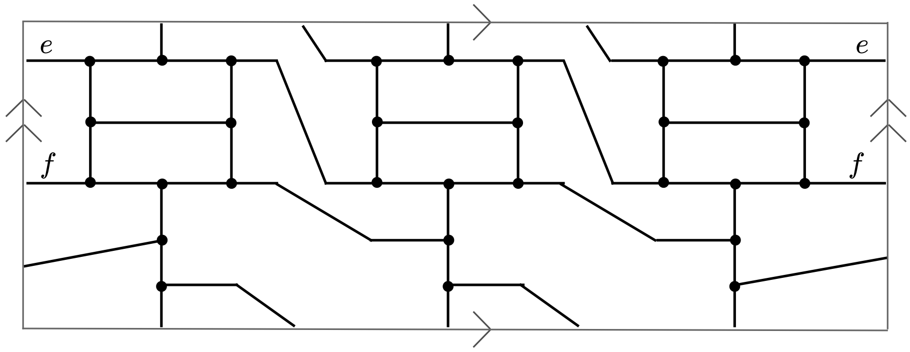

Figure 9. Three copies of the Figure 8 gadget form a toroidal embedding dual to Figure 7.

with color b assigned to edge f. This forces the upper-right edges emanating from each gadget to be assigned color b, which means that both edge e and edge f must be assigned color b. Therefore this graph's dual (Figure 7, top) cannot have a strong Grünbaum coloring.

This same drawing, but with the top-left partial edge joined to the bottom-right partial edge and the bottom-left partial edge joined to the top-right partial edge, gives an embedding on the Klein bottle. Again assigning color b to edge f, we find that the upper-right partial edge must also be color b. This eliminates the possibility that the dual (Figure 7, bottom) could have a strong Grünbaum coloring. □

Note that Proposition 2.9 gives no information about the existence of a K_5-free triangulation T embedded on the projective plane such that T has at least one noncontractible 3-cycle and has a Grünbaum coloring but not a strong Grünbaum coloring.

At this point it seems that perhaps strong Grünbaum colorings are unusual on most surfaces, even for triangulations with a sparse distribution of noncontractible 3-cycles. Yet, the next proposition gives strong Grünbaum colorings for any nonspherical surface.

Proposition 2.10. *Given $g \geq 1$, there exists a triangulation T_g embedded on $\mathbb{S}_g$ and there exists a triangulation T'_g embedded on $\mathbb{N}_g$ such that T_g, T'_g has a 3-cycle corresponding to every homotopy generator and has a strong Grünbaum coloring.*

Proof. Consider the triangulations of the torus and projective plane given in Figure 10; each has a strong Grünbaum coloring. Note that the Grünbaum coloring of the triangulated cylinder shown in Figure 4 is a strong Grünbaum coloring; thus we may use the same construction as given in the proof of Proposition 2.7 for each of the triangulations given in Figure 10 to produce the desired examples. □

In the opposite direction, we may consider a triangulation with a single non-contractible 3-cycle. In this mildest case, must every such triangulation with a Grünbaum coloring also have a strong Grünbaum coloring? It turns out that the answer is 'no.'

Proposition 2.11. *There exists a triangulation embedded on the torus with exactly one non-contractible 3-cycle that has a Grünbaum coloring but not a strong Grünbaum coloring.*

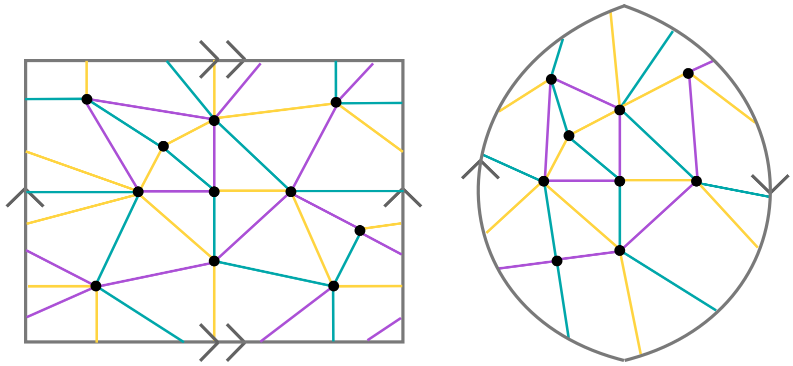

Figure 10. Strong Grünbaum colored triangulations on torus(left) and projective plane (right).

