



Relaxing the injectivity constraint on arithmetic and harmonious labelings

Christian Barrientos^a, Maged Z. Youssef^{fb}

^aDepartment of Mathematics, Valencia College, Orlando, FL 32832, U.S.A.

^bDepartment of Mathematics and Statistics, College of Sciences, Al Imam Mohammad Ibn Saud Islamic University, Riyadh 11623, Saudi Arabia, and

Department of Mathematics, Faculty of Sciences, Ain Shams University, Cairo 11566, Egypt

chr_barrientos@yahoo.com, mzyoussef11566@yahoo.com

Abstract

Several of the most studied graph labelings are injective functions, this constraint precludes some graphs from admitting such labelings; a well-known example is given by the family of trees that cannot be harmoniously labeled. In order to study the existence of these labelings for certain graphs, the injectivity constraint is often dropped. In this work we eliminate this condition for two different, but related, additive vertex labelings such as the harmonious and arithmetic labelings. The new labelings are called semi harmonious and semi arithmetic. We consider some families of graphs that do not admit the injective versions of these labelings, among the graphs considered here we have cycles and other cycle-related graphs, including the analysis of some operations like the Cartesian product and the vertex or edge amalgamation; in addition, we prove that all trees admit a semi harmonious labeling. Something similar is done with the concept of arithmetic labeling, studying finite unions of semi arithmetic graphs together with some general results.

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1. Introduction

In this work we follow the notation and terminology used in [6] and [8]. The set of integers is denoted by \mathbb{Z} while the set of nonnegative integers is denoted by \mathbb{N} and the additive group of integers (mod n) by \mathbb{Z}_n . The arithmetic sequence of length q with first element a and difference d is denoted by $[a, a + (q - 1)d]_d$, except when $d = 1$, where we do not use the subindex. By a (p, q) -graph we understand a graph with p vertices and q edges, i.e., a graph of order p and size q .

An *additive vertex labeling* of a (p, q) -graph G is a function $f : V(G) \rightarrow S$, where S is a set of nonnegative integers, that induces for each edge uv of G a *weight* defined as $f(u) + f(v)$. In this work, the word labeling is used in the context of an additive vertex labeling. In [10], Graham and Sloane introduced the concept of harmonious labeling to study modular versions of additive bases problems stemming from error-correcting codes; these labelings are one of the most study additive vertex labelings. Let G be a (p, q) -graph such that $q \geq p$ and f be a labeling of G , we say that f and G are *harmonious* if f is an injective function with codomain \mathbb{Z}_q , and the set of induced weights is \mathbb{Z}_q , after each weight has been reduced (mod q). Let w_1, w_2, \dots, w_q be consecutive integers, then $\{w_1, w_2, \dots, w_q\} \equiv \mathbb{Z}_q$ modulo q . Therefore, assuming that $f : V(G) \rightarrow \mathbb{Z}_q$ is injective and the induced weights form a set of q consecutive integers, then f is harmonious. Trees constitute an important family of graphs that fail the condition $q \geq p$. This type of graph was also considered in [10]; there, Graham and Sloane dropped the injectivity condition allowing one label to be used twice. In this work we allow multiple repetitions of multiple labels. The following result was proved in [10], we included it here because it is used in some of the results in Section 2.

Theorem 1.1. *If G is a harmonious graph of even size q where the degree of every vertex is divisible by 2^k , for some positive integer k , then q is divisible by 2^{k+1} .*

The following result, due to Youssef [15], proves the existence of a harmonious labeling for graphs obtained with multiple copies of the same harmonious graph. By nG we understand the disjoint union of n copies of a graph G and by $G^{(n)}$, the one-point union of n copies of G .

Theorem 1.2. *If G is a harmonious graph, then both nG and $G^{(n)}$ are harmonious provided that n is a positive odd integer.*

Some variations of the concept of harmonious labeling have been studied by several authors. Lee et al. [12] generalized the concept of harmonious graph in the following terms: a graph G with q edges is called *felicitous* if there exists an injective function $f : V(G) \rightarrow \mathbb{Z}_{q+1}$ such that the set of induced weights is \mathbb{Z}_q , when each weight is reduced (mod q). Among other results, they proved that the cycle C_n is felicitous if and only if $n \not\equiv 2 \pmod{4}$. Chang et al. [5] said that a graph G with q edges is *strongly k -harmonious* if there exists an injective function $f : V(G) \rightarrow \mathbb{Z}_q$ such that the set of induced weights is $[k, k + q - 1]$. A little less restrictive is the concept of (k, d) -arithmetic graph introduced by Acharya and Hegde [1]; a graph G with q edges is called (k, d) -*arithmetic* if there exists an injective function $f : V(G) \rightarrow \mathbb{N}$ such that the set of induced weights is $[k, k + (q - 1)d]_d$

for two positive integers k and d . Lourdsamy and Seenivasan [13] said that a graph G with q edges is *vertex equitable* if there exists a function $f : V(G) \rightarrow [0, \lceil \frac{q}{2} \rceil]$ such that the set of induced weights is $[1, q]$ and for every pair of elements i and j in the range of f , the number of vertices labeled i and the number of vertices labeled j differ by at most one unit. They characterized the cycles that are vertex equitable.

In this work we generalize the concepts of harmonious and (k, d) -arithmetic graphs by relaxing the injectivity constraint of the corresponding labeling. We refer to the new versions of these labelings as semi harmonious and semi (k, d) -arithmetic. In Section 2 we show the existence of a semi harmonious labeling for several types of cycle-related graphs; we characterize the cycles that admit such a labeling; using semi harmonious cycles we study the finite union, the one-point union, the edge amalgamation, and the Cartesian product with the path P_m . We conclude that section proving that the complete bipartite graph is semi harmonious as well. We continue in Section 3 proving that all trees are semi harmonious. In Section 4 we study semi (k, d) -arithmetic labelings, we prove that if G is semi (k, d) -arithmetic, then it is also semi (rk, rd) -arithmetic for every $r \geq 1$ and that nG is both semi (k, d) -arithmetic and semi $(k + d, d)$ -arithmetic; we also study the complete bipartite graph proving that any graph which components are complete bipartite graphs is semi (k, d) -arithmetic for any ordered pair (k, d) of positive integers; in the last result of this section we prove that if G_i is a semi (k_i, d) -arithmetic graph of size q_i for each $i = 1, 2$, then $G_1 \cup G_2$ is semi (k_1, d) -arithmetic.

2. Semi Harmonious Labelings

A (p, q) -graph G is said to be *semi harmonious* if there exists a labeling $f : V(G) \rightarrow \mathbb{Z}_q$ such that $\{(f(u) + f(v)) \pmod{q} : uv \in E(G)\} = \mathbb{Z}_q$. Every labeling f that satisfies this definition is called *semi harmonious*.

Suppose that f is a semi harmonious labeling of a (p, q) -graph G . Let c be a positive constant, we said that the labeling g of G is a *shifting* of f in c units if $g(u) = f(u) + c$, for every $u \in V(G)$. It is important to note that if uv is the edge of G of weight m , then $g(u) + g(v) = m + 2c$; in other terms, the set of edge labels induced by g consists of q consecutive integers.

