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γ-Paired dominating graphs of lollipop, umbrella and coconut graphs

Pannawat Eakawinrujee, Nantapath Trakultraipruk*

Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Pathum Thani 12120, Thailand

p.eakawinrujee@gmail.com, n.trakultraipruk@yahoo.com

**corresponding author*

Abstract

A paired dominating set of a graph G is a dominating set whose induced subgraph has a perfect matching. The paired domination number $\gamma_{pr}(G)$ of G is the minimum cardinality of a paired dominating set. A paired dominating set D is a $\gamma_{pr}(G)$-set if $|D| = \gamma_{pr}(G)$. The γ-paired dominating graph $PD_\gamma(G)$ of G is the graph whose vertex set is the set of all $\gamma_{pr}(G)$-sets, and two $\gamma_{pr}(G)$-sets D_1 and D_2 are adjacent in $PD_\gamma(G)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. This paper determines the paired domination numbers of lollipop graphs, umbrella graphs, and coconut graphs. We also consider the γ-paired dominating graphs of those three graphs.

Keywords: paired dominating graph, paired domination number, gamma graph, lollipop graph, umbrella graph, coconut graph
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1. Introduction

We in general follow the graph theory notation and terminology from [22]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *open* and *closed neighborhoods* of a vertex $v \in V(G)$ are $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The *open* and *closed neighborhoods* of a set $D \subseteq V(G)$ are $N(D) = \bigcup_{v \in D} N(v)$ and $N[D] = N(D) \cup D$, respectively. We use P_k and C_k to denote a path and a cycle, respectively, with k vertices.

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A set $D \subseteq V(G)$ is a *dominating set* of G if every vertex $v \in V(G)$ which does not belong to D has a neighbor in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets. A dominating set D is a $\gamma(G)$*-set* if $|D| = \gamma(G)$. For more details on domination and its variants in graphs, see [2, 5, 11, 12, 14].

Subramanian and Sridharan [21] defined the *gamma graph* of G, denoted by $\gamma.G$, to be the graph whose vertex set is the set of all $\gamma(G)$-sets, and two $\gamma(G)$-sets D_1 and D_2 are adjacent in $\gamma.G$ if they satisfy the following condition: for some $u \in D_1$ and $v \notin D_1$,

$$D_2 \;=\; (D_1 \setminus \{u\}) \cup \{v\}, \tag{1}$$

or $|D_1 \setminus D_2| = 1 = |D_2 \setminus D_1|$. Fricke et al. [9] defined the γ*-graph* $G(\gamma)$ of G, which is the graph with $V(G(\gamma)) = V(\gamma.G)$, and two $\gamma(G)$-sets D_1 and D_2 are adjacent in $G(\gamma)$ if they satisfy the condition (1) and $uv \in E(G)$. Observe that $G(\gamma)$ is a spanning subgraph of $\gamma.G$. For additional results on gamma graphs or γ-graphs, see [3, 4, 15, 16, 17].

The k*-dominating graph* $D_k(G)$ of G, studied by Haas and Seyffarth [10], is the graph whose vertex set is the set of all dominating sets of G having cardinality at most k, and two vertices of $D_k(G)$ are adjacent if they differ by either adding or deleting a single vertex. The authors determined conditions for $D_k(G)$ to be connected. For additional results on k-dominating graph, see [18], and for other variations of the k-dominating graph, see [1, 8].

Wongsriya and Trakultraipruk [23] defined the γ*-total dominating graph* $TD_\gamma(G)$ of G to be the graph whose vertex set is the set of all $\gamma_t(G)$-sets (minimum total dominating sets). Two $\gamma_t(G)$-sets D_1 and D_2 are adjacent in $TD_\gamma(G)$ if they satisfy the condition (1). They studied $TD_\gamma(P_k)$ and $TD_\gamma(C_k)$. The γ*-independent dominating graph* [19] and the γ*-induced-paired dominating graph* [20] are defined similarly.

A set $D \subseteq V(G)$ is a *paired dominating set* of G if it is a dominating set and the subgraph of G induced by D contains a perfect matching. The *paired domination number* $\gamma_{pr}(G)$ of G is the minimum cardinality among all paired dominating sets. A paired dominating set D is a $\gamma_{pr}(G)$*-set* if $|D| = \gamma_{pr}(G)$. Let D be a paired dominating set of G with a perfect matching M. We say that a vertex $v \in D$ *dominates* a vertex u if they are adjacent in G. If an edge $uv \in M$, then we call the set $\{u, v\}$ a *pair*. The concept of paired domination was introduced by Haynes and Slater [13].

In [6], we defined the γ*-paired dominating graph* $PD_\gamma(G)$ of G to be the graph whose vertices are $\gamma_{pr}(G)$-sets, and two $\gamma_{pr}(G)$-sets D_1 and D_2 are adjacent in $PD_\gamma(G)$ if they satisfy the condition (1). We studied $PD_\gamma(P_k)$ in [6] and $PD_\gamma(C_k)$ in [7]. This paper determines the paired domination numbers of lollipop graphs, umbrella graphs, and coconut graphs. We also determine the γ-paired dominating graphs of those graphs.

2. Preliminary Results

In this section, we recall some definitions, notations, and results used in the proofs of our main results.

A *support vertex* is a vertex adjacent to a vertex of degree one. Haynes and Slater [13] provided a couple of useful lemmas.

Lemma 2.1 ([13]). *If v is a support vertex of a graph G, then v is in every paired dominating set of G.*

Lemma 2.2 ([13]). *Let $k \geq 2$ be an integer. Then $\gamma_{pr}(P_k) = 2\lceil \frac{k}{4} \rceil$.*

The *Cartesian product* of graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ where vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

Let $P_p : u_1u_2u_3 \cdots u_p$ and $P_q : v_1v_2v_3 \cdots v_q$ be the paths, where p and q are positive integers. Fricke et al. [9] defined a *stepgrid* $SG_{p,q}$ to be the subgraph of $P_p \square P_q$ induced by $\{(u_x, v_y) : 1 \leq x \leq p, 1 \leq y \leq q, x - y \leq 1\}$. We call the vertex (u_x, v_y) in the stepgrid as the *vertex at the position* (x, y). The stepgrids $SG_{2,2}$ and $SG_{4,3}$ are shown in Figure 1.

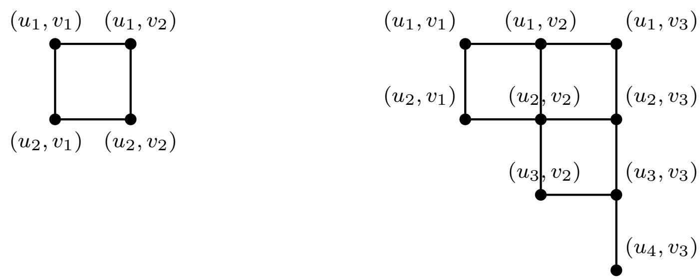

Figure 1: The stepgrids $SG_{2,2}$ (left) and $SG_{4,3}$ (right)

Let $P_p : u_1u_2u_3 \cdots u_p$, $P_q : v_1v_2v_3 \cdots v_q$, and $P_r : w_1w_2w_3 \cdots w_r$ be the paths, where p, q, and r are positive integers. In [6], we defined a *stepgrid* $SG_{p,q,r}$ be the graph with vertex set

$$\begin{aligned} V(SG_{p,q,r}) &= \{(u_x, v_y, w_z) \in V(P_p \square P_q \square P_r) : 1 \leq x \leq p, 1 \leq y \leq q, 1 \leq z \leq r, \\ &\quad x - y \leq 0, x - z \leq 1, y - z \geq 0\} \end{aligned}$$

and edge set

$$E(SG_{p,q,r}) = E(P_p \square P_q \square P_r) \cup \{(u_x, v_x, w_x)(u_{x+1}, v_{x+1}, w_x) : 1 \leq x \leq p - 1\}.$$

The vertex (u_x, v_y, w_z) is called the *vertex at the position* (x, y, z) in $SG_{p,q,r}$. The stepgrid $SG_{4,4,3}$ is shown in Figure 2, where we write (x, y, z) for (u_x, v_y, w_z).