Proof. We will show that the dual to the properly edge 3-colored graph in Figure 11 has the desired property. This graph has exactly one triple of edges dual to a non-contractible 3-cycle. The proper

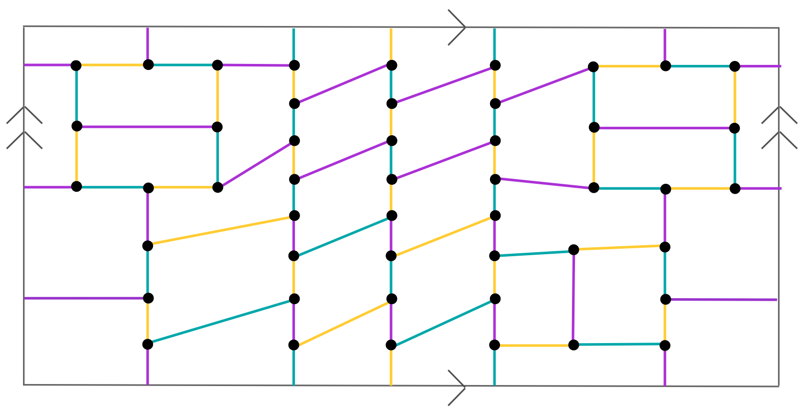

Figure 11. A proper edge coloring of a 3-regular toroidal graph.

edge 3-coloring immediately gives a Grünbaum coloring of the dual triangulation that is not strong Grünbaum, so it remains to show that no other proper edge 3-coloring of the Figure 11 embedded graph can correspond to a strong Grünbaum coloring of the dual triangulation.

As shown in Figure 14, this graph is constructed by combining two (uncolored) copies of the embedded gadget shown in Figure 12 with three (uncolored) copies of the embedded gadget shown at left in Figure 13 and one (uncolored) copy of the gadget shown at right in Figure 13. For ease of reference, for each gadget we will call the left-hand pendant edges the *input* of the gadget and the right-hand pendant edges the *output* of the gadget.

Consider the embedded gadget shown at left in Figure 12. If it is part of the dual to a strong Grünbaum coloring, then it must have three different colors on the left-hand triple of pendant edges. Checking possible edge colorings by brute force produces only the six proper edge colorings shown at right in Figure 12. Note the five different colorings of the output.

Next consider the left part of Figure 13, which shows the colorings of an embedded stick gadget where the input colorings match the patterns of the output colorings from Figure 12. At right in