Note that the necessary condition for the existence of a harmonious labeling for a graph of even size, given in Theorem 1.1, is still valid for semi harmonious labelings. In addition we have that for every $n \geq 3$, the complete graph K_n is semi harmonious if and only if K_n is harmonious. While the cycle C_n is harmonious if and only if n is odd, we can prove that the cycle C_n is semi harmonious except when $n \equiv 2 \pmod{4}$. The structure of the proof is similar to the proofs in [7] and [13] for the corresponding labelings.

Theorem 2.1. *The cycle C_n is semi harmonious if and only if $n \not\equiv 2 \pmod{4}$.*

Proof. The necessity follows from Theorem 1.1. To prove the sufficiency, we analyze two cases, in either case we assume that $V(G) = \{u_1, u_2, \dots, u_n\}$ and $E(G) = \{u_1u_2, u_2u_3, \dots, u_nu_1\}$.

Case 1. When $n \equiv 0$ or $3 \pmod{4}$. Consider the labeling $f : V(G) \rightarrow \mathbb{Z}_n$ defined for every vertex u_i of C_n as follows:

$$f(u_i) = \begin{cases} \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \leq \frac{n}{2} \text{ is odd,} \\ \frac{i+1}{2}, & \text{if } i > \frac{n}{2} \text{ is odd.} \end{cases}$$

Observe that the range of f is $[0, \frac{n}{2}]$ when n is even and $[0, \frac{n+1}{2}]$ when n is odd. Now we want to determine the weight of the edge $u_i u_{i+1}$, that is, $f(u_i) + f(u_{i+1})$. Let $i < \lfloor \frac{n}{2} \rfloor$; if i is even, then $f(u_i) + f(u_{i+1}) = \frac{i}{2} + \frac{i}{2} = i$; if i is odd, then $f(u_i) + f(u_{i+1}) = \frac{i-1}{2} + \frac{i+1}{2} = i$. Therefore, we have $\lfloor \frac{n-1}{2} \rfloor$ edges which weights are $1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$. Suppose now that $\lfloor \frac{n}{2} \rfloor \leq i < n$; if i is even, then $f(u_i) + f(u_{i+1}) = \frac{i}{2} + \frac{i+2}{2} = i + 1$; if i is odd, then $f(u_i) + f(u_{i+1}) = \frac{i+1}{2} + \frac{i+1}{2} = i + 1$. Thus, we have $\lfloor \frac{n}{2} \rfloor$ edges which weights are $\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \dots, n$. Since the edge $v_n v_1$ has weight $f(u_n) + f(u_1) = \lfloor \frac{n}{2} \rfloor + 0 = \lfloor \frac{n}{2} \rfloor$, the set of weights induced by f is $\{1, 2, \dots, n\}$. Consequently, f is semi harmonious.

Case 2. When $n \equiv 1 \pmod{4}$. In this case the labeling f is defined as:

$$f(u_i) = \begin{cases} \frac{i-2}{2}, & \text{if } i \text{ is even,} \\ \frac{i-1}{2}, & \text{if } i \leq \frac{n+1}{2} \text{ is odd,} \\ \frac{i+1}{2}, & \text{if } i \geq \frac{n+3}{2} \text{ is odd.} \end{cases}$$

Now, the range of f is $[0, \frac{n+1}{2}]$. Suppose that $1 \leq i \leq \frac{n+1}{2}$; if i is odd, then $f(u_i) + f(u_{i+1}) = \frac{i-1}{2} + \frac{i+1-2}{2} = i - 1$; if i is even, then $f(u_i) + f(u_{i+1}) = \frac{i-2}{2} + \frac{i+1-1}{2} = i - 1$. Thus, we get $\frac{n+1}{2}$ edges which weights are $0, 1, \dots, \frac{n-1}{2}$. Assuming that $\frac{n+3}{2} \leq i \leq n - 1$, the weight of $u_i u_{i+1}$ is $f(u_i) + f(u_{i+1}) = \frac{i-2}{2} + \frac{i+2}{2} = i$ when i is even and $f(u_i) + f(u_{i+1}) = \frac{i+1}{2} + \frac{i+1-2}{2} = i$ when i is odd. Thus, we get $\frac{n-3}{2}$ edges which weights are $\frac{n+3}{2}, \frac{n+5}{2}, \dots, n - 1$. The edge $u_n u_1$ has weight $\frac{n+1}{2} + 0 = \frac{n+1}{2}$. Hence, the set of edge labels is \mathbb{Z}_n . Therefore, f is a semi harmonious labeling of C_n . \square

We show in Figure 1 three examples of the semi harmonious labeling of C_n described within the proof of Theorem 2.1 corresponding to the congruences of $n \pmod{4}$.

Theorem 2.2. Suppose that for each $i \in \{1, 2, \dots, s\}$, the cycle C_{n_i} is semi harmonious. For each $k \in \{0, 1, 3\}$, we define the super set $S_k = \{C_{n_i} : n_i \equiv k \pmod{4}\}$, where If $|S_3| - |S_1|$ is either 0 or 1, then the 2-regular graph $G = \cup_{i=1}^s C_{n_i}$ is semi harmonious.

Proof. If G is connected, then G is semi harmonious as was proven in Theorem 2.1. Assume that G is disconnected; we arrange the components of G in such a way that for each $1 \leq j \leq r$, $n_j \equiv 0 \pmod{4}$, and for each $r + 1 \leq j < s$, whenever $n_j \equiv 3 \pmod{4}$ then $n_{j+1} \equiv 1 \pmod{4}$. Thus, if $|S_3| = |S_1|$, then the size of G is divisible by 4 and when $|S_3| = |S_1| + 1$, the size of G is congruent to $3 \pmod{4}$ and $n_s \equiv 3 \pmod{4}$. Each component of G is initially labeled using the corresponding semi harmonious labeling in Theorem 2.1. Most of these initial labelings will be shifted; thus, the shifted labeling will be the final labeling of that component.

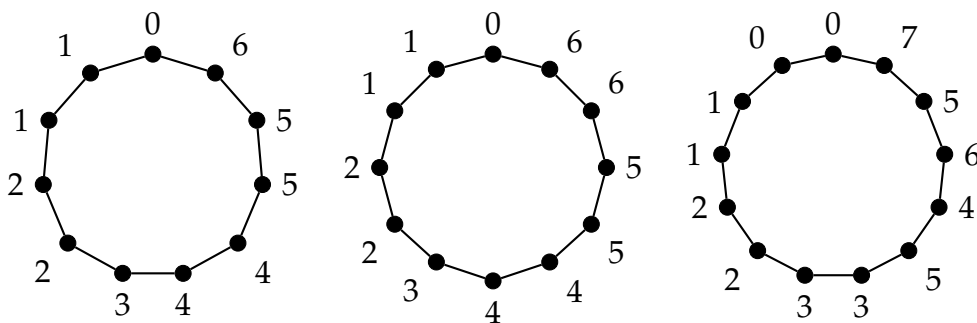


Figure 1. Semi harmonious labeling of C_{11} , C_{12} , and C_{13}

Suppose that $1 \leq j \leq r$; when $j = 1$, the final labeling of C_{n_1} is its initial labeling; for $j > 1$, the final labeling of C_{n_j} is obtained by shifting its initial labeling $\frac{1}{2} \sum_{i=1}^{j-1} n_i$ units. Since $n_j \equiv 0 \pmod{4}$ for each $1 \leq j \leq r$, the initial labeling of C_{n_j} induces the weights $1, 2, \dots, n_j$; therefore, its final labeling induces the weights $1 + \sum_{i=1}^{j-1} n_i, 2 + \sum_{i=1}^{j-1} n_i, \dots, n_j + \sum_{i=1}^{j-1} n_i = \sum_{i=1}^j n_i$. Thus, the weights on the edges of $\cup_{i=1}^r C_{n_i}$ are $1, 2, \dots, \sum_{i=1}^r n_i$. Note that $\sum_{i=1}^r n_i$ is an even number.

