Electronic Journal of Graph Theory and Applications **12** (2) (2024), 273–288

Electronic Journal of
Graph Theory and Applications

1-well-covered graphs containing a clique of size $n/3$

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Abstract

A graph is *well-covered* if all of its maximal independent sets have the same size. A graph that remains well-covered upon the removal of any vertex is called a *1-well-covered* graph. These graphs, when they have no isolated vertices, are also known as $\mathbf{W}_2$ graphs. It is well-known that every graph $G \in \mathbf{W}_2$ has two disjoint maximum independent sets. In this paper, we investigate connected $\mathbf{W}_2$ graphs with n vertices that contain a clique of size $n/3$. We prove that if the removal of two disjoint maximum independent sets from a graph $G \in \mathbf{W}_2$ leaves a clique of size at least 3, then G contains a clique of size $n/3$. Using this result, we provide a complete characterization of these graphs, based on eleven graph families.

Keywords: independent set, clique, matching, well-covered
Mathematics Subject Classification : 05C69,05C70, 05C99
DOI: 10.5614/ejgta.2024.12.2.8

1. Introduction

We consider only simple, finite, and undirected graphs, and use standard terminology. A set of vertices in a graph is called *independent* if none of its vertices share an edge. An independent set that has the largest possible size is referred to as a *maximum independent set*. The number of vertices in the largest independent set of a graph G is known as the *independence number*, denoted by $\alpha(G)$. The problem of identifying graphs where every maximal independent set is also a maximum independent set was introduced by M.D. Plummer in 1970, who referred to such

Received: 17 February 2022, Revised: 18 August 2024, Accepted: 23 August 2024.

graphs as *well-covered*. Since then, numerous studies have been conducted on this topic. Identifying well-covered graphs is generally a co-NP-complete problem [4, 19]. However, certain subclasses of well-covered graphs can be recognized in polynomial time [5, 8, 3, 11].

Staples introduced $\mathbf{W}_2$ graphs in 1979 as graphs in which any two disjoint independent sets are contained within two disjoint maximum independent sets [20]. These graphs are also referred to as *1-well-covered* graphs without isolated vertices, meaning they remain well-covered even after the removal of any vertex. Hence, a graph G belongs to $\mathbf{W}_2$ if and only if G is 1-well-covered without isolated vertices [20]. After the initial exploration of fundamental properties of 1-well-covered graphs in [20], various studies focused on specific subclasses. Pinter characterized two categories of planar 1-well-covered graphs: those that are 4-regular and 3-connected [15], and those with girth 4 [17]. He also developed constructions for infinite families of 1-well-covered graphs with girth 4 [18]. Subsequently, Hartnell provided a characterization of 1-well-covered graphs without 4-cycles in [12]. Hoang and Trung [13] gave a characterization of the $\mathbf{W}_2$ graphs satisfying the condition that every triangle is also a dominating set for the graph. Recently, Deniz and Ekim investigated edge stable equimatchable graphs which actually coincide 1-well-covered line graphs [7]. Also, Levit and Mandrescu gave some characterizations of 1-well-covered graphs in terms of the existence of special independent sets [14]. More recently, Deniz [6] gave a detailed study on a classification of 1-well-covered graphs with respect to their independence and matching numbers.

A vertex x of a graph G is called *shedding* if for every independent set S in $G - N_G[x]$, there is a vertex $v \in N_G(x)$ so that $S \cup \{v\}$ is independent. $\mathbf{W}_2$ graphs are also known as graphs in which every vertex is a shedding vertex. In fact, Levit and Mandrescu showed in [14] that for a vertex v in a well-covered graph G without isolated vertices $G - v$ is well-covered if and only if v is shedding. Shedding vertices are closely connected to independence complexes of graphs in combinatorial topology. Specifically, they are crucial in determining vertex-decomposable graphs, as there must be an ordering of shedding vertices in a graph G to classify it as vertex-decomposable [2, 21].

In this paper, we study 1-well-covered graphs with n vertices that contain a clique of size $n/3$. Note that every graph $G \in \mathbf{W}_2$, where $\mathbf{W}_2$ is the class of 1-well-covered graph without isolated vertices, has two disjoint maximum independent sets. We show that for a graph $G \in \mathbf{W}_2$ if $G - (I_1 \cup I_2)$ is a clique of size t for disjoint maximum independent sets I_1 and I_2, then G has at most $3t$ vertices.