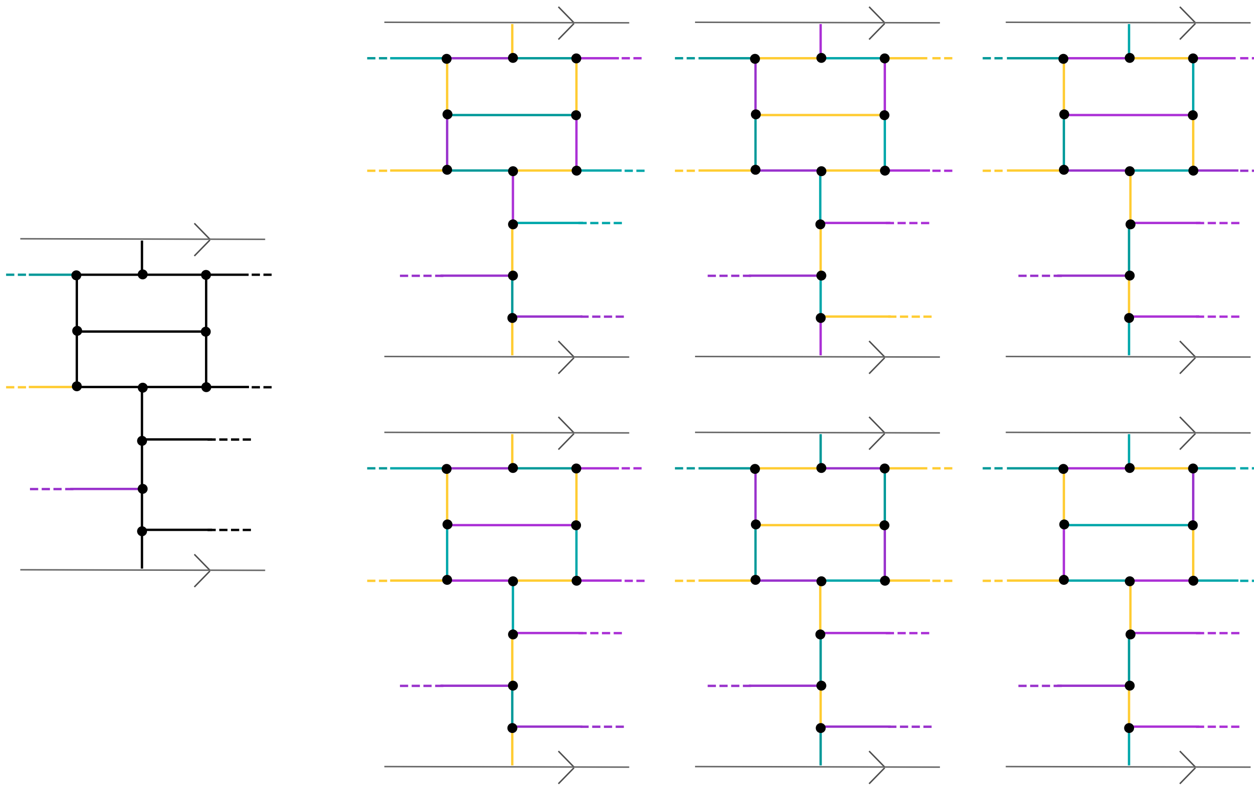

Figure 12. A gadget and its possible "strong" proper 3-colorings.

Figure 13 is our third, small gadget: if the input edges are the same color, then the output edges must be a different color; if the input edges are different colors, then the output has the same two colors with positions swapped.

Now examine the toroidal 3-regular graph shown in Figure 14. Suppose that the left-hand triple of edges is assigned three different colors in a "strong" proper edge 3-coloring (so that it will be dual to a strong Grünbaum coloring). No matter which coloring the left (Figure 12) gadget has, this coloring will propagate across the three copies of the Figure 13 stick gadget to output one of the five output colorings of the Figure 12 gadget. Then the Figure 13 small gadget changes the coloring: the four pendant edges leading into the reflected Figure 12 gadget on the right must have a coloring that is *not* one of the five possible colorings that will allow the triple of edges at right to receive three different colors.

Therefore the Figure 14 graph's dual cannot have a strong Grünbaum coloring. This completes the proof. □

2.4. *Revealing the challenge: a hypergraph reformulation of the problem*

So far in this paper we have built a catalog of results about conditions when a strong Grünbaum coloring does and does not exist, but a structural classification seems to be out of reach. This difficulty motivates a reframing of the problem.