Suppose now that $j > r$; the final labeling of C_{n_j} is obtained in essentially the same way, but the shifting applied to the initial labeling of C_{n_j} depends of the parity of $\sum_{i=1}^{j-1} n_i$. We start with $j = r + 1$, since $\sum_{i=1}^r n_i$ is an even number and the initial labeling of $C_{n_{r+1}}$ induces the weights $1, 2, \dots, n_{r+1}$, where n_{r+1} is odd, shifting it $\frac{1}{2} \sum_{i=1}^r n_i$ units, induces the weights $1, 2, \dots, \sum_{i=1}^{r+1} n_i$ on the edges of $\cup_{i=1}^{r+1} C_{n_i}$, but now $\sum_{i=1}^{r+1} n_i$ is odd, however the initial labeling of $C_{n_{r+2}}$ induces the weights $0, 1, \dots, n_{r+2} - 1$; consequently, if it is shifted $\frac{1}{2} (1 + \sum_{i=1}^{r+1} n_i)$ units, the induced weights on the edges of $\cup_{i=1}^{r+2} C_{n_i}$ are $1, 2, \dots, \sum_{i=1}^{r+2} n_i$, where $\sum_{i=1}^{r+2} n_i$ is an even number. We continue with this process until each component has its final labeling. The process guarantees that the induced weights on the edges of G are $1, 2, \dots, \sum_{i=1}^s n_i$. In other terms, the final labeling of G is semi harmonious. \square

In Figure 2 we show an example of this labeling where $G = 3C_4 \cup C_8 \cup C_7 \cup C_5$.

If F and G are connected graphs, a *one-point union* of them is any of the graphs obtained by amalgamating a vertex of F with a vertex of G ; when F and G are vertex transitive, the outcome of the one-point union is unique. An immediate consequence of Theorem 2.1 is that the one-point union of C_m with C_n is semi harmonious when both cycles are semi harmonious and $m + n \not\equiv 2 \pmod{4}$. We prove this claim in the following corollary.

Corollary 2.1. *Let C_m and C_n be semi harmonious. If $m + n \not\equiv 2 \pmod{4}$, then the one-point union of C_m with C_n is a semi harmonious graph.*

Proof. Suppose that the cycles C_m and C_n are semi harmonious and that $m + n \not\equiv 2 \pmod{4}$. We assume that both cycles are labeled using the function f described in Theorem 2.1. We

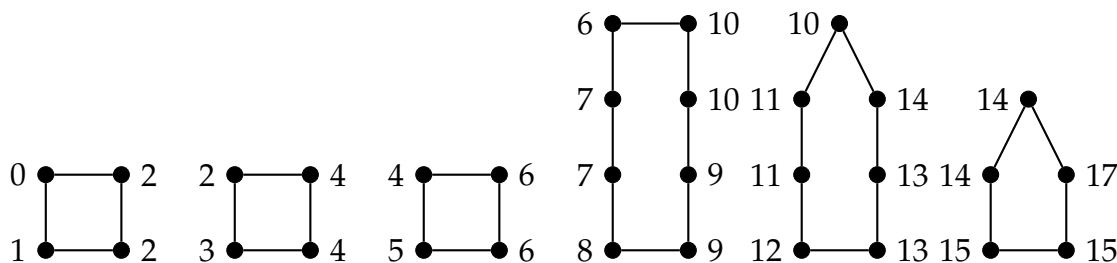


Figure 2. Semi harmonious labeling of a 2-regular graph with six components

analyze three cases determined by the equivalence class of $m + n \pmod{4}$.

Case 1. When $m + n \equiv 0 \pmod{4}$. In this case, either $m, n \equiv 0 \pmod{4}$ or $m \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{4}$. Suppose first that $m, n \equiv 0 \pmod{4}$, then the largest label used on C_m is $\frac{m}{2}$; the initial labeling of C_n is shifted $\frac{m}{2}$ units, thus the labels on this cycle are in the range $[\frac{m}{2}, \frac{m+n}{2}]$ and the induced weights are in $[m + 1, m + n]$. Since both cycles have a vertex labeled $\frac{m}{2}$, we amalgamate them to produce the one-point union which is semi harmonious. Suppose now that $m \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{4}$; the labeling of C_n is shifted $\frac{m+1}{2}$ units, the labels on C_n are in the range $[\frac{m+1}{2}, \frac{m+n+2}{2}]$ and the induced weights are in $[m + 1, m + n]$. Amalgamating the vertices labeled $\frac{m+1}{2}$ we get the one-point union of these cycles together with a semi harmonious labeling.

Case 2. When $m \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$. As we did before, the labeling of C_n is shifted $\frac{m}{2}$ units; in this way the labels on C_n are in the range $[\frac{m}{2}, \frac{m+n+1}{2}]$ and the induced weights are in $[m + 1, m + n]$. We proceed with the amalgamation of a vertex of C_m labeled $\frac{m}{2}$ with the vertex of C_n with this label. The resulting graph is semi harmonious.

Case 3. When $m \equiv 1 \pmod{4}$ and $n \equiv 0 \pmod{4}$. The largest label used on C_m is $\frac{m+1}{2}$, so the labeling of C_n is shifted $\frac{m+1}{2} - 1 = \frac{m-1}{2}$ units, then the labels on C_n are in the range $[\frac{m-1}{2}, \frac{m+n-1}{2}]$ and the induced weights are in $[m, m + n - 1]$. Since the weights on the edges of C_m are in $[0, m - 1]$, the amalgamation of these labeled cycles, using the vertex labeled $\frac{m+1}{2}$ in C_m with one of the vertices labeled $\frac{m-1}{2}$ in C_n , results in the one-point union with the appropriate semi harmonious labeling. \square

The labeling of C_n given in Theorem 2.1 can be also used to produce a semi harmonious labeling of a subfamily of outerplanar graphs. This subfamily is obtained via edge amalgamation of two cycles, say C_m and C_n , where both m and n are odd or m is odd and $n \equiv 0 \pmod{4}$. The edge amalgamation of C_m and C_n is designated with the symbol $C_m \square C_n$. In order to present a broader spectrum of semi harmonious graphs that can be built with the labelings in Theorem 2.1, we need to introduce another semi harmonious labeling for the case where $m \equiv 1 \pmod{4}$. We present this labeling in the following lemma; since the proof is basically the same that the proof of Theorem 2.1 we omit some details but include the most significant steps.