Eakawinrujee and Trakultraipruk [6] determined the γ-paired dominating graphs of paths and their properties. At this point, we denote $P_k : v_1v_2v_3 \cdots v_k$ to be the path with k vertices.

Lemma 2.3 ([6]). *Let $k \geq 0$ be an integer. Then there is exactly one $\gamma_{pr}(P_{4k+3})$-set containing the pair $\{v_{4k+2}, v_{4k+3}\}$ and this set has degree one in $PD_\gamma(P_{4k+3})$.*

Lemma 2.4 ([6]). *Let $k \geq 1$ be an integer. All $\gamma_{pr}(P_{4k+2})$-sets containing the pair $\{v_{4k+1}, v_{4k+2}\}$ form a path with $k+1$ vertices in $PD_\gamma(P_{4k+2})$, where one endpoint contains the pair $\{v_{4k-2}, v_{4k-1}\}$ and the others contain the pair $\{v_{4k-3}, v_{4k-2}\}$.*

Lemma 2.5 ([6]). *Let $k \geq 1$ be an integer. Then all $\gamma_{pr}(P_{4k+1})$-sets containing the pair $\{v_{4k}, v_{4k+1}\}$ form a stepgrid $SG_{k+1,k}$ in $PD_\gamma(P_{4k+1})$ (see Figure 3), where $D_{1,k}, D_{2,k}, \ldots, D_{k,k}$ contain the pair $\{v_{4k-3}, v_{4k-2}\}$, $D_{k+1,k}$ contains the pair $\{v_{4k-2}, v_{4k-1}\}$, and the others contain the pair $\{v_{4k-4}, v_{4k-3}\}$. Moreover, $D_{1,1}, D_{2,1}, D_{1,k}$ have degree three, $D_{2,k}, D_{3,k}, \ldots, D_{k,k}$ have degree four, and $D_{k+1,k}$ has degree two in $PD_\gamma(P_{4k+1})$.*

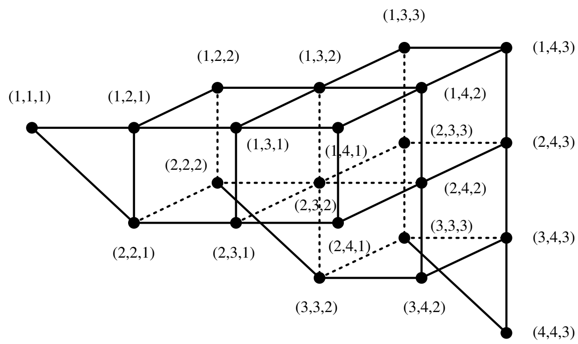

Figure 2: The stepgrid $SG_{4,4,3}$

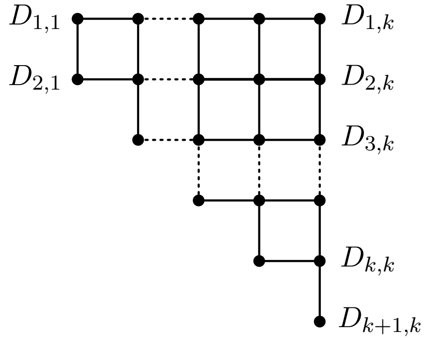

Figure 3: The stepgrid $SG_{k+1,k}$ in $PD_\gamma(P_{4k+1})$

Theorem 2.1 ([6]). *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k}) \cong P_1$.*

Theorem 2.2 ([6]). *Let $k \geq 0$ be an integer. Then $PD_\gamma(P_{4k+3}) \cong P_{k+2}$.*

Theorem 2.3 ([6]). *Let $k \geq 0$ be an integer. Then $PD_\gamma(P_{4k+2}) \cong SG_{k+1,k+1}$.*

Theorem 2.4 ([6]). *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k+1}) \cong SG_{k+1,k+1,k}$.*

From the proof of Theorem 2.2, we get the following result.

Corollary 2.1. *Let $k \geq 1$ be an integer and $PD_\gamma(P_{4k-1}) \cong P_{k+1} \cong D_1 D_2 \cdots D_{k+1}$, where D_x is a $\gamma_{pr}(P_{4k-1})$-set for all $x \in \{1, 2, \ldots, k+1\}$. If D_{k+1} contains the pair $\{v_{4k-2}, v_{4k-1}\}$, then $D_x = S_x \cup \{v_{4k-3}, v_{4k-2}\}$, where S_x is a $\gamma_{pr}(P_{4k-5})$-set for all $x \in \{1, 2, \ldots, k\}$ and especially S_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and $D_{k+1} = S_k \cup \{v_{4k-2}, v_{4k-1}\}$.*

The following corollary can be obtained from the proofs of Lemma 2.5 and Theorem 2.4.

Corollary 2.2. *Let $k \geq 1$ be an integer and $D_{x,y,z}$ the $\gamma_{pr}(P_{4k+1})$-set at the position (x, y, z) in $PD_\gamma(P_{4k+1}) \cong SG_{k+1,k+1,k}$ for all $x, y \in \{1, 2, \ldots, k+1\}$, $z \in \{1, 2, \ldots, k\}$ with $x - y \leq 0, x - z \leq 1, y - z \geq 0$. If either $x = 1$ or $y = k+1$, then $D_{x,y,z}$ contains the pair $\{v_{4k}, v_{4k+1}\}$. Moreover, if $D_{x,k+1,z}$ contains the pair $\{v_{4k}, v_{4k+1}\}$, then*

(1) $D_{x,k+1,z} = (D_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{v_{4k+1}\}$ *for all* $x, z \in \{1, 2, \ldots, k\}$, *and* $D_{k+1,k+1,k} = (D_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{v_{4k+1}\}$,

(2) $D_{x,k+1,k} = D_x \cup \{v_{4k-3}, v_{4k-2}, v_{4k}, v_{4k+1}\}$, *where* D_x *is a* $\gamma_{pr}(P_{4k-5})$*-set for all* $x \in \{1, 2, \ldots, k\}$, D_k *contains the pair* $\{v_{4k-6}, v_{4k-5}\}$, *and* $D_{k+1,k+1,k} = D_k \cup \{v_{4k-2}, v_{4k-1}, v_{4k}, v_{4k+1}\}$,

(3) $D_{x,k+1,z}$ *contains the pairs* $\{v_{4k-4}, v_{4k-3}\}, \{v_{4k}, v_{4k+1}\}$ *for all* $z < k$.

Let G_1 and G_2 be complete graphs with p vertices, where $V(G_1) = \{u_1, u_2, \ldots, u_p\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_p\}$. We define A_p to be the graph with vertex set $V(A_p) = \{(u_x, v_y) \in V(G_1 \Box G_2) : 1 \leq x \leq y \leq p\}$ and edge set $E(A_p) = E(G_1 \Box G_2) \cup \{(u_x, v_y)(u_{y+1}, v_z) : 1 \leq x \leq y < z \leq p\}$. We illustrate the graph A_3 as shown in Figure 4.