Theorem 1.1. *Let $G \in \mathbf{W}_2$ with n vertices, and suppose that I_1 and I_2 are disjoint maximum independent sets. If $S = V(G) - (I_1 \cup I_2)$ induces a clique of size at least 3 in G, then $n \leq 3|S|$.*

Notice that if G is a graph as described in Theorem 1.1, then G has at most $3|S|$ vertices. Since G has two disjoint maximum independent sets, we have $\alpha(G) \leq |S|$. This implies that G has a clique of size at least $n/3$. Hence, for a connected graph $G \in \mathbf{W}_2$, if $S = V(G) - (I_1 \cup I_2)$ induces a clique of size at least 3 for a pair of disjoint maximum independent sets I_1 and I_2, then G has a clique of size at least $n/3$.

For a given graph in $\mathbf{W}_2$, we show how to construct an infinite family of $\mathbf{W}_2$ graphs. We then divide categorize the graphs for which $G - (I_1 \cup I_2)$ is a clique for disjoint maximum independent sets I_1 and I_2, into three subclasses with respect to their independence numbers.

These results allow us to achieve a complete characterization of such graphs, presented as a list of eleven graph families.

Theorem 1.2. *A connected graph G is in $\mathbf{W}_2$ such that the removal of two disjoint maximum independent sets from G leaves a clique if and only if G belongs to one of the graph classes $\mathcal{C}(G_2), \mathcal{C}(G_3), \ldots, \mathcal{C}(G_9), \mathcal{C}(K_2), \mathcal{C}(C_5)$ and $\mathcal{C}(K_t \circ K_2)$ for $t \geq 2$ (see Figures 3 and 6).*

The remainder of this paper is organized as follows. In Section 2, we begin with definitions and preliminary results related to 1-well-covered graphs. In Section 3, we introduce the graph $G(u; w)$ for a given 1-well-covered graph G and a vertex $u \in V(G)$. Finally, in Section 4, we consider the graph G belonging to $\mathbf{W}_2$ for which $S = G - (I_1 \cup I_2)$ induces a clique in G, where I_1 and I_2 are two disjoint maximum independent sets.

2. Preliminaries

Let $G = (V, E)$ be graph and given a subset of vertices S, the subgraph induced by S in G is denoted by $G[S]$, and $G \setminus S$ represents the subgraph induced by $V \setminus S$, i.e., $G[V \setminus S]$. When S consists of a single vertex v, we denote $G \setminus S$ by $G - v$. The graph $G - S$ thus corresponds to the subgraph $G[V(G) - S]$. For a vertex v, the *open neighborhood* of v in a subgraph H is denoted by $N_H(v)$, and the *closed neighborhood* of v, denoted by $N_H[v]$, is $N_H(v) \cup v$. If the subgraph H is clear from context, the subscript H is omitted. For a subset $S \subseteq V$, $N_H(S)$ (resp. $N_H[S]$) represents the union of the open (resp. closed) neighborhoods of the vertices in S. We say that S is *complete* to T for $S, T \subset V(G)$ if every vertex in S is adjacent to all vertices in T. Additionally, we use the notation $[k]$ to refer the set $1, 2, \ldots, k$.

We use the notation K_n, C_n, and P_n to represent the complete graph, cycle, and path on n vertices, respectively. Additionally, $K_{r,s}$ denotes the complete bipartite graph for any $r, s \geq 1$. The notation rK_2 refers to a graph consisting of r components, each being K_2. A graph G is said to be F-free if none of its induced subgraphs is isomorphic to F. The notations $d_G(x)$, $\Delta(G)$, and $\delta(G)$ represent the degree of a vertex x, the maximum and minimum degrees of a graph G, respectively. A vertex with degree one is called *leaf*, while a vertex with degree zero is called *leaf*. A subgraph of G that is isomorphic to a complete graph is referred to as a *clique*. The *clique number* of a graph G, denoted by $\omega(G)$, represents the number of vertices in the largest clique in G. A matching is a collection of edges in G such that no two edges share a common endpoint. The maximum size of a matching in G is known as the *matching number* of G, denoted by $\mu(G)$. A matching M *saturates* a vertex v if v is an endpoint of an edge in M; otherwise, the vertex v is considered *unsaturated* by M. A vertex u in a graph G is said to be *dominated* by another vertex $v \in V(G) \setminus u$ if $N_G[u] \subseteq N_G[v]$. A subset $S \subseteq V(G)$ *dominates* a set of vertices T if every vertex in T is adjacent to at least one vertex in S. Recall that each graph in $\mathbf{W}_2$ has two disjoint maximum independent sets. For simplicity, we will refer to these as *DMI sets*.