Lemma 2.1. *The cycle C_m is semi harmonious when $m \equiv 1(\text{mod } 4)$.*

Proof. Consider the following labeling of the vertices of C_m :

$$f(u_i) = \begin{cases} i - 1, & \text{if } 1 \leq i \leq \frac{m+1}{2} \text{ is odd,} \\ i - 2, & \text{if } 2 \leq i \leq \frac{m-1}{2} \text{ is even,} \\ m + 1 - i, & \text{if } i \geq \frac{m+3}{2}. \end{cases}$$

The range of f is $[0, \frac{m-1}{2}]$. If $2 \leq i \leq \frac{m+1}{2}$, the edge $u_i u_{i-1}$ has weight $2i - 4$ independently of the parity of i . If $\frac{m+3}{2} \leq i \leq m$, the edge $u_i u_{i+1}$ (understanding that $m + 1 \equiv 1(\text{mod } n)$) has weight $2(m - i) + 1$. The edge $u_{\frac{m+1}{2}} u_{\frac{m+3}{2}}$ has weight $m - 1$. Therefore, the set of induced weights is $[0, m - 1]$; consequently, f is a semi harmonious labeling of C_m . \square

Theorem 2.3. *The edge amalgamation $C_m \sqcap C_n$ is a semi harmonious graph when any of the following conditions holds:*

- (1) $m \equiv 3(\text{mod } 4)$ and $n \equiv 0(\text{mod } 4)$ or $n \equiv 3(\text{mod } 4)$,
- (2) $m \equiv 1(\text{mod } 4)$ and $n \equiv 0(\text{mod } 4)$ or $n \equiv 1(\text{mod } 4)$.

Proof. Suppose that $m \equiv 3(\text{mod } 4)$ and $n \equiv 0(\text{mod } 4)$ or $n \equiv 3(\text{mod } 4)$. We use on both cycles the labeling described in Theorem 2.1, but shifting $\frac{m-1}{2}$ units the labeling of C_n ; in this way both cycles have an edge with end-vertices labeled $\frac{m-1}{2}$ and $\frac{m+1}{2}$. The labels on C_m are in the range $[0, \frac{m+1}{2}]$ and the induced weights are in $[1, m]$; the labels on C_n are in the range $[\frac{m-1}{2}, \frac{m+n-1}{2}]$ when n is even or $[\frac{m-1}{2}, \frac{m+n}{2}]$ when n is odd; the weights on the edges of C_n are in $[m, m + n - 1]$. Amalgamating the edges of weight m we obtain $C_m \sqcap C_n$ with a semi harmonious labeling.

Suppose now that $m \equiv 1(\text{mod } 4)$ and $n \equiv 0(\text{mod } 4)$ or $n \equiv 1(\text{mod } 4)$. If m and n are both odd, the initial labeling of C_m and C_n is the labeling in Lemma 2.1; we shift $\frac{n-1}{2}$ units the labeling of C_m , in this way the labels on C_m are in the range $[\frac{n-1}{2}, \frac{m+n-2}{2}]$ and the induced weights are in $[n - 1, m + n - 2]$. The vertices u_1 and u_2 of C_m are now labeled $\frac{n-1}{2}$. Since the labeling of C_n has not been modified, the vertices $u_{\frac{n+1}{2}}$ and $u_{\frac{n+3}{2}}$ are labeled $\frac{n-1}{2}$. Then, both cycles have an edge of weight $n - 1$ which end-vertices are labeled $\frac{n-1}{2}$. Amalgamating these edges we obtain $C_m \sqcap C_n$ with a semi harmonious labeling. If m and n have different parity, the labeling of C_m is coming from Lemma 2.1 and the labeling of C_n from Theorem 2.1; we shift $\frac{n}{2}$ units the labeling of C_m , in this way the labels on C_m are in the range $[\frac{n}{2}, \frac{m+n-1}{2}]$ and the induced weights are in $[n, m + n - 1]$. The vertices u_1 and u_2 of C_m are labeled $\frac{n}{2}$, this is also the label on the vertices u_{n-1} and u_n of C_n . Thus, both cycles have an edge of weight n which end-vertices are labeled $\frac{n}{2}$. The amalgamation of these edges produces $C_m \sqcap C_n$ with a semi harmonious labeling. \square

In the next result we work with the Cartesian product of any of the semi harmonious cycles and a path, proving that $C_n \times P_2$ is semi harmonious except when $n \equiv 2(\text{mod } 4)$.

Lemma 2.2. *If C_n is semi harmonious, then the Cartesian product of C_n and P_2 is a semi harmonious graph.*

Proof. When $n \equiv 0, 3 \pmod{4}$, we use on C_n the labeling described in Theorem 2.1; when $n \equiv 1 \pmod{4}$, we use on C_n the labeling described in Lemma 2.1. Since $C_n \times P_2$ contains two copies of C_n , we label both copies with its corresponding semi harmonious labeling and shift the labels on the second copy n units. When $n \equiv 0, 3 \pmod{4}$, the labels on the first copy of C_n are in $\left[0, \left\lceil \frac{n}{2} \right\rceil\right]$ with induced weights in $[1, n]$ and on the second copy the labels are in $\left[n, n + \left\lceil \frac{n}{2} \right\rceil\right]$ with induced weights in $[2n + 1, 3n]$. When $n \equiv 1 \pmod{4}$, the labels of the first copy are in $\left[0, \frac{n-1}{2}\right]$ and the induced weights in $[0, n - 1]$, the labels of the second copy are in $\left[n, \frac{3n-1}{2}\right]$ and the weights in $[2n, 3n - 1]$.

Independently of the value of n , the Cartesian product $C_n \times P_2$ is obtained by connecting u_i^1 with u_{i+1}^2 , where the sum $i + 1$ is taken mod n .

Case 1. When $n \equiv 0 \pmod{4}$. Suppose that $1 \leq i \leq \frac{n-2}{2}$; if i is odd, then the edge $u_i^1 u_{i+1}^2$ has weight $\frac{i-1}{2} + \frac{i+1}{2} + n = n + i$; if i is even, this edge has weight $\frac{i}{2} + \frac{i+1-1}{2} + n = n + i$. Thus, we have $\frac{n-2}{2}$ edges whose weights are $n + 1, n + 2, \dots, n + \frac{n-2}{2} = \frac{3n-2}{2}$. The edge $u_n^1 u_1^2$ has weight $n + \frac{n}{2} = \frac{3n}{2}$. Suppose now that $\frac{n}{2} \leq i \leq n - 1$; if i is odd, then the edge $u_i^1 u_{i+1}^2$ has weight $\frac{i+1}{2} + \frac{i+1}{2} + n = n + i + 1$; if i is even, this edge has weight $\frac{i}{2} + \frac{i+1+1}{2} + n = n + i + 1$. Thus, we have $\frac{n}{2}$ edges which weights are $\frac{3n}{2} + 1, \frac{3n}{2} + 2, \dots, 2n$. Therefore, the weights of the edges of $C_n \times P_2$ are the integers in $[1, 3n]$; i.e., this graph is semi harmonious when $n \equiv 0 \pmod{4}$.