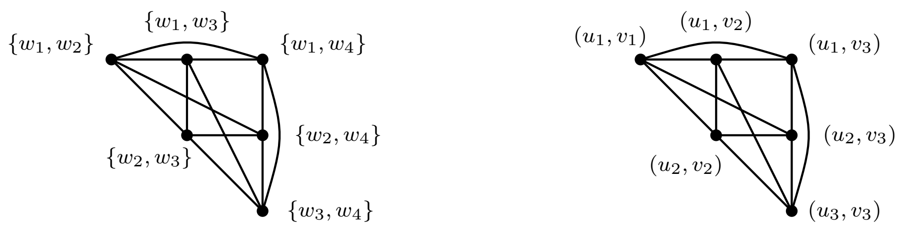

Figure 4: The graphs $PD_\gamma(K_4)$ (left) and A_3 (right)

Theorem 2.5. *Let* $k \geq 2$ *be an integer. Then* $PD_\gamma(K_k) \cong A_{k-1}$.

Proof. Let $V(K_k) = \{w_1, w_2, \ldots, w_k\}$. Note that $\gamma_{pr}(K_k) = 2$, so $V(PD_\gamma(K_k)) = \{\{w_m, w_n\} : 1 \leq m < n \leq k\}$. Let $V(A_{k-1}) = \{(u_x, v_y) : 1 \leq x \leq y \leq k-1\}$. Define $f : V(PD_\gamma(K_k)) \to V(A_{k-1})$ by $f(\{w_m, w_n\}) = (u_m, v_{n-1})$. Clearly, f is bijection, and preserve edges and non-edges. The theorem follows. □

3. Paired Domination Numbers of Lollipop Graphs, Umbrella Graphs, and Coconut Graphs

In this section, we give the definitions of a lollipop graph, a umbrella graph, and a coconut graph. We then determine the paired domination numbers of those graphs.

A *lollipop graph* $L_{p,q}$ is obtained by appending an endpoint of a path P_p to a vertex of a complete graph K_q. For convenence, we label the vertices of the path as $v_1, v_2, \ldots, v_p$ and the vertices of the complete graph as $u_1, u_2, \ldots, u_q$, where v_p is adjacent to u_1. For example, the lollipop graph $L_{7,6}$ is shown in Figure 5.

A *umbrella graph* $U_{p,q}$ is obtained by joining an endpoint of a path P_p to the central vertex of a fan graph $F_q \cong K_1 \vee P_{q-1}$. A *coconut graph* $C_{p,q}$ is obtained by joining an endpoint of a path P_p to the support vertex of a star graph $S_q \cong K_{1,q-1}$. We label the vertices of $U_{p,q}$ and $C_{p,q}$ as shown in Figures 6 and 7, respectively.

Let p be a positive integer. If $q = 1$, then $L_{p,q} \cong U_{p,q} \cong C_{p,q} \cong P_{p+1}$, so $\gamma_{pr}(L_{p,q}) = \gamma_{pr}(U_{p,q}) = \gamma_{pr}(C_{p,q}) = 2\lceil \frac{p+1}{4} \rceil$ by Lemma 2.2. If $q \geq 2$, then we get the following theorem.

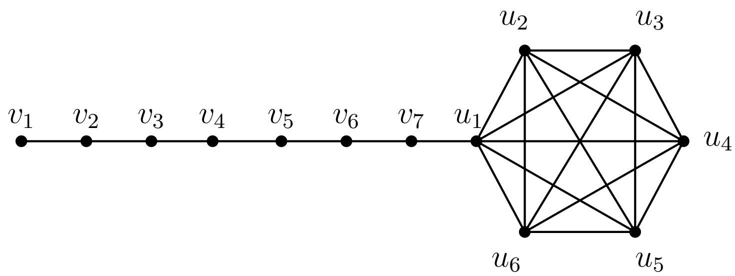

Figure 5: The lollipop graph $L_{7,6}$

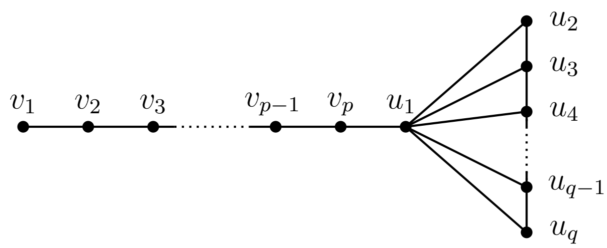

Figure 6: The umbrella graph $U_{p,q}$

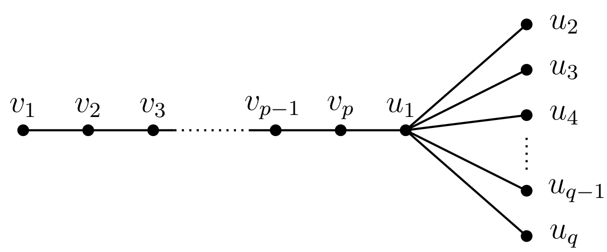

Figure 7: The coconut graph $C_{p,q}$

Theorem 3.1. *Let $p \geq 1$ and $q \geq 2$ be integers. Then $\gamma_{pr}(L_{p,q}) = \gamma_{pr}(U_{p,q}) = \gamma_{pr}(C_{p,q}) = 2\lceil \frac{p+2}{4} \rceil$.*

Proof. If $q = 2$, then $L_{p,q}$ is a path with $p+2$ vertices. By Lemma 2.2, we get $\gamma_{pr}(L_{p,2}) = 2\lceil \frac{p+2}{4} \rceil$. Let $q \geq 3$ and $\widehat{P}_{u_2}$ be the graph obtained from $L_{p,q}$ by deleting the vertices $u_3, u_4, \ldots, u_q$. Clearly, $\widehat{P}_{u_2}$ is a path with $p+2$ vertices, and $\gamma_{pr}(\widehat{P}_{u_2}) = 2\lceil \frac{p+2}{4} \rceil$. Let D be a $\gamma_{pr}(L_{p,q})$-set. To prove $\gamma_{pr}(L_{p,q}) \geq 2\lceil \frac{p+2}{4} \rceil$, we show that $|D| \geq \gamma_{pr}(\widehat{P}_{u_2})$. If $u_1 \in D$, then D contains either the pair $\{v_p, u_1\}$ or, without loss of generality, $\{u_1, u_2\}$. In both cases, D is a paired dominating set of $\widehat{P}_{u_2}$, so $|D| \geq \gamma_{pr}(\widehat{P}_{u_2})$. Thus, we assume that $u_1 \notin D$. Since D is a $\gamma_{pr}(L_{p,q})$-set, D must contain exactly two vertices from $\{u_2, u_3, \ldots, u_q\}$. Without loss of generality, we may assume that D contains the pair $\{u_2, u_3\}$. Hence, $D' = (D \setminus \{u_3\}) \cup \{u_1\}$ is a paired dominating set of $\widehat{P}_{u_2}$, so $|D| = |D'| \geq \gamma_{pr}(\widehat{P}_{u_2})$. Now, we get $\gamma_{pr}(L_{p,q}) \geq 2\lceil \frac{p+2}{4} \rceil$. Note that $U_{p,q}$ and $C_{p,q}$ are spanning subgraphs of $L_{p,q}$, so $\gamma_{pr}(U_{p,q}) \geq \gamma_{pr}(L_{p,q})$ and $\gamma_{pr}(C_{p,q}) \geq \gamma_{pr}(L_{p,q})$.