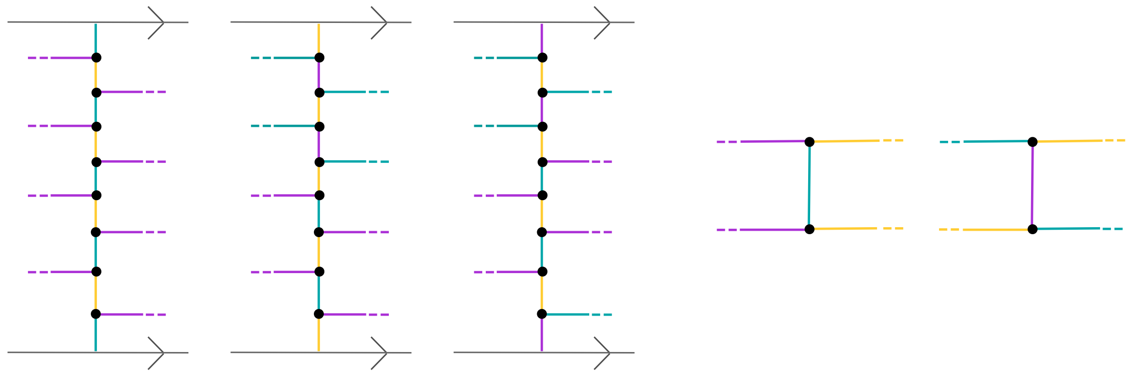

Figure 13. At left, propagation of colorings from Figure 12 across an embedded gadget; at right, the possible colorings of an unembedded gadget.

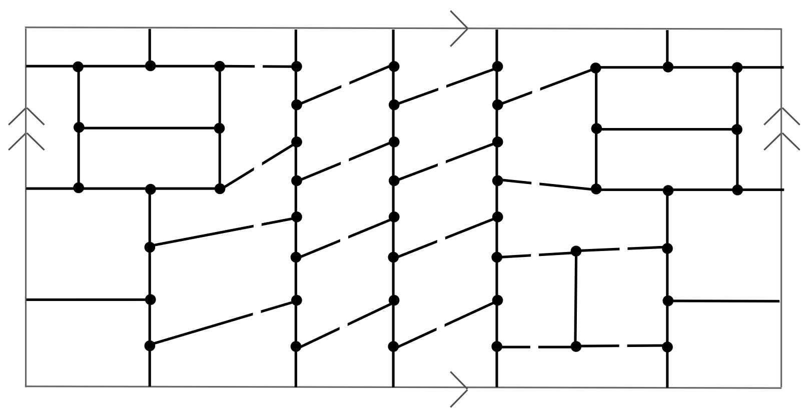

Figure 14. The decomposition of the Figure 11 graph into gadgets.

So far we have described a strong Grünbaum coloring of a triangulation T as dual to a proper edge 3-coloring of an embedded cubic graph G that also has some special triples of edges that must receive three different colors. Another way of conceptualizing this structure is as a hypergraph H constructed from our embedded cubic graph G, where we add a vertex v for each dual non-contractible 3-cycle of T, and v is incident to the three corresponding edges in G. In H, every vertex is incident to three edges but may have much higher degree—in particular, an added vertex has degree at least 5 because it is adjacent to the two vertices on each of its three incident edges (and at most two of those edges share a vertex). The strong Grünbaum coloring now corresponds to a proper edge coloring of H.

We may require H to be linear, i.e. having no two edges that share more than one vertex, by not allowing T to have multiple edges. Note that H is not uniform: most edges have size 2 (as they are not dual to edges in non-contractible 3-cycles), and some have size 3 or more. An edge of H that is dual to an edge of T in exactly one non-contractible 3-cycle will have size 3, but an edge of H that is dual to an edge of T in multiple non-contractible 3-cycles will have larger size.

Unfortunately, as of this writing there are no known results on proper edge coloring of non-uniform hypergraphs. In some sense this explains why we are not easily able to obtain stronger results, especially about which triangulations have (or do not have) strong Grünbaum colorings.

3. Extra colors

A natural extension of determining which triangulations have strong Grünbaum colorings is: For a triangulation with no strong Grünbaum coloring, how many additional colors are needed for both facial and non-contractible triangles to be assigned three colors each?