Case 2. When $n \equiv 3 \pmod{4}$. Suppose that $1 \leq i \leq \frac{n-1}{2}$; if i is odd, then the edge $u_i^1 u_{i+1}^2$ has weight $\frac{i-1}{2} + \frac{i+1}{2} + n = n + i$; if i is even, this edge has weight $\frac{i}{2} + \frac{i+1-1}{2} + n = n + i$. Thus, we have $\frac{n-1}{2}$ edges whose weights are $n + 1, n + 2, \dots, \frac{3n-1}{2}$. The edge $u_n^1 u_1^2$ has weight $\frac{n+1}{2} + n = \frac{3n+1}{2}$. Suppose now that $\frac{n+1}{2} \leq i \leq n - 1$; if i is odd, then the edge $u_i^1 u_{i+1}^2$ has weight $\frac{i+1}{2} + \frac{i+1}{2} + n = n + i + 1$; if i is even, this edge has weight $\frac{i}{2} + \frac{i+1+1}{2} + n = n + i + 1$. Thus, we have $\frac{n-1}{2}$ edges which weights are $\frac{3n+1}{2} + 1, \frac{3n+1}{2} + 2, \dots, 2n$. Therefore, the weights of the edges of $C_n \times P_2$ are the integers in $[1, 3n]$; i.e., this graph is semi harmonious when $n \equiv 3 \pmod{4}$.

Case 3. When $n \equiv 1 \pmod{4}$. Suppose that $1 \leq i \leq \frac{n+1}{2}$; if i is odd, then the edge $u_i^1 u_{i+1}^2$ has weight $i - 1 + i + 1 - 2 + n = n + 2i - 2$; if i is even, this edge has weight $i_2 + i + 1 - 1 + n = n + 2i - 2$. Thus, we have $\frac{n+1}{2}$ edges which weights are $n, n + 2, \dots, 2n - 1$. Suppose now that $\frac{n+3}{2} \leq i \leq n$ the edge $u_i^1 u_{i+1}^2$ has weight $3n - 2i + 1$. Thus, we have $\frac{n-1}{2}$ edges whose weights are $n + 1, n + 3, \dots, 2n - 2$. Therefore, the weights of the edges of $C_n \times P_2$ are the integers in $[0, 3n - 1]$; which implies that this graph is semi harmonious when $n \equiv 1 \pmod{4}$. \square

With this result we are ready to prove that $C_n \times P_m$ is semi harmonious for each $m \geq 1$ and every $n \not\equiv 2 \pmod{4}$. In order to simplify the proof of this result we must observe first that Lemma 2.2 is still valid if the vertex u_i^1 is connected with u_{i-1}^2 , where the difference $i - 1$ is taken (mod n).

Theorem 2.4. *If C_n is semi harmonious, then the Cartesian product of C_n and P_m is a semi harmonious graph for every $m \geq 1$.*

Proof. The cases where $m = 1, 2$ were considered in Theorem 2.1 or Lemma 2.1, and Lemma 2.2, respectively; so, we are assuming that $m \geq 3$. Let R_1, R_2, \dots, R_m be the copies of C_n in $C_n \times P_m$. Each of these copies is initially labeled following the criteria described in Lemma 2.2. For $i \geq 2$, the labeling on R_i is shifted $n(i - 1)$ units. Thus, the labels on R_j and R_{j+1} are n units apart as in Lemma 2.2. We proceed to connect R_j and R_{j+1} in the following form: if j is odd the vertex u_i^j of R_j is connected with the vertex u_{i+1}^{j+1} of R_{j+1} , if j is even the vertex u_i^j of R_j is connected with the vertex u_{i-1}^{j+1} of R_{j+1} , where the addition $i + 1$ and the difference $i - 1$ are taken (mod n). By Lemma 2.2 we know that the edges connecting R_j and R_{j+1} have weights with the suitable values to complement the weights on the copies of C_n , therefore, $C_n \times P_m$ is a semi harmonious graph. \square

In Figure 3 we show an example of this semi harmonious labeling for the case $C_9 \times P_3$.

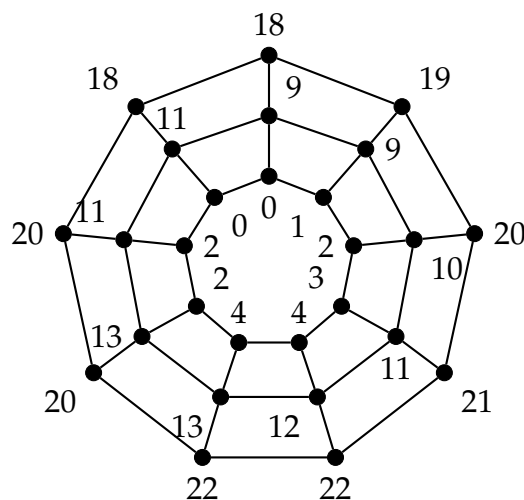


Figure 3. Semi harmonious labeling of $C_9 \times P_3$

Now we turn our attention to the complete bipartite graph $K_{m,n}$. Let A and B be the stable sets of $K_{m,n}$, Graham and Sloane [10] proved that this graph is harmonious if and only if at least one of the stable sets is a singleton. In the next result we show that $K_{m,n}$ is semi harmonious for all values of m and n . But first, we note that if f is semi harmonious labeling of $K_{m,n}$, then f restricted to A or B must be injective, otherwise, we obtain at least two edges with the same weight. This observation can be extended to any complete multipartite graph. Furthermore, for the same reason, i.e., duplication of weights, there are no two labels used on both stable sets or in any two stable sets of a multipartite graph.

Theorem 2.5. *The complete bipartite graph $K_{m,n}$ is semi harmonious.*

Proof. Assume that A and B are the stable sets of $K_{m,n}$, where $A = \{u_1, u_2, \dots, u_m\}$ and $B = \{v_1, v_2, \dots, v_n\}$. Consider the labeling f defined on the elements of A as $f(u_i) = i - 1$ and on the elements of B as $f(v_j) = m(j - 1) + 1$. Note that the range of f is $[0, mn - m + 1]$, and that f restricted to either A or B is an injective function. For a fixed value of j , the weights of the edges $u_i v_j$ form the interval $W_j = [m(j - 1) + 1, mj]$. Since $\max W_{j-1} = m(j - 1) < \min W_j = m(j - 1) + 1$ and $\cup_{j=1}^n W_j = [1, mn]$, we have that f is indeed a semi harmonious labeling of $K_{m,n}$. \square

3. All Trees are Semi Harmonious

In this section we prove that all trees are semi harmonious. This result is a consequence of other known results in the area of difference vertex labelings, these labelings are out of the scope of this work and are not discussed here. In order to keep the result self-contained, we describe the labeling of a tree without using these existing results.