Next, we show the upper bounds of $\gamma_{pr}(L_{p,q}), \gamma_{pr}(U_{p,q})$, and $\gamma_{pr}(C_{p,q})$. If $p \equiv 1, 2 \pmod 4$, let $D = \{v_i, v_{i+1} : i \equiv 2 \pmod 4, i \leq p-3\} \cup \{v_p, u_1\}$; otherwise, let $D = \{v_i, v_{i+1} : i \equiv 2 \pmod 4, i \leq p-5\} \cup \{v_{p-2}, v_{p-1}, v_p, u_1\}$. Then D is a paired dominating set of $L_{p,q}$ with cardinality $2\lceil \frac{p+2}{4} \rceil$, so $\gamma_{pr}(L_{p,q}) \leq 2\lceil \frac{p+2}{4} \rceil$. Since D is also a paired dominating set of $U_{p,q}$ and $C_{p,q}$, $\gamma_{pr}(U_{p,q}) \leq 2\lceil \frac{p+2}{4} \rceil$ and $\gamma_{pr}(C_{p,q}) \leq 2\lceil \frac{p+2}{4} \rceil$. □

4. γ-Paired Dominating Graphs of Lollipop Graphs

In this section, we determine the γ-paired dominating graph of a lollipop graph $L_{p,q}$. If $q = 1$, then we get the γ-paired dominating graph of $L_{p,q} \cong P_{p+1}$ from Theorems 2.1 - 2.4. For $q \geq 2$, we consider the value of p into four cases and then we obtain the following results.

Theorem 4.1. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $PD_\gamma(L_{4k+2,q}) \cong P_1$.*

Proof. By Theorem 3.1, we have $\gamma_{pr}(L_{4k+2,q}) = 2k+2$. It is easy to check that there is exactly one $\gamma_{pr}(L_{4k+2,q})$-set, which is $D = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\} \cup \{v_{4k+2}, u_1\}$, so the theorem holds. □

Lemma 4.1. *Let $k \geq 0$ and $q \geq 2$ be integers. Then each $\gamma_{pr}(L_{4k+1,q})$-set contains the vertex u_1. Moreover, if a $\gamma_{pr}(L_{4k+1,q})$-set contains the pair $\{u_1, u_i\}$ for some i, then this set does not contain v_{4k+1}.*

Proof. If $q = 2$, then u_1 is a support vertex of $L_{4k+1,q}$, so this lemma holds by Lemma 2.1. Let $q \geq 3$ and suppose on the contrary that there is a $\gamma_{pr}(L_{4k+1,q})$-set D such that $u_1 \notin D$. Then D must contain exactly two vertices from $\{u_2, u_3, \ldots, u_q\}$. Since $|D| = 2k+2$, the other $2k$ vertices of D must dominate all vertices in P_{4k+1}. This contradicts the fact that $2k$ vertices can dominate at most $4k$ vertices in P_{4k+1}.

Next, we suppose that there is a $\gamma_{pr}(L_{4k+1,q})$-set D containing the pairs $\{v_{4k}, v_{4k+1}\}, \{u_1, u_i\}$ for some i. Then $v_{4k-1} \notin D$. Recall that $|D| = 2k+2$, so the other $2k-2$ vertices must dominate all vertices in P_{4k-2}, which is impossible. □

Theorem 4.2. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $PD_\gamma(L_{4k+1,q}) \cong L_{k,q}$.*

Proof. By Lemma 4.1, each $\gamma_{pr}(L_{4k+1,q})$-set must contain either the pair $\{v_{4k+1}, u_1\}$ or $\{u_1, u_i\}$ where $i \neq 1$. We first find all $\gamma_{pr}(L_{4k+1,q})$-sets containing the pair $\{v_{4k+1}, u_1\}$. Note that these sets do not contain $u_2, u_3, \ldots, u_q$. Let P be the subgraph of $L_{4k+1,q}$ induced by $\{v_1, v_2, \ldots, v_{4k+1}, u_1\}$. Clearly, P is a path with $4k+2$ vertices. Then $\gamma_{pr}(L_{4k+1,q}) = 2k+2 = \gamma_{pr}(P)$, and every $\gamma_{pr}(L_{4k+1,q})$-set containing the pair $\{v_{4k+1}, u_1\}$ is a $\gamma_{pr}(P)$-set containing the pair $\{v_{4k+1}, u_1\}$ and vice versa. By Lemma 2.4, we get that all $\gamma_{pr}(L_{4k+1,q})$-sets containing the pair $\{v_{4k+1}, u_1\}$ form a path $D_1D_2 \cdots D_{k+1}$ in $PD_\gamma(L_{4k+1,q})$ where, without loss of generality, D_{k+1} contains the pair $\{v_{4k-2}, v_{4k-1}\}$ and the others contain the pair $\{v_{4k-3}, v_{4k-2}\}$.

We next find all $\gamma_{pr}(L_{4k+1,q})$-sets containing the pair $\{u_1, u_i\}$ where $i \in \{2, 3, \ldots, q\}$. By Lemma 4.1, these sets do not contain v_{4k+1}. Then such a $\gamma_{pr}(L_{4k+1,q})$-set is a union of a $\gamma_{pr}(P_{4k})$-set and $\{u_1, u_i\}$. Theorem 2.1 shows that, for each i, there is only one $\gamma_{pr}(L_{4k+1,q})$-set containing the pair $\{u_1, u_i\}$. For each $i \in \{2, 3, \ldots, q\}$, let

$$D_{k+i} = (D_{k+1} \setminus \{v_{4k+1}\}) \cup \{u_i\}.$$

Thus, for each i, D_{k+i} is the only $\gamma_{pr}(L_{4k+1,q})$-set containing the pair $\{u_1, u_i\}$. It is clear that $D_{k+1}, D_{k+2}, \ldots, D_{k+q}$ are pairwise adjacent. We can check that, for all $x \in \{1, 2, \ldots, k\}$ and $i \in \{2, 3, \ldots, q\}$, $(D_x \setminus \{v_{4k+1}\}) \cup \{u_i\}$ is not a dominating set, and thus D_x is not adjacent to all $D_{k+2}, D_{k+3}, \ldots, D_{k+q}$. Therefore, all $\gamma_{pr}(L_{4k+1,q})$-sets form a lollipop graph $L_{k,q}$. □

Let p and q be positive integers. We define $A_{p,q}$ to be the graph with $V(A_{p,q}) = V(SG_{p,q})$ and $E(A_{p,q}) = E(SG_{p,q}) \cup \{(u_x, v_y)(u_x, v_{y'}) : p-1 \le y < y'-1 \le q-1\}$. We also define $B_{p,q}$ to be the graph with

$$V(B_{p,q}) = V(A_{p,q}) \cup \{(u_x, v_y) : p+1 \le x \le y \le q\}$$

and

$$\begin{aligned} E(B_{p,q}) \quad = \quad & E(A_{p,q}) \cup \{(u_x, v_y)(u_x, v_{y'}) : p+1 \le x \le q-1, x \le y < y' \le q\} \cup \\ & \{(u_x, v_y)(u_{x'}, v_y) : p+1 \le y \le q, p \le x < x' \le y\} \cup \\ & \{(u_x, v_y)(u_{y+1}, v_z) : p \le x \le y < z \le q\}. \end{aligned}$$

Figure 8 shows the graphs $A_{3,4}$ and $A_{4,6}$ and Figure 9 shows the graphs $B_{3,4}$ and $B_{4,6}$, where we use (x, y) instead of (u_x, v_y). Note that if $p \ge q$, then $A_{p,q} \cong B_{p,q} \cong SG_{p,q}$.