We begin by stating some established results related to well-covered graphs, which will be used in the remainder of the paper.

Theorem 2.1. *[1] Let S be an independent set in a graph G. Then, every independent set disjoint from S can be matched into S if and only if S is maximum.*

Theorem 2.2. *[14] Let G be a well-covered graph and let v be a non-isolated vertex. Then v is a shedding vertex. if and only if $G - v$ is well-covered.*

It directly follows from Theorem 2.2 that every vertex in a graph $G \in \mathbf{W}_2$ is a shedding vertex.

Theorem 2.3. *[20] The graph $G \in \mathbf{W}_2$ if and only if $\alpha(G - v) = \alpha(G)$ and $G - v$ is well-covered, for every $v \in V(G)$.*

Theorem 2.3 shows that $\mathbf{W}_2$ graphs and 1-well-covered graphs are equivalent when the graph has no isolated vertices. Therefore, 1-well-covered graphs without isolated vertices are the same as $\mathbf{W}_2$ graphs. Additionally, when the graph is connected, these two graph families coincide. Hence, we typically use the $\mathbf{W}_2$ notation instead of referring to connected 1-well-covered graphs.

Recall that while every vertex of a graph in $\mathbf{W}_2$ is a shedding vertex, the converse is not true; that is, a graph where each vertex is a shedding vertex does not necessarily belong to $\mathbf{W}_2$. Indeed, the graph H_1 in Figure 1 has the property that each of its vertices is a shedding vertex, yet H_1 is not in $\mathbf{W}_2$. Consider the other graphs in Figure 1. The graph H_2 is a well-covered but does not belong to $\mathbf{W}_2$. The graph H_3 belongs to $\mathbf{W}_2$ and is also well-covered. Finally, the graph H_4 is neither well-covered nor a member of $\mathbf{W}_2$.

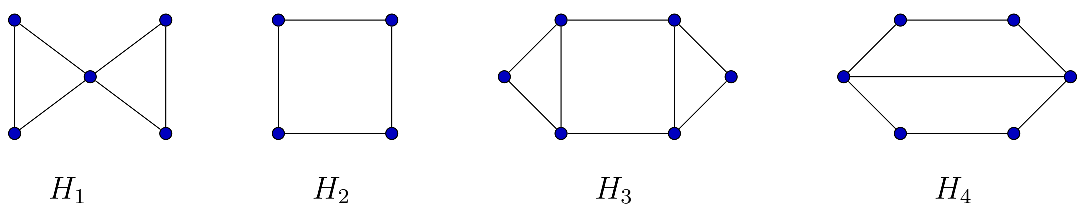


Figure 1: The graphs H_1, H_2, H_3, and H_4.

Proposition 2.1. *[14]*

(i) *If G is a connected graph in $\mathbf{W}_2$ with n vertices such that $\alpha(G) + \mu(G) = n$, then G is isomorphic to K_2.*

(ii) *The only connected bipartite graph belonging to $\mathbf{W}_2$ is K_2.*

Lemma 2.1. *[16] Let G be a graph in $\mathbf{W}_2$. Then, the graph $G - N_G[S]$ is in $\mathbf{W}_2$ for every independent set S in G. In particular, $\alpha(G) = \alpha(G - N_G[S]) + |S|$.*

We note that if v is a shedding vertex in a graph G, then it follows from the definition of shedding vertex that there is no independent set S in $G - N_G[v]$ that dominates $N_G(v)$. This, in particular, implies that G does not have any dominated vertices.

Corollary 2.1. *If G is a connected graph with at least 3 vertices, then no shedding vertex in G can be a leaf vertex. In particular, when $G \in \mathbf{W}_2$, it follows that $\delta(G) \geq 2$.*

According to [14, Corollary 2.12], the only connected graphs in $\mathbf{W}_2$ with order $2\alpha(G)+1$ are C_3 and C_5. From this, the following observation can be made.

Corollary 2.2. *Let $G \in \mathbf{W}_2$. Then $G - x$ is a bipartite well-covered graph for some $x \in V(G)$ if and only if G is C_3 or C_5.*

By definition, a graph G belongs to $\mathbf{W}_2$ if any two disjoint independent sets in G can be extended to two DMI sets. Thus, in a graph $G \in \mathbf{W}_2$, every pair of disjoint independent sets can be expanded to form two DMI sets. We often use this property of $\mathbf{W}_2$ in order to show that a graph belongs to the class $\mathbf{W}_2$.