Theorem 3.1. *All trees are semi harmonious.*

Proof. Suppose that T is a tree of size q with stable sets A and B such that $|A| = a$ and $|B| = b$. When T is represented as a rooted tree, with root $v_1^0 \in A$, its vertices are distributed in $h + 1$ levels; we denote these levels by L_0, L_1, \dots, L_h , being the root the only vertex on level L_0 . Let $v_1^i, v_2^i, \dots, v_{h_i}^i$ be the vertices of T on L_i , which are organized from left to right, that is, v_j^i is placed to the left of v_{j+1}^i for each $i \geq 1$.

Let f be the labeling of T that satisfies the following conditions, where v_x^{i-1} and v_y^{i-1} are the parents of v_j^i and v_{j+1}^i , respectively.

- (1) $f(v_1^0) = 0$,
- (2) $f(v_1^1) = a$,
- (3) $f(v_j^i) \leq f(v_{j+1}^i)$ for every $1 \leq i \leq h$ and every $1 \leq j \leq h_i$,
- (4) if $f(v_x^{i-1}) = f(v_y^{i-1})$, then $f(v_{j+1}^i) = f(v_j^i) + 1$,
- (5) if $f(v_y^{i-1}) = f(v_x^{i-1}) + 1$, then $f(v_{j+1}^i) = f(v_j^i)$,
- (6) if $f(v_{h_{i-2}}^{i-2}) + f(v_{h_{i-1}}^{i-1}) = w$, then $f(v_1^{i-1}) + f(v_1^i) = w + 1$.

We claim that f is a semi harmonious labeling of T . In order to prove this claim, we just need to show that the set of induced weights is $\{a, a + 1, \dots, a + q - 1\}$, i.e., a set of q consecutive integers.

Condition (3) tells us that for every $0 \leq i \leq h$ the sequence $f(v_1^i), f(v_2^i), \dots, f(v_{h_i}^i)$ is non-decreasing.

Note that condition (4) implies two different, but related things. If v_j^i and v_{j+1}^i are siblings, that is, if $v_x^{i-1} = v_y^{i-1}$, then the edges $v_x^{i-1} v_j^i$ and $v_x^{i-1} v_{j+1}^i$ have weights that are

consecutive integers because $f(v_{j+1}^i) = f(v_j^i) + 1$. The same result is obtained when v_j^i and v_{j+1}^i are not siblings because $f(v_x^{i-1}) = f(v_y^{i-1})$ independently of the fact that $v_x^{i-1} \neq v_y^{i-1}$.

Condition (5) guarantees that in the event that v_j^i and v_{j+1}^i are not siblings but $f(v_y^{i-1}) = f(v_x^{i-1}) + 1$, the weights of the edges $v_x^{i-1}v_j^i$ and $v_x^{i-1}v_{j+1}^i$ are consecutive integers because v_j^i and v_{j+1}^i have the same label.

As a conclusion of (3), (4), and (5) we get that the sequence formed by the weights of the edges connecting the vertices of L_{i-1} with the vertices of L_i is an arithmetic sequence of difference 1. Condition (6) says that the largest weight on an edge between vertices of L_{i-2} and L_{i-1} is exactly one unit less than the smallest weight of an edge between vertices of L_{i-1} and L_i . Consequently, the weights induced on the edges of T form a set of exactly q consecutive integers. Since $f(v_1^0) = 0$ and $f(v_1^1) = a$, the smallest of these weights is a . \square

This labeling is a modification of the labeling used in [3] to prove that every tree of size q is a spanning tree of an α -graph of size $q + \epsilon(T)$, where $\epsilon(T)$ is the excess of T (this parameter was originally introduced in [2], but its analysis is out of the scope of this work). In [11] Jungreis and Reid presented a method to transform an α -labeling into a sequential labeling, which can be transformed into a harmonious labeling. The semi harmonious labeling of T is obtained combining the results in [3] and [11].

In Figure 4 we show the semi harmonious labeling obtained with the procedure described in Theorem 3.1 for a tree of size $q = 31$ with stable sets of cardinality $a = b = 16$.

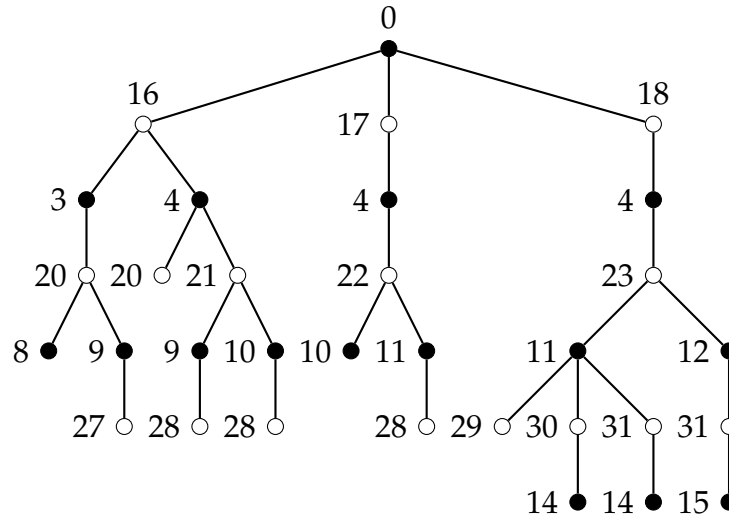


Figure 4. Semi harmonious labeling of a tree of size 31

4. Semi Arithmetic Graphs

Let k, d be a pair of positive integers, a (p, q) -graph G is said to be *semi (k, d) -arithmetic* if there exists a labeling $f : V(G) \rightarrow \mathbb{N}$ such that the induced weights are $k, k+d, k+2d, \dots, k+$

$(q - 1)d$, i.e., an arithmetic sequence with first element k and difference d . As before, the labeling f is called *semi (k, d) -arithmetic*.

Note that all the semi harmonious labelings discussed in the previous section can be seen as semi (k, d) -arithmetic labelings where $k = 0$ and $d = 1$.

In [1], Acharya and Hegde said that the injection $f : V(G) \rightarrow \{0, 1, \dots, p - 1\}$ is *strongly (k, d) -indexable* if the set of induced weights is $W = \{k, k + d, \dots, k + d(q - 1)\}$. When $d = 1$, both the labeling and the graph are said to be *k -indexable*, if in addition $k = 1$, they are simply called *strongly indexable*.

Let G be a bipartite graph of order p and size q , such that G is k -indexable where $k \geq 2$. Then, for each $w \in W = \{k, k + 1, \dots, k + q - 1\}$ there exists $uv \in E(G)$ such that $f(u) + f(v) = w$. If A and B are the stable sets of G , we may assume that $u \in A$ and $v \in B$, and that the vertex labeled 0 belongs to A . Consider the labeling g of G defined for each $v \in V(G)$ as

$$g(v) = \begin{cases} f(v), & \text{if } v \in A, \\ f(v) + i - k, & \text{if } v \in B, \end{cases}$$

where $i \geq 1$. Note that $k = \min\{f(v) : v \in B\}$; this implies that $f(v) + i - k \geq i$ for every $v \in B$. Let $uv \in E(G)$ such that $f(u) + f(v) = w$, since G is bipartite u and v are in different stable sets, therefore

$$g(u) + g(v) = f(u) + f(v) + i - k = w + i - k.$$

This implies that the set of weights induced by g is $\{i, i + 1, \dots, i + q - 1\}$. Consequently, g is a semi $(i, 1)$ -arithmetic labeling of G . In this way we have proven the next theorem.