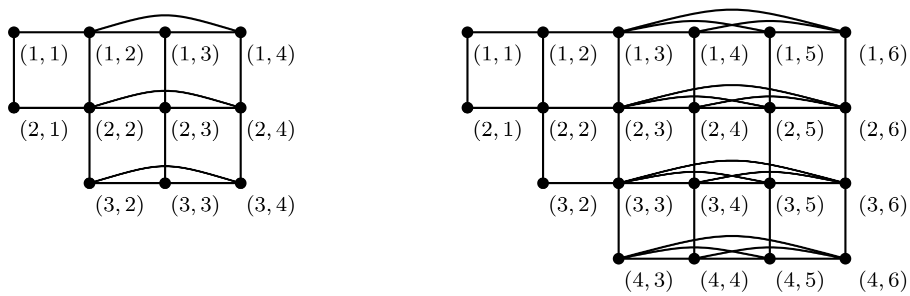

Figure 8: The graphs $A_{3,4}$ (left) and $A_{4,6}$ (right)

Theorem 4.3. *Let $k \ge 1$ and $q \ge 2$ be integers. Then $PD_\gamma(L_{4k,q}) \cong B_{k+1,k+q-1}$.*

Proof. Note that $L_{4k,2} \cong P_{4k+2}$. By Theorem 2.3, we get $PD_\gamma(L_{4k,2}) \cong SG_{k+1,k+1} \cong B_{k+1,k+1}$. Let $q \ge 3$. If a $\gamma_{pr}(L_{4k,q})$-set contains the vertex u_1, then it contains either the pair $\{v_{4k}, u_1\}$ or $\{u_1, u_i\}$ where $i \ne 1$. We first find all $\gamma_{pr}(L_{4k,q})$-sets containing the pair $\{v_{4k}, u_1\}$. Let P be the subgraph of $L_{4k,q}$ induced by $\{v_1, v_2, \ldots, v_{4k}, u_1\}$. Then each $\gamma_{pr}(L_{4k,q})$-set containing the pair $\{v_{4k}, u_1\}$ is a $\gamma_{pr}(P)$-set containing the pair $\{v_{4k}, u_1\}$ and vice versa. By Lemma 2.5, all $\gamma_{pr}(L_{4k,q})$-sets containing the pair $\{v_{4k}, u_1\}$ form a stepgrid $SG_{k+1,k}$ in $PD_\gamma(L_{4k,q})$. For all $x \in \{1, 2, \ldots, k+1\}$ and $y \in \{1, 2, \ldots, k\}$ with $x - y \le 1$, let $D_{x,y}$ be the $\gamma_{pr}(L_{4k,q})$-set containing the pair $\{v_{4k}, u_1\}$ at the position (x, y) in $SG_{k+1,k}$. Then $D_{1,k}, D_{2,k}, \ldots, D_{k,k}$ contain the pair $\{v_{4k-3}, v_{4k-2}\}$, $D_{k+1,k}$ contains the pair $\{v_{4k-2}, v_{4k-1}\}$, and $D_{x,y}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$ for all $y \ne k$.

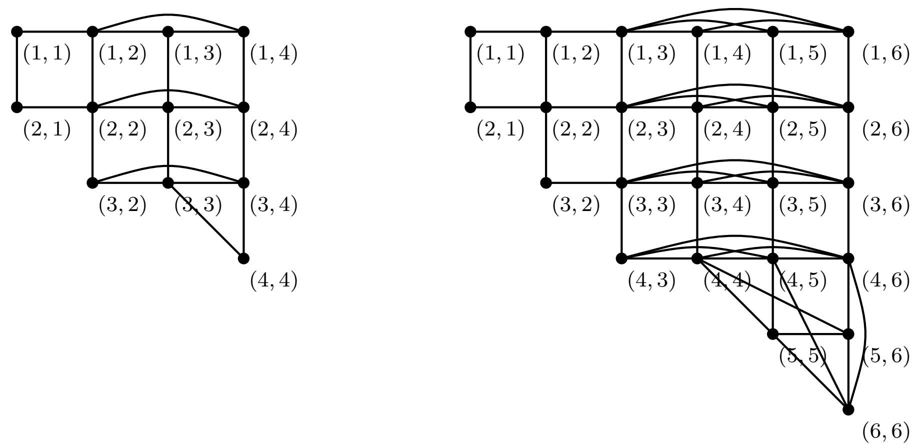

Figure 9: The graphs $B_{3,4}$ (left) and $B_{4,6}$ (right)

We next find all $\gamma_{pr}(L_{4k,q})$-sets containing the pair $\{u_1, u_i\}$ where $i \in \{2, 3, \ldots, q\}$. Similar to Lemma 4.1, these sets do not contain v_{4k}. Then such a $\gamma_{pr}(L_{4k,q})$-set is a union of a $\gamma_{pr}(P_{4k-1})$-set and $\{u_1, u_i\}$. By Theorem 2.2, for each i, there are $k + 1$ $\gamma_{pr}(L_{4k,q})$-sets containing the pair $\{u_1, u_i\}$ and they form a path in $PD_\gamma(L_{4k,q})$. Recall that $D_{1,k}, D_{2,k}, \ldots, D_{k,k}$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}, \{v_{4k}, u_1\}$, and $D_{k+1,k}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}, \{v_{4k}, u_1\}$. For each $x \in \{1, 2, \ldots, k+1\}$ and $i \in \{2, 3, \ldots, q\}$, let

$$D_{x,k+i-1} = (D_{x,k} \setminus \{v_{4k}\}) \cup \{u_i\}.$$

Hence, for each i, the sets $D_{1,k+i-1}, D_{2,k+i-1}, \ldots, D_{k+1,k+i-1}$ are the only $\gamma_{pr}(L_{4k,q})$-sets containing the pair $\{u_1, u_i\}$ and they form a path. We also get that, for each x, $D_{x,k}, D_{x,k+1}, \ldots, D_{x,k+q-1}$ are pairwise adjacent. Note that $D_{x,y}$ with $y < k$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}, \{v_{4k}, u_1\}$, so $(D_{x,y} \setminus \{v_{4k}\}) \cup \{u_i\}$ is not a dominating set for all i. This means that $D_{x,y}$ with $y < k$ is not adjacent to every $\gamma_{pr}(L_{4k,q})$-set containing the pair $\{u_1, u_i\}$. Now, all $\gamma_{pr}(L_{4k,q})$-sets containing u_1 form a graph $A_{k+1,k+q-1}$ in $PD_\gamma(L_{4k,q})$ (see Figure 10).

We finally find all $\gamma_{pr}(L_{4k,q})$-sets that do not contain u_1. Then these sets contain exactly two vertices from $\{u_2, u_3, \ldots, u_q\}$. Note that such a $\gamma_{pr}(L_{4k,q})$-set is a union of a $\gamma_{pr}(P_{4k})$-set and $\{u_i, u_j\}$ for some distinct $i, j \in \{2, 3, \ldots, q\}$. Clearly, $D = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-1\}$ is a unique $\gamma_{pr}(P_{4k})$-set. Thus, $D \cup \{u_i, u_j\}$ is the only $\gamma_{pr}(L_{4k,q})$-set containing the pair $\{u_i, u_j\}$. Recall that, for each $i \in \{2, 3, \ldots, q\}$, $D_{k+1,k+i-1}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}, \{u_1, u_i\}$. Then $D_{k+1,k+i-1}$ is a union of a $\gamma_{pr}(P_{4k-4})$-set and $\{v_{4k-2}, v_{4k-1}, u_1, u_i\}$, and thus $D_{k+1,k+i-1} = \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k-2\} \cup \{v_{4k-2}, v_{4k-1}, u_1, u_i\} = D \cup \{u_1, u_i\}$. For all $1 \leq i < j \leq q$, let

$$D^{i,j} = D \cup \{u_i, u_j\}.$$

Theorem 2.5 implies that all $D^{i,j}$'s form a graph A_{q-1} in $PD_\gamma(L_{4k,q})$ (see Figure 10). Note that $D_{x,y}$ with $y \leq k$ does not contain $u_2, u_3, \ldots, u_q$, so it is not adjacent to $D^{i,j}$ for all $2 \leq i < j \leq q$. Recall that, for each $i \in \{2, 3, \ldots, q\}$, $D_{x,k+i-1}$ with $x \leq k$ contains the pairs