Intuitively we expect that sparser non-contractible 3-cycles will indicate fewer additional colors are needed and denser or overlapping non-contractible 3-cycles will indicate more additional colors are required. For example, consider a triangulation that has only a few non-contractible 3-cycles that are far apart. At worst, each non-contractible 3-cycle could be forced to be monochromatic in any Grünbaum coloring. In this case, at most 2 additional colors will be needed overall, for a total of 5 colors. On the other hand, if a triangulation has many non-contractible 3-cycles with edges shared pairwise, we expect more colors will be needed. Still, K_6 embedded on $\mathbb{N}_1$ (with many non-contractible 3-cycles) appears to need only 5 colors, so perhaps only a few extra colors often suffices.

Encouraged by such examples, we attempt to find upper bounds on the number of extra colors needed to strong Grünbuam color an embedded triangulation T. To quantify, let $b(T)$ denote the least number of edge colors needed so that every facial triangle and every non-contractible 3-cycle uses exactly three colors.

3.1. A potential hypergraph-based upper bound

While there are no results on edge coloring non-uniform hypergraphs, there is a conjecture that generalizes Vizing's theorem, namely that $\chi'(H) \leq \Delta(H) + 1$. This holds for $\Delta(H) \leq 5$ [9], but is open for $\Delta(H) > 5$. (The generalized Vizing bound is sometimes achieved by hypergraphs not generated from embedded graphs: Consider a hypergraph with vertices a, b, c, d, e, f, g and edges $abc, ade, afg, bdf, beg, cef, cdg$, which has $\Delta(H) = 6, \chi'(H) = 7$.)

In the case of a single non-contractible 3-cycle, we see that this bound is not useful. Such a triangulation with a Grünbaum coloring but no strong Grünbaum coloring needs at most 2 additional colors for the non-contractible 3-cycle. As noted in Section 2.4, here $\Delta(H) \geq 5$, so the generalized Vizing conjecture gives an upper bound for $b(H)$ of at least 6 in this case. Examining Figure 3, we see that with non-contractible 3-cycles that share edges, we may still only need 2 additional colors, but the associated hypergraph has $\Delta(H) = 10$, for a conjectured hypergraph Vizing upper bound of 11. More generally, non-contractible 3-cycles that are not disjoint drive up $\Delta(H)$ and thereby make the use of a hypergraph version of the Vizing bound unlikely.

Suppose instead that results on proper edge coloring of non-uniform hypergraphs existed. To use such results would require encoding a given embedding as a hypergraph. In turn, this means identifying every non-contractible 3-cycle, which is exactly why generating precise results—or even computing examples!—is so difficult.

3.2. A chromatic-number-based upper bound

Consider a complete graph: in this case, every three edges form a 3-cycle and so a strong Grünbaum coloring is also a proper edge coloring. Thus, $b(K_n) = n-1$ if n is even and $b(K_n) = n$ if n is odd. We can use this to find an upper bound for $b(T)$ in terms of the chromatic number of T, but in fact this holds for graphs in general, where $b(G)$ denotes the least number of edge colors needed so that every triangle in a proper edge coloring receives three colors:

Theorem 3.1 (by Mike Albertson)**.** *For any simple graph G, $b(G) \leq b(K_{\chi(G)}) = \chi(G)$ if $\chi(G)$ is odd and $= \chi(G) - 1$ if $\chi(G)$ is even.*

Proof. We properly vertex color G using $\chi(G)$ colors, and consider an edge coloring of $K_{\chi(G)}$. Given an edge e of G with vertices v_1, v_2 that have colors c_1, c_2, we assign e the same color as the edge of $K_{\chi(G)}$ with incident vertices of colors c_1, c_2. Then any triangle of G will have three different edge colors. □

3.3. Comparing/contrasting the two bounds

The conjectured hypergraph version of Vizing's theorem depends on the degree of the hypergraph, which in turn depends on the structure of the non-contractible 3-cycles. The chromatic number of a triangulation is bounded above by the Heawood number of the embedding surface. These parameters are completely unrelated, so we expect one bound to be better for some embeddings, and the other to be better for other embeddings. For example, a graph with large maximum degree will likely have a lower Heawood number. In any event, neither of these two approaches appears to be helpful for any large class of cases.

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