Theorem 4.1. *If G is a bipartite k -indexable graph with $k \geq 2$, then G admits a semi $(i, 1)$ -arithmetic labeling for each $i \geq 1$.*

Germina [9] proved that for each odd value of n , the ladder $L_n = P_n \times P_2$ is k -indexable when $k = \frac{n+1}{2}$. Therefore, we can apply the result of Theorem 4.1 to these graphs. In Figure 5 we use the labeling of L_7 given in [9] to show the semi $(i, 1)$ -arithmetic labeling described before, for each $i \in \{1, 2, 3, 4, 5\}$, being the case $i = 4$ the one given in [9].

Next we present another general property of the semi (k, d) -arithmetic labelings.

Proposition 4.1. *If G is a semi (k, d) -arithmetic graph, then G is a semi (rk, rd) -arithmetic graph for each $r \geq 2$.*

Proof. Let G be a graph of size q and f be a semi (k, d) -arithmetic labeling of G . Thus, the set of weights induced by f is $W = \{k, k + d, \dots, k + d(q - 1)\}$. Then, for every $w \in W$ there exists $uv \in E(G)$ such that $f(u) + f(v) = w$. Let g be the labeling of G defined, for every $v \in V(G)$, by $g(v) = rf(v)$ for some $r \geq 2$. The weight induced by g on the edge uv is:

$$g(u) + g(v) = r(f(u) + f(v)) = rw.$$

Since $w = k + di$, for some $i \in \{0, 1, \dots, q - 1\}$, we have that the new weight of uv is $rk + rdi$. In other terms, the set of weights induced by g is $W' = \{rk, rk + rd, \dots, rk + rd(q - 1)\}$, i.e., these weights form an arithmetic sequence of difference rd and first element rk . Therefore, g is a semi (rk, rd) -arithmetic labeling of G . \square

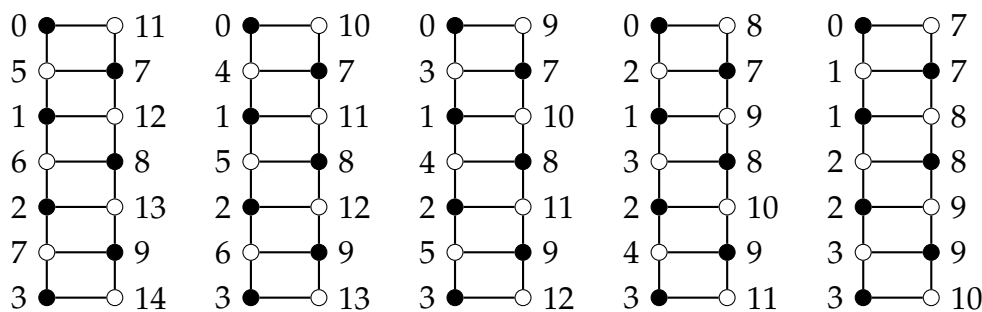


Figure 5. Semi $(i, 1)$ -arithmetic labelings of $L_7 = P_7 \times P_2$ for each $i \in \{1, 2, 3, 4, 5\}$

Recall that the union of n copies of a graph G is denoted by nG . In the following results we work with semi (k, d) -arithmetic labelings of graphs of the form nG .

Theorem 4.2. *Let G be a (p, q) -graph of even size and k, d be a pair of positive integers. If G is semi (k, d) -arithmetic, then for every positive integer n the graph nG is semi (k, d) -arithmetic.*

Proof. Let G be a semi (k, d) -arithmetic graph of order p and even size q with $V(G) = \{u_1, u_2, \dots, u_p\}$. For each $i \in [1, n]$, let G_i be a copy of G , where $V(G_i) = \{u_1^i, u_2^i, \dots, u_p^i\}$ and u_j^i be the replica of u_j . Suppose that f is a semi (k, d) -arithmetic labeling of G . This labeling can be extended to a semi (k, d) -arithmetic labeling g of nG . For each vertex u_j^i of nG , where $j \in [1, p]$ and $i \in [1, n]$, $g(u_j^i) = f(u_j) + \frac{qd(i-1)}{2}$.

Since G is a semi (k, d) -arithmetic graph, the set of weights induced on the edges of G_i is $W_i = [k + qd(i - 1), k + qd(i - 1) + (q - 1)d]_d = [k + (i - 1)qd, k + iqd - d]_d$. Thus,

$$\bigcup_{i=1}^n W_i = [k, k + (nq - 1)d]_d.$$

Hence, g is a semi (k, d) -arithmetic labeling of nG as we claimed. □

Theorem 4.3. *Let G be a (p, q) -graph of odd size and k, d be a pair of positive integers. If G is semi (k, d) - and $(k + d, d)$ -arithmetic, then for every positive integer n the graph nG is also semi (k, d) - and $(k + d, d)$ -arithmetic.*

Proof. Let G be a semi (k, d) -arithmetic and semi $(k + d, d)$ -arithmetic graph of order p and odd size q with $V(G) = \{u_1, u_2, \dots, u_p\}$. For each $i \in [1, n]$, let G_i be a copy of G , where $V(G_i) = \{u_1^i, u_2^i, \dots, u_p^i\}$ and u_j^i be the replica of u_j . Suppose that f and g are a semi (k, d) -arithmetic labeling and a semi $(k + d, d)$ -arithmetic labeling of G , respectively.

We show first that nG is a semi (k, d) -arithmetic graph. Consider the following labeling of the vertices of nG :

$$h(u_i) = \begin{cases} f(u_j) + \frac{dq(i-1)}{2}, & \text{if } i \text{ is odd,} \\ g(u_j) + \frac{d(q(i-1)-1)}{2}, & \text{if } i \text{ is even.} \end{cases}$$

Note that all the labels assigned by h are nonnegative integers. Let W_i be the set formed by the weights induced by h on the edges of G_i . Independently of the parity of i , we get that $W_i = [k + dq(i - 1), k + d(qi - 1)]_d$. Since,

$$\min W_i - \max W_{i-1} = (k + dq(i - 1)) - (k + d(q(i - 1) - 1)) = d,$$

we conclude that the weights induced by h on the edges of nG form an arithmetic sequence of difference d which first and last elements are k and $k + d(qn - 1)$, respectively. Based on the fact that this sequence has qn terms, we have that there are no repeated weights. Consequently, h is a semi (k, d) -arithmetic labeling of nG .