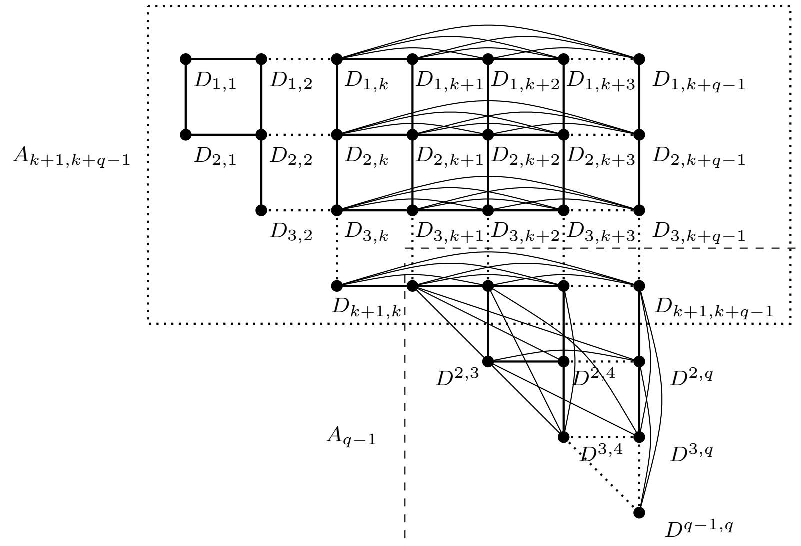

Figure 10: The graph $B_{k+1,k+q-1}$

$\{v_{4k-3}, v_{4k-2}\}, \{u_1, u_i\}$, so $(D_{x,k+i-1} \setminus \{u_1\}) \cup \{u_j\}$ is not a dominating set for $j \neq 1$, and thus $D_{x,k+i-1}$ is not adjacent to $D^{i,j}$ for all $2 \leq i < j \leq q$. This completes the proof. $\square$

Let p, q and r be positive integers. Let $A_{p,q,r}$ be the graph with $V(A_{p,q,r}) = V(SG_{p,q,r})$ and

$$\begin{aligned} E(A_{p,q,r}) &= E(SG_{p,q,r}) \cup \{(u_x, v_y, w_z)(u_x, v_{y'}, w_z) : r+2 \leq y+2 \leq y' \leq q\} \cup \\ &\quad \{(u_r, v_r, w_r)(u_{r+1}, v_{y'}, w_r) : r+2 \leq y' \leq q\}. \end{aligned}$$

Let $B_{p,q,r}$ be the graph with

$$V(B_{p,q,r}) = V(A_{p,q,r}) \cup \{(u_x, v_y, w_z) : 1 \leq x \leq p, r+1 \leq z < y \leq q\}$$

and

$$\begin{aligned} E(B_{p,q,r}) &= E(A_{p,q,r}) \cup \{(u_x, v_y, w_z)(u_x, v_y, w_{z'}) : r+2 \leq y \leq q, r \leq z < z' \leq y-1\} \cup \\ &\quad \{(u_x, v_y, w_z)(u_x, v_{y'}, w_z) : r+1 \leq z \leq q-2, z+1 \leq y < y' \leq q\} \cup \\ &\quad \{(u_x, v_y, w_z)(u_x, v_{y'}, w_y) : r \leq z < y < y' \leq q\} \cup \\ &\quad \{(u_x, v_y, w_z)(u_{x+1}, v_y, w_z) : r < z < q\}. \end{aligned}$$

The graphs $A_{4,5,3}$ and $A_{3,5,2}$ are shown in Figure 11, while the graphs $B_{4,5,3}$ and $B_{3,5,2}$ are shown in Figure 12, where we write (x, y, z) instead of (u_x, v_y, w_z). We observe that if $q = r$ or $q = r+1$, then $A_{p,q,r} \cong B_{p,q,r} \cong SG_{p,q,r}$.

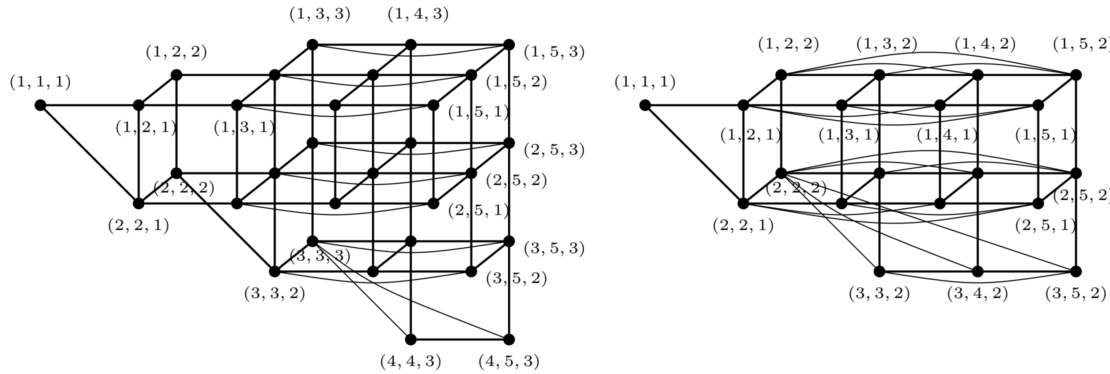

Figure 11: The graphs $A_{4,5,3}$ (left) and $A_{3,5,2}$ (right)

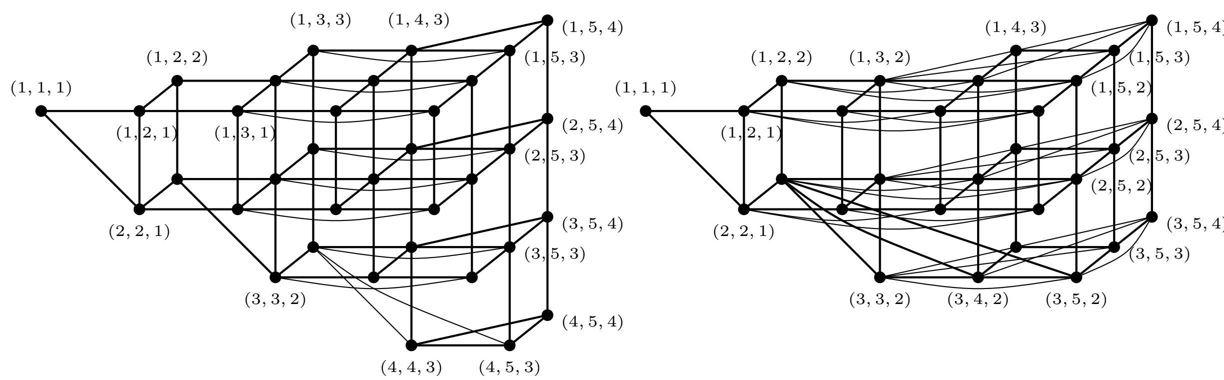

Figure 12: The graphs $B_{4,5,3}$ (left) and $B_{3,5,2}$ (right)

Theorem 4.4. *Let $k \geq 1$ and $q \geq 2$ be integers. Then $PD_\gamma(L_{4k-1,q}) \cong B_{k+1,k+q-1,k}$.*