3. Insertion and deletion of vertices in 1-well-covered graphs

In a 1-well-covered graph, by definition, the removal of any vertex does not change its well-coveredness property while it may not to be 1-well-covered. In this section, we investigate these graphs of when it is possible to add (or delete) a vertex in the graph under preserving its 1-well-covered property.

Definition 3.1. Any two vertices u, v in a graph G are said to be *twins* if u and v have the same set of neighbours, that is, if $N_G(u) = N_G(v)$. We make a slight modification to this definition as follows; a pair u, v in G is called a *c-twin* if $N_G[u] = N_G[v]$.

Given a connected graph G and a vertex $u \in V(G)$. We define the graph $G(u : w)$ as a graph obtained from G by adding a new vertex w to G and make adjacent w with all vertices of $N_G[u]$. Namely, $V(G(u : w)) = V(G) \cup \{w\}$ and $E(G(u : w)) = E(G) \cup \{wv : v \in N_G[u]\}$. Observe that u and w are c-twin vertices in the graph $G(u : w)$. For instance, if $G = C_5$, and u is any vertex in G, then $G(u : w)$ is the graph depicted in Figure 2.

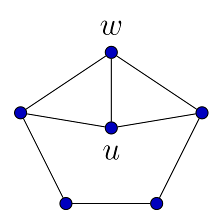


Figure 2: The graph $G(u : w)$.

It can be easily observed that $\alpha(G) = \alpha(G(u : w))$ for every graph G and $u \in V(G)$. We next show that $G(u : w)$ preserves its 1-well-covered property.

Theorem 3.1. *Let $G \in \mathbf{W}_2$ and $u \in V(G)$. Then $G(u : w)$ is in $\mathbf{W}_2$ as well.*

Proof. We pick two disjoint independent sets T_1, T_2 in $G(u : w)$, and we extend them to two DMI sets in $G(u : w)$ so that $G(u : w)$ belongs to $\mathbf{W}_2$.

First, if $w \notin T_1 \cup T_2$, then there exist two DMI sets in $G(u : w)$ containing T_1 and T_2, since $G \in \mathbf{W}_2$. Therefore, we further assume that $w \in T_1 \cup T_1$. Note that w cannot be in both T_1

and T_2, since they are disjoint. Assume without loss of generality that $w \in T_1$. Notice that $N_G[u] = N_{G(u:w)}[u] - w$, and $T_1 \cap N_G[u] = \{w\}$.

Let $u \in T_2$. Obviously $T_2 \cap N_G[u] = \{u\}$. By Lemma 2.1, $G - N_G[u]$ is in $\mathbf{W}_2$, which implies that there exist two DMI sets S_1, S_2 in $G - N_G[u]$ containing $T_1 - w$ and $T_2 - u$, respectively. Then the sets $S_1 \cup \{w\}$ and $S_2 \cup \{u\}$ are two DMI sets in $G(u : w)$ containing T_1 and T_2, respectively, as claimed.

Let $u \notin T_2$. Consider the sets $(T_1 - w) \cup \{u\}$ and T_2, they are clearly disjoint. Since $G \in \mathbf{W}_2$, we can extend $(T_1 - w) \cup \{u\}$ and T_2 to two DMI sets S_1 and S_2 in G, respectively. Thus, the sets $S' = (S_1 - u) \cup \{w\}$ and S_2 are DMI sets in $G(u : w)$ containing T_1 and T_2, respectively, as claimed. Hence, G is in $\mathbf{W}_2$. □

Corollary 3.1. *If G is well-covered, and $u \in V(G)$, then $G(u : w)$ is well-covered as well.*

In a well-covered graph G, a vertex $w \in V(G)$ is said to be *extendable* if $G - w$ is well-covered and $\alpha(G) = \alpha(G - w)$. Extendable vertices were used in [9, 10] in order to construct some families of well-covered graphs.

Following Theorem 3.1, it turns out that the vertices u and w in the graph $G(u : w)$ are extendable.

Corollary 3.2. *If G is a well-covered graph and $u \in V(G)$, then u and w are extendable vertices in the graph $G(u : w)$.*

The converse of Theorem 3.1 is not generally true since the graph $G_1 - w$ for a vertex w of degree 2 is not 1-well-covered although $G_1 \in \mathbf{W}_2$ (see Figure 3).