Now we prove that nG is semi $(k + d, d)$ -arithmetic. Let h' be the labeling of nG defined on the vertices of G_i as:

$$h'(u_i) = \begin{cases} g(u_j) + \frac{dq(i-1)}{2}, & \text{if } i \text{ is odd,} \\ f(u_j) + \frac{d(q(i-1)+1)}{2}, & \text{if } i \text{ is even.} \end{cases}$$

As in the previous case, regardless the parity if i , the set formed by the weights on the edges of G_i is $W_i = [k + d(q(i - 1) + 1), k + dq(i)]_d$. Again,

$$\min W_i - \max W_{i-1} = (k + dq(i + 1)) - (k + d(q(i - 1) - 1)) = d,$$

which implies that the weights on the edges of nG form an arithmetics sequence of difference d which first and last elements are $k + d$ and $k + dqn$, respectively. Thus, h' is a semi $(k + d, d)$ -arithmetic labeling of nG . \square

Now, we turn our attention to the complete bipartite graphs. In [10], Graham and Sloane proved that $K_{m,n}$ is harmonious if and only if it is acyclic. Bu and Shi [4] proved that $K_{m,n}$ is (k, d) -arithmetic if $k \neq id$ for each positive $i \leq n - 1$. Lu et al. [14] proved that $K_{1,m} \cup K_{p,qn}$ is (k, d) -arithmetic if $d \neq 1$ and $k > d(q - 1) + 1$. In the following lemma we show that $K_{m,n}$ is semi (k, d) -arithmetic for any pair k, d of positive integers.

Lemma 4.1. *The complete bipartite graph $K_{m,n}$ is semi (k, d) -arithmetic for every pair k, d of positive integers.*

Proof. Let $A = \{u_1, u_2, \dots, u_m\}$ and $B = \{v_1, v_2, \dots, v_n\}$ be the stable sets of $K_{m,n}$. Consider the following labeling of the vertices of $K_{m,n}$:

- $f(u_i) = nd(i - 1)$ for each $1 \leq i \leq m$,
- $f(v_j) = k + d(j - 1)$ for each $1 \leq j \leq n$.

Since all the parameters and variables involved in the definition of f are positive integers, we conclude that the range of f is a subset of \mathbb{N} . Note that

$$f(u_i) + f(v_j) = nd(i - 1) + k + d(j - 1) = k + d(n(i - 1) + j - 1).$$

Thus, for a fixed value of i , the set of weights induced on the edges $u_i v_j$ is

$$W_i = \{k + dn(i - 1), k + dn(i - 1) + d, \dots, k + dn(i - 1) + d(n - 1)\}.$$

Since $i \in \{1, 2, \dots, m\}$, the union of these sets is $\cup_{i=1}^m W_i = \{k, k + d, \dots, k + d(mn - 1)\}$.

Therefore, the labeling f is semi (k, d) -arithmetic. □

In the following result we prove that this labeling of $K_{m,n}$ can be extended to an arbitrary union of complete bipartite graphs.

Theorem 4.4. *If G is a graph which components are complete bipartite graphs, then G is semi (k, d) -arithmetic.*

Proof. Let $G = \cup_{t=1}^r K_{m_t, n_t}$ and $A_t = \{u_1^t, u_2^t, \dots, u_{m_t}^t\}$ and $B_t = \{v_1^t, v_2^t, \dots, v_{n_t}^t\}$ be the stable sets of K_{m_t, n_t} . Basically, the labeling of each component of G follows the pattern of the labeling f described in the previous lemma. In particular, the labeling of K_{m_1, n_1} is f itself. For $t \geq 2$, the labeling of K_{m_t, n_t} depends on the labeling of $K_{m_{t-1}, n_{t-1}}$; if xd is the label on $u_{m_{t-1}}^{t-1}$, then the label on u_1^t is $(x - 1)d$, if $k + yd$ is the label on $v_{n_{t-1}}^{t-1}$, then the label on v_1^t is $k + (y + 2)d$. Therefore, the largest weight on an edge of $K_{m_{t-1}, n_{t-1}}$ is $k + d(x + y)$ and the smallest weight on an edge of K_{m_t, n_t} is $(x - 1)d + k + (y + 2)d = k + d(x + y + 1)$.

Consequently, the weights on the edges of K_{m_t, n_t} form an arithmetic sequence of difference d which first element is $k + d(x + y + 1)$. As a result of this, we have that the weights on the edges of G form an arithmetic sequence of difference d which first element is k . Hence, G is a semi (k, d) -arithmetic graph. □

In Figure 6 we show an example of this labeling for a graph G which components are $K_{3,3}$, $K_{3,4}$, and $K_{4,3}$.

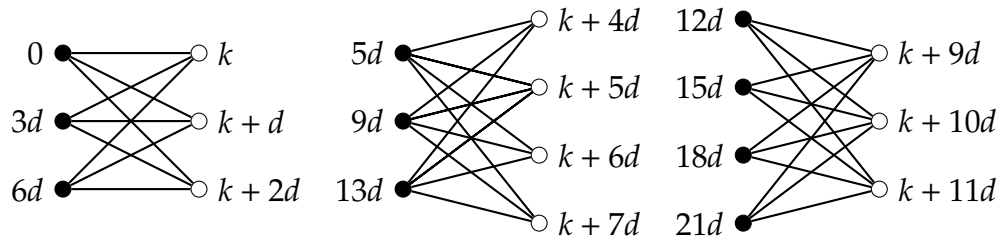


Figure 6. Semi (k, d) -arithmetic labeling of $K_{3,3} \cup K_{3,4} \cup K_{4,3}$

Theorem 4.5. *For $i = 1, 2$, let G_i be a semi (k_i, d) -arithmetic graph of size q_i , where $dq_1 \geq k_2 - k_1 > 0$. The graph $G_1 \cup G_2$ is semi (k_1, d) -arithmetic if one of the following conditions holds:*

- (1) k_1 and k_2 have the same parity and dq_1 is even, or
- (2) k_1 and k_2 have different parity and dq_1 is odd.

Proof. Suppose that f_i is a semi (k_i, d) -arithmetic labeling of G_i . Consider the labeling $h : V(G_1 \cup G_2) \rightarrow \mathbb{N}$ defined as

$$h(v) = \begin{cases} f_1(v), & \text{if } v \in V(G_1), \\ f_2(v) + \frac{k_1 + dq_1 - k_2}{2}, & \text{if } v \in V(G_2). \end{cases}$$

We claim that h is a semi (k_1, d) -arithmetic labeling of $G_1 \cup G_2$. In order to prove our claim, we note first that the fraction $\frac{k_1 + dq_1 - k_2}{2}$ is a nonnegative integer because of conditions (1) or (2). Observe now that the weights induced by h of the edges of G_1 are $k_1, k_1 + d, \dots, k_1 + d(q_1 - 1)$. Since f_2 is a semi (k_2, d) -arithmetic labeling of G_2 , it induces the weights $k_2, k_2 + d, \dots, k_2 + d(q_2 - 1)$. This implies that the weights induced by h on the edges of G_2 are the same numbers but shifted $k_1 + dq_1 - k_2$ units, that is, $k_1 + dq_1, k_1 + d(q_1 + 1), \dots, k_1 + d(q_1 + q_2 - 1)$. Therefore, the weights induced by h on the edges of $G_1 \cup G_2$ form an arithmetic sequence of difference d and first element k_1 . Consequently, h is a semi (k_1, d) -arithmetic labeling as we claimed. \square

References

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