Proof. If $q = 2$, then $L_{4k-1,q} \cong P_{4k+1}$, so $PD_\gamma(L_{4k-1,2}) \cong SG_{k+1,k+1,k} \cong B_{k+1,k+1,k}$ by Theorem 2.4. Let $q \geq 3$. We first find all $\gamma_{pr}(L_{4k-1,q})$-sets containing the vertex u_1. For each $i \in \{2, 3, \ldots, q\}$, let P^i be the subgraph of $L_{4k-1,q}$ induced by $\{v_1, v_2, \ldots, v_{4k-1}, u_1, u_i\}$, and then $PD_\gamma(P^i) \cong SG_{k+1,k+1,k}$ by Theorem 2.4. For all $x, y \in \{1, 2, \ldots, k+1\}$, $z \in \{1, 2, \ldots, k\}$ with $x - y \leq 0, x - z \leq 1, y - z \geq 0$ and for each $i \in \{2, 3, \ldots, q\}$, let $D^i_{x,y,z}$ be the $\gamma_{pr}(P^i)$-set at the position (x, y, z) in $SG_{k+1,k+1,k}$. By Corollary 2.2, without loss of generality, we may assume that $D^i_{x,k+1,z}$ contains the pair $\{u_1, u_i\}$ and $D^i_{x,y,z}$ contains the pair $\{v_{4k-1}, u_1\}$ for all $y \neq k+1$. Note that, for $y \neq k+1$, we have $D^i_{x,y,z} = D^j_{x,y,z}$ for all $i, j \in \{2, 3, \ldots, q\}$, and then we let $D_{x,y,z} = D^i_{x,y,z}$. Note that $\gamma_{pr}(P^i) = 2k + 2 = \gamma_{pr}(L_{4k-1,q})$. Hence, each $\gamma_{pr}(P^i)$-set is a $\gamma_{pr}(L_{4k-1,q})$-set for all $i \in \{2, 3, \ldots, q\}$. Therefore, $D_{x,y,z}$ with $y \neq k+1$ is a $\gamma_{pr}(L_{4k-1,q})$-set containing the pair $\{v_{4k-1}, u_1\}$, and $D^i_{x,k+1,z}$ is a $\gamma_{pr}(L_{4k-1,q})$-set containing the pair $\{u_1, u_i\}$ for each $i \in \{2, 3, \ldots, q\}$. We claim that $D^i_{x,k+1,z}$ is adjacent to $D^j_{x,k+1,z}$ for all $i \neq j$. By Corollary 2.2(1), for $x, z \in \{1, 2, \ldots, k\}$, $D^i_{x,k+1,z} = (D_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{u_i\} = [(D_{x,k,z} \setminus \{v_{4k-1}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{x,k+1,z} \setminus \{u_j\}) \cup \{u_i\}$, and $D^i_{k+1,k+1,k} = (D_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{u_i\} =$

$[(D_{k,k,k} \setminus \{v_{4k-3}\}) \cup \{u_j\}] \setminus \{u_j\} \cup \{u_i\} = (D^j_{k+1,k+1,k} \setminus \{u_j\}) \cup \{u_i\}$. The claim holds. For each $i \in \{2, 3, \ldots, q\}$, let $D_{x,k+i-1,z} = D^i_{x,k+1,z}$. Note that every $\gamma_{pr}(L_{4k-1,q})$-set containing u_1 is a $\gamma_{pr}(P^i)$-set for some $i \in \{2, 3, \ldots, q\}$, so all $D_{x,y,z}$'s are the only $\gamma_{pr}(L_{4k-1,q})$-sets containing u_1 and they form a graph $A_{k+1,k+q-1,k}$ in $PD_\gamma(L_{4k-1,q})$ (see Figure 11 (left) for $k = 3$ and $q = 2$).

We next find all $\gamma_{pr}(L_{4k-1,q})$-sets that do not contain the vertex u_1. Then such a $\gamma_{pr}(L_{4k-1,q})$-set is a union of a $\gamma_{pr}(P_{4k-1})$-set and $\{u_i, u_j\}$ for some distinct $i, j \in \{2, 3, \ldots, q\}$. By Theorem 2.2, $PD_\gamma(P_{4k-1}) \cong P_{k+1} \cong D_1 D_2 \cdots D_{k+1}$, where D_x is a $\gamma_{pr}(P_{4k-1})$-set for all $x \in \{1, 2, \ldots, k+1\}$. By Lemma 2.3, without loss of generality, we may assume that D_{k+1} contains the pair $\{v_{4k-2}, v_{4k-1}\}$. For all $x \in \{1, 2, \ldots, k+1\}$ and $2 \le i < j \le q$, let $D^{i,j}_x = D_x \cup \{u_i, u_j\}$. Thus, for each pair of i and j, the sets $D^{i,j}_1, D^{i,j}_2, \ldots, D^{i,j}_{k+1}$ are the only $\gamma_{pr}(L_{4k-1,q})$-sets containing the pair $\{u_i, u_j\}$ and they form a path in $PD_\gamma(L_{4k-1,q})$. By Corollary 2.1, for all $x \in \{1, 2, \ldots, k\}$ and $2 \le i < j \le q$,

$$D^{i,j}_x = D_x \cup \{u_i, u_j\} = S_x \cup \{v_{4k-3}, v_{4k-2}, u_i, u_j\},$$

where S_x is a $\gamma_{pr}(P_{4k-5})$-set and especially S_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and

$$D^{i,j}_{k+1} = D_{k+1} \cup \{u_i, u_j\} = S_k \cup \{v_{4k-2}, v_{4k-1}, u_i, u_j\}.$$

For all $x \in \{1, 2, \ldots, k+1\}$ and $i \in \{2, 3, \ldots, q\}$, let $D^{1,i}_x = D_{x,k+i-1,k} = D^i_{x,k+1,k}$. By Corollary 2.2(2), for all $x \in \{1, 2, \ldots, k\}$ and $i \in \{2, 3, \ldots, q\}$, we have

$$D^{1,i}_x = D^i_{x,k+1,k} = S'_x \cup \{v_{4k-3}, v_{4k-2}, u_1, u_i\},$$

where S'_x is a $\gamma_{pr}(P_{4k-5})$-set and particularly S'_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and

$$D^{1,i}_{k+1} = D^i_{k+1,k+1,k} = S'_k \cup \{v_{4k-2}, v_{4k-1}, u_1, u_i\}.$$

By Lemma 2.3, we get $S_k = S'_k$. Theorem 2.2 shows that $S_x = S'_x$ for all $x \in \{1, 2, \ldots, k\}$. Therefore, for each $x \in \{1, 2, \ldots, k+1\}$, all $D^{i,j}_x$'s with $1 \le i < j \le q$ form a graph A_{q-1} in $PD_\gamma(L_{4k-1,q})$ (see Figure 13).

Let $D = \{D^{i,j}_x : 1 \le x \le k+1, 2 \le i < j \le q\}$. Note that $D_{x,y,z}$ with $y \le k$ does not contain $u_2, u_3, \ldots, u_q$, so it is not adjacent to any set in D. By Corollary 2.2(3), for each $i \in \{2, 3, \ldots, q\}$, $D_{x,k+i-1,z} = D^i_{x,k+1,z}$ with $z < k$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}, \{u_1, u_i\}$, so $(D_{x,k+i-1,z} \setminus \{u_1\}) \cup \{u_j\}$ is not a dominating set for all $j \ne 1$. This implies that $D_{x,k+i-1,z}$ is not adjacent to any set in D. Therefore, all $\gamma_{pr}(L_{4k-1,q})$-sets form a graph $B_{k+1,k+q-1,k}$. □

5. γ-Paired Dominating Graphs of Umbrella Graphs and Coconut Graphs

Let p and q be positive integers. If $q = 1$, then $U_{p,q} \cong P_{p+1} \cong C_{p,q}$, and thus $PD_\gamma(U_{p,q})$ and $PD_\gamma(C_{p,q})$ can be obtained from Theorems 2.1 - 2.4. Let $q \ge 2$. If $p = 4k+2$ for some $k \ge 0$, then it is easy to check that $\{v_{4i+2}, v_{4i+3} : 0 \le i \le k-1\} \cup \{v_{4k+2}, u_1\}$ is the only $\gamma_{pr}(U_{p,q})$-set and the only $\gamma_{pr}(C_{p,q})$-set, so we get the following theorem immediately.