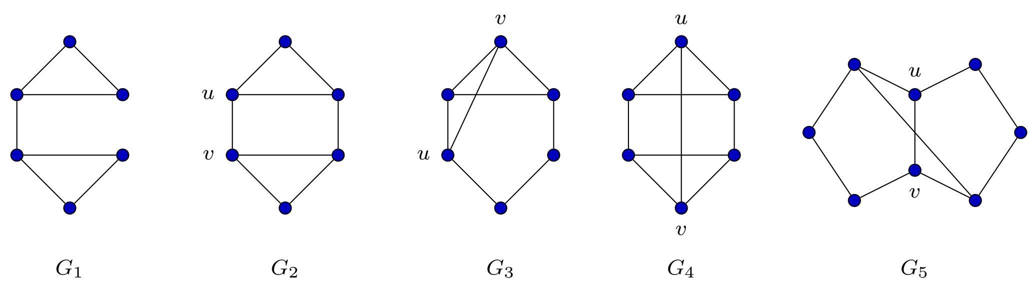


Figure 3: The graphs $G_1, G_2, \ldots, G_5$.

4. 1-well-covered graphs containing a clique of size $n/3$

A graph G belonging to $\mathbf{W}_2$ can be partitioned into three sets I_1, I_2, S where I_1 and I_2 are two disjoint independent sets in G, and $S = V(G) - (I_1 \cup I_2)$. In this section, we first bound the size of G by $3|S|$ when $G - (I_1 \cup I_2)$ is a clique for DMI sets I_1 and I_2. By using this result, we further obtain a complete characterization of those graphs.

Notice that a graph G is in $\mathbf{W}_2$ if and only if every connected component of G is in $\mathbf{W}_2$. Therefore, we will focus exclusively on connected graphs in $\mathbf{W}_2$ for the remainder of the paper.

Proposition 4.1. *Let $G \in \mathbf{W}_2$, and suppose that I_1 and I_2 are DMI sets. If $S = V(G) - (I_1 \cup I_2)$ induces a clique in G, then every vertex in S has exactly one neighbour in each of I_1, I_2.*

Proof. Suppose that $S = V(G) - (I_1 \cup I_2)$ induces a clique in G for DMI sets I_1 and I_2 with $|I_i| = r$ for $i = 1, 2$. Since G is well-covered, every vertex in S has a neighbour in each of I_1, I_2. Indeed, if there exists $u \in S$ having no neighbour in I_1, then $I_1 \cup \{u\}$ would be a maximal independent set of size $r + 1$, a contradiction.

Let $u \in S$ be given. By Lemma 2.1, $G - N_G[u]$ is in $\mathbf{W}_2$ with $\alpha(G - N_G[u]) = r - 1$. Note that the graph $G - N_G[u]$ is bipartite since S induces a clique in G. Then, by Proposition 2.1 that $G - N_G[u]$ is isomorphic to $(r - 1)K_2$. Then, we conclude that any vertex $u \in S$ cannot have more than one neighbour in I_i for $i \in \{1, 2\}$ since otherwise the graph $G - N_G[u]$ would have at most $2r - 3$ vertices, a contradiction. Consequently, every vertex in S has exactly one neighbour in each of I_1, I_2. □

Proposition 4.2. *Let $G \in \mathbf{W}_2$. Suppose $S = V(G)-(I_1 \cup I_2)$ for DMI sets $I_1 = \{x_1, x_2 \ldots, x_r\}$ and $I_2 = \{y_1, y_2 \ldots, y_r\}$ with $\{x_1y_1, x_2y_2, \ldots, x_ry_r\} \subset E(G)$. Then, for each $i \in [r]$, at least one endpoint of the edge x_iy_i is adjacent to a vertex in S.*

Proof. Assume for a contradiction that there exists an index $i \in [r]$ such that $N_G(S) \cap \{x_i, y_i\} = \emptyset$. We then deduce that $N_G(x_i) \subseteq I_2$ and $N_G(y_i) \subseteq I_1$. Recall also that, by Corollary 2.1, the minimum degree of a graph belonging to $\mathbf{W}_2$ is at least 2, so $|N_G(y_i) \cap I_1| \geq 2$. Therefore, $N_G(x_i)$ is dominated by $I_1 - x_i$. Nevertheless, this gives a contradiction since x_i is a shedding vertex. □

Notice that if G is in $\mathbf{W}_2$ with n vertices such that $G - (I_1 \cup I_2)$ is a clique for DMI sets I_1 and I_2, then G contains a clique of size $n - 2\alpha(G)$. Next let us show that G cannot contain a clique of size $n - 2\alpha(G) + 2$ when $G \neq K_n$ for $n \in \mathbb{N}$.

Proposition 4.3. *Let $G \in \mathbf{W}_2$ with n vertices, and $G \neq K_n$. For DMI sets I_1 and I_2, if $S = V(G) - (I_1 \cup I_2)$ induces a clique in G, then $|S| \leq \omega(G) \leq |S| + 1$.*

Proof. Suppose that $S = V(G) - (I_1 \cup I_2)$ induces a clique in G for DMI sets I_1 and I_2. Then, G contains a clique of size $n - 2\alpha(G) = |S|$, so $\omega(G) \geq |S|$.

Let $\alpha(G) = r$, $I_1 = \{x_1, x_2 \ldots, x_r\}$ and $I_2 = \{y_1, y_2 \ldots, y_r\}$. We may assume $\{x_1y_1, x_2y_2, \ldots, x_ry_r\} \subset E(G)$ by Theorem 2.1. Clearly $r \geq 2$ since $G \neq K_n$. Assume for a contradiction that there exist $x_i \in I_1$ and $y_j \in I_2$ such that $\{x_i, y_j\}$ is complete to S. Then $i \neq j$ by Proposition 4.2 together with Proposition 4.1. This means that x_i has at least two neighbours in I_2, which are $y_i, y_j \in I_2$. Also, $G - N_G[x_i]$ is bipartite since x_i is complete to S. Then, by Proposition 2.1 that $G - N_G[x_i]$ is isomorphic to $(r - 1)K_2$. However, $G - N_G[x_i]$ has at most $2r - 3$ vertices since x_i has at least two neighbours in I_2, a contradiction. Thus, there are no such $x_i \in I_1$ and $y_j \in I_2$. Hence, G has no clique of size $n - 2\alpha(G) + 2$. Consequently, $|S| \leq \omega(G) \leq |S| + 1$. □

We next state some technical results related to $\mathbf{W}_2$ graphs with the partition I_1, I_2 and S.

Proposition 4.4. *Let $G \in \mathbf{W}_2$. Suppose that $G - (I_1 \cup I_2)$ is a clique for DMI sets I_1 and I_2. Then every vertex in I_1 (resp. I_2) has at most two neighbours in I_2 (resp. I_1).*

Proof. We assume to the contrary that there exists $x \in I_1$ such that it has at least three neighbours in I_2. Then $\alpha(G - N_G[x]) = \alpha(G) - 1$ and $|I_2 - N_G(x)| \leq \alpha(G) - 3$. However, we cannot extend the independent sets $I_1 - x$ and $(I_2 - N_G(x)) \cup \{x\}$ into two DMI sets in G as $G - (I_1 \cup I_2)$ is a clique, a contradiction that $G \in \mathbf{W}_2$. By symmetry, the claim follows when a vertex y of I_2 has more than two neighbours in I_1. □

Lemma 4.1. *Let $G \in \mathbf{W}_2$. Suppose that for DMI sets I_1 and I_2, the set $S = V(G) - (I_1 \cup I_2)$ induces a clique of size at least 3 in G. If $\alpha(G) \geq 4$, then every vertex in $I_1 \cup I_2$ has a neighbour in S.*

Proof. Suppose that $S = V(G) - (I_1 \cup I_2)$ induces a clique of size $|S| \geq 3$ in G for DMI sets I_1 and I_2. Let $\alpha(G) = r \geq 4$, $I_1 = \{x_1, x_2 \ldots, x_r\}$ and $I_2 = \{y_1, y_2 \ldots, y_r\}$. Clearly, G has $n = 2r + |S|$ vertices. By Theorem 2.1, we may assume $\{x_1y_1, x_2y_2, \ldots, x_ry_r\} \subset E(G)$.