Theorem 5.1. *Let $k \ge 0$ and $q \ge 2$ be integers. Then $PD_\gamma(U_{4k+2,q}) \cong P_1 \cong PD_\gamma(C_{4k+2,q})$.*

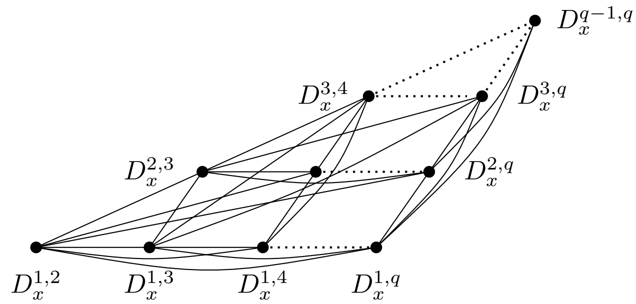

Figure 13: The graph A_{q-1} formed by all $D_x^{i,j}$'s with $1 \leq i < j \leq q$

Lemma 5.1. *Let $k \geq 0$ and $q \geq 2$ be integers. Then each $\gamma_{pr}(U_{4k+1,q})$-set contains the vertex u_1.*

Proof. If $q = 2$, then u_1 is a support vertex of $U_{4k+1,q}$, so this lemma holds by Lemma 2.1. Let $q \geq 3$ and suppose that there is a $\gamma_{pr}(U_{4k+1,q})$-set D such that $u_1 \notin D$. Then D must contain at least two vertices from $\{u_2, u_3, \ldots, u_q\}$. Recall that $|D| = 2k + 2$, so at most $2k$ vertices of D must dominate all vertices in P_{4k+1}, which is impossible. □

Theorem 5.2. *Let $k \geq 0$ and $q \geq 2$ be integers. Then $PD_\gamma(U_{4k+1,q}) \cong L_{k,q} \cong PD_\gamma(C_{4k+1,q})$.*

Proof. By Theorem 3.1, $\gamma_{pr}(U_{4k+1,q}) = \gamma_{pr}(L_{4k+1,q}) = \gamma_{pr}(C_{4k+1,q})$. Lemmas 2.1 and 5.1 imply that every $\gamma_{pr}(C_{4k+1,q})$-set and every $\gamma_{pr}(U_{4k+1,q})$-set contains either the pair $\{v_{4k+1}, u_1\}$ or $\{u_1, u_i\}$ where $i \neq 1$. We follow the steps in the proof of Theorem 4.2, so we are done. □

Let $k \geq 1$ be an integer. If $q \in \{2, 3\}$, then $U_{4k,q} \cong L_{4k,q}$, and hence $PD_\gamma(U_{4k,q}) \cong B_{k+1,k+q-1}$ by Theorem 4.3. Let $q \geq 4$. Note that every $\gamma_{pr}(U_{4k,q})$-set is a $\gamma_{pr}(L_{4k,q})$-set, but the converse need not be true for some $\gamma_{pr}(L_{4k,q})$-set that does not contain u_1. From the proof of Theorem 4.3, we know that each $\gamma_{pr}(L_{4k,q})$-set that does not contain u_1 is $D^{i,j} = D \cup \{u_i, u_j\}$, where D is a $\gamma_{pr}(P_{4k})$-set and $2 \leq i < j \leq q$. Similarly, each $\gamma_{pr}(U_{4k,q})$-set that does not contain u_1 is of the form $D \cup \{u_i, u_j\}$ for some $2 \leq i < j \leq q$. For $q = 4$, we have $D^{2,4}$ is a $\gamma_{pr}(L_{4k,4})$-set but not a $\gamma_{pr}(U_{4k,4})$-set, so $PD_\gamma(U_{4k,4}) \cong PD_\gamma(L_{4k,4}) - \{D^{2,4}\}$. For $q = 5$, only $D^{3,4}$ is a $\gamma_{pr}(U_{4k,5})$-set among all $\gamma_{pr}(L_{4k,5})$-sets containing the pair $\{u_i, u_j\}$ where $2 \leq i < j \leq 5$, and thus $PD_\gamma(U_{4k,5}) \cong PD_\gamma(L_{4k,5}) - \{D^{2,3}, D^{2,4}, D^{2,5}, D^{3,5}, D^{4,5}\}$.

Corollary 5.1. *Let $k \geq 1$ and $q \geq 6$ be integers. Then $PD_\gamma(U_{4k,q}) \cong A_{k+1,k+q-1}$.*

Proof. Recall that $\gamma_{pr}(U_{4k,q}) = \gamma_{pr}(L_{4k,q})$. Similar to Lemma 5.1, we can prove that each $\gamma_{pr}(U_{4k,q})$-set contains u_1, and then it contains either the pair $\{v_{4k}, u_1\}$ or $\{u_1, u_i\}$ where $i \neq 1$. Then we follow the first two paragraphs of the proof in Theorem 4.3. □

By Lemma 2.1, each $\gamma_{pr}(C_{4k,q})$-set contains u_1. Again, we follow the first two paragraphs of the proof in Theorem 4.3, so we get the following corollary.

Corollary 5.2. *Let $k \geq 1$ and $q \geq 2$ be integers. Then $PD_\gamma(C_{4k,q}) \cong A_{k+1,k+q-1}$.*

Let $k \geq 1$ be an integer. By Theorem 4.4, we get that $PD_\gamma(U_{4k-1,q}) \cong PD_\gamma(L_{4k-1,q}) \cong B_{k+1,k+q-1,k}$ for $q \in \{2,3\}$. Let $q \geq 4$. In the proof of Theorem 4.4, we know $D_1^{i,j}, D_2^{i,j}, \ldots, D_{k+1}^{i,j}$ are the only $\gamma_{pr}(L_{4k-1,4})$-sets containing the pair $\{u_i, u_j\}$ where $2 \leq i < j \leq q$. Note that $D_1^{2,4}, D_2^{2,4}, \ldots, D_{k+1}^{2,4}$ are not $\gamma_{pr}(U_{4k-1,4})$-sets, so $PD_\gamma(U_{4k-1,4}) \cong PD_\gamma(L_{4k-1,4}) - \{D_x^{2,4} : 1 \leq x \leq k+1\}$. Among all $\gamma_{pr}(L_{4k-1,5})$-sets containing the pair $\{u_i, u_j\}$ for $2 \leq i < j \leq 5$, only $D_1^{3,4}, D_2^{3,4}, \ldots, D_{k+1}^{3,4}$ are $\gamma_{pr}(U_{4k-1,5})$-sets, so we get that $PD_\gamma(U_{4k-1,5}) \cong PD_\gamma(L_{4k-1,5}) - \{D_x^{2,3}, D_x^{2,4}, D_x^{2,5}, D_x^{3,5}, D_x^{4,5} : 1 \leq x \leq k+1\}$.

We can easily check that $\gamma_{pr}(U_{4k-1,q}) = \gamma_{pr}(L_{4k-1,q}) = \gamma_{pr}(C_{4k-1,q})$, every $\gamma_{pr}(U_{4k-1,q})$-set contains u_1 for $q \geq 6$, and every $\gamma_{pr}(C_{4k-1,q})$-set contains u_1 for $q \geq 2$. We can obtain the following results by repeating the steps of proof in Theorem 4.4 (first paragraph).

Corollary 5.3. *Let $k \geq 1$ and $q \geq 6$ be integers. Then $PD_\gamma(U_{4k-1,q}) \cong A_{k+1,k+q-1,k}$.*

Corollary 5.4. *Let $k \geq 1$ and $q \geq 2$ be integers. Then $PD_\gamma(C_{4k-1,q}) \cong A_{k+1,k+q-1,k}$.*

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