Assume for a contradiction that there exists $x_i \in I_1$ for $i \in [r]$ such that it has no neighbour in S. By Corollary 2.1 and Proposition 4.4, x_i has exactly two neighbours in I_2. Then, we claim that every vertex in S is adjacent to one of the neighbours of x_i in I_2. Indeed, if $u \in S$ is adjacent to none of the neighbours of x_i in I_2, then x_i and its two neighbours would survive in $G - N_G[u]$. However, $G - N_G[u]$ must consist of K_2 components by Proposition 2.1 and Lemma 2.1, a contradiction. Thus, by Proposition 4.1, there exists $y_j \in I_2$ having no neighbours in S due to $r \geq 4$. Clearly, we have $i \neq j$ by Proposition 4.2. Similarly as before, y_j has exactly two neighbours in I_1, also every vertex in S is adjacent to one of the two neighbours of y_j in I_1. Since for each vertex $s \in S$, the vertex s has a unique neighbour in I_ℓ for $\ell = 1, 2$ by Proposition 4.1 and $r \geq 4$, we deduce that $|N_G(S) \cap I_t| = 2$. It then follows from Proposition 4.2 that we have $\alpha(G) = r = 4$, and exactly one endpoint of each edge $x_\ell y_\ell$ has a neighbour in S for $\ell \in [4]$. We may then assume without loss of generality that $N_G(x_i) = \{y_3, y_4\}$, $N_G(y_j) = \{x_1, x_2\}$, and let $i = 4$, $j = 1$.

Let $u, v \in S$ such that $u \in N_G(x_1)$ and $v \in N(x_2)$. Obviously, x_1 (resp. x_2) is the unique neighbour of u (resp. v) in I_1 by Proposition 4.1. Notice also that the graph $G - N_G[u]$ is in $\mathbf{W}_2$ by Lemma 2.1, and so $G - N_G[u]$ consists of K_2 components by Proposition 2.1. However, x_2 and its two neighbours y_1, y_2 belong to $G - N_G[u]$, a contradiction. □

Result 4.1. *Let $G \in \mathbf{W}_2$. Suppose that for DMI sets I_1 and I_2, the set $S = V(G) - (I_1 \cup I_2)$ induces a clique of size at least 3 in G. If $\alpha(G) \geq 4$, then $\alpha(G) \leq |S|$.*

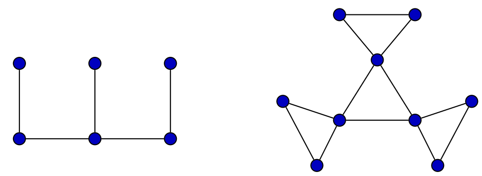
Proof. By Lemma 4.1, every vertex in $I_1 \cup I_2$ has a neighbour in S. Moreover, each vertex of S has exactly one neighbour in I_i for $i = 1, 2$ by Proposition 4.1. Thus, we conclude that $\alpha(G) \leq |S|$ as claimed. □

Let us now prove one of our main results, which will be essential for the proof of Theorem 1.2.

Theorem 1.1. *Let $G \in \mathbf{W}_2$ with n vertices, and suppose that I_1 and I_2 are DMI sets. If $S = V(G) - (I_1 \cup I_2)$ induces a clique of size at least 3 in G, then $n \leq 3|S|$.*

Proof. Suppose that I_1, I_2 are DMI sets in G, and let $S = V(G) - (I_1 \cup I_2)$ induce a clique of size at least 3 in G. By Result 4.1, if $\alpha(G) \geq 4$, then $\alpha(G) \leq |S|$. It then follows that $n \leq 3|S|$ since G has $n = 2r + |S|$ vertices. Also, if $\alpha(G) \leq 3$, then $n \leq 3|S|$ since $|S| \geq 3 \geq r$. □

Given two graphs H_1 and H_2. The *corona* $H_1 \circ H_2$ is the graph obtained by taking each vertex of H_1 and connecting it to all vertices of a copy of H_2 (see, for instance, Figure 4). Clearly, the graph G_1 in Figure 3 corresponds to the graph $K_2 \circ K_2$.



(a) (b)

Figure 4: (a) The graph $P_3 \circ K_1$. (b) The graph $K_3 \circ K_2$.

The provided upper bound in Theorem 1.1 is sharp since the graph $K_t \circ K_2$ attains the bound for each $t \geq 3$. We next state an easy consequence of Theorem 1.1.

Corollary 4.1. *Let $G \in \mathbf{W}_2$ with n vertices. For DMI sets I_1 and I_2, if $G - (I_1 \cup I_2)$ is a clique of size at least 3, then $\alpha(G) \leq \frac{n}{3} \leq \omega(G)$.*

Proposition 4.5. *Suppose that $G \in \mathbf{W}_2$ with n vertices. For DMI sets I_1 and I_2, if $G - (I_1 \cup I_2)$ is isomorphic to K_2, then $\alpha(G) \leq 3$.*

Proof. Suppose that $S = V(G) - (I_1 \cup I_2)$ induces a K_2 in G for DMI sets I_1 and I_2. Assume for a contradiction that $\alpha(G) \geq 4$. By Propositions 4.1 and 4.2, we deduce that $\alpha(G) = 4$. Then G has $|S| + 8 = 10$ vertices.

Let $I_1 = \{x_1, x_2, x_3, x_4\}$, $I_2 = \{y_1, y_2, y_3, y_4\}$, and $S = \{u, v\}$. By Theorem 2.1, we may assume $\{x_1y_1, x_2y_2, x_3y_3, x_4y_4\} \subset E(G)$. Recall that for each vertex $s \in S$, the vertex s has a unique neighbour in I_ℓ for $\ell = 1, 2$ by Proposition 4.1, also for each $i \in [4]$, at least one endpoint of the edge x_iy_i is adjacent to S by Proposition 4.2. Thus, we deduce that S has exactly two neighbours in each of I_1, I_2, and so the remaining two vertices of each I_1, I_2 have no neighbour in S. Without loss of generality, we may assume that $N_G(S) = \{x_1, x_2, y_3, y_4\}$, and $N_G(u) \cap I_2 = \{y_3\}$. By Corollary 2.1 and Proposition 4.4, x_3 has exactly two neighbours in I_2. Then, by applying the same process as in the proof of Lemma 4.1, we claim that every vertex in S is adjacent to one of the two neighbours of x_3 in I_2. Indeed, if there exists $u \in S$ having no neighbour in $N_G(x_3)$, then x_3 and its two neighbours would survive in $G - N_G[u]$. However, $G - N_G[u]$ must consist of K_2 components by Lemma 2.1 and Proposition 2.1, a contradiction. This forces that $N_G(x_3) = \{y_3, y_4\}$. By the same reason, every vertex in S is adjacent to one of two neighbours of x_4 in I_2. Therefore $N_G(x_3) = \{y_3, y_4\} = N_G(x_4)$.

On the other hand, the graph $G - N_G[u]$ is in $\mathbf{W}_2$ by Lemma 2.1, and so $G - N_G[u]$ consists of K_2 components by Proposition 2.1. However, y_4 and its two neighbours x_3, x_4 belong to $G - N_G[u]$, a contradiction. □

For a connected graph $G \in \mathbf{W}_2$ with n vertices, suppose that $G - (I_1 \cup I_2)$ is a clique of size t for DMI sets I_1 and I_2. If $t \geq 3$, then, by Corollary 4.1, G has at most $\frac{n}{3}$ vertices. On the other hand, if $t \leq 2$, then G has at most $3t + 2$ vertices by Corollary 2.2 and Proposition 4.5.

Corollary 4.2. *Let $G \in \mathbf{W}_2$ with n vertices. For DMI sets I_1 and I_2, if $G - (I_1 \cup I_2)$ is a clique of size t with $t \leq 2$, then $n \leq 3t + 2 \leq 8$, and $\omega(G) \geq \frac{n-2}{3}$.*

By combining Corollaries 4.1 and 4.2, we obtain the following.

Result 4.2. *Let $G \in \mathbf{W}_2$ with n vertices. If the removing of two DMI sets from G leaves a clique, then G has a clique of size $\frac{n-2}{3}$.*

We now consider the $\mathbf{W}_2$ graphs obtained from another one by attaching some c-twin vertices. Actually, we have already shown in Theorem 3.1 that if $G \in \mathbf{W}_2$ and $u \in V(G)$, then $G(u : w)$ is in $\mathbf{W}_2$ as well. We now consider the case of adding more than one c-twin consecutively.

Given a connected graph $H \in \mathbf{W}_2$ such that $S = V(H) - (I_1 \cup I_2)$ induces a clique in H for DMI sets I_1 and I_2. We define a graph family $\mathcal{C}(H)$ whose members consist of the graph obtained from H by adding a vertex set T into S and making all vertices of T as c-twin with some vertices of S so that $T \cup S$ induces a clique in the resulting graph. In other words, a graph G belongs to $\mathcal{C}(H)$ if there exists a set of c-twin vertices $T \subset S$ such that $G - T$ is isomorphic to H where $S = V(G) - (I_1 \cup I_2)$ induces a clique in G for DMI sets I_1 and I_2. Clearly, $H \in \mathcal{C}(H)$. For instance, if $H = C_3$, then $\mathcal{C}(C_3) = \mathcal{C}(K_1 \circ K_2)$ consists of complete graphs having at least three vertices. Also, a member of $\mathcal{C}(K_3 \circ K_2)$ is depicted in Figure 5 where the vertices u, v are added into the graph $K_3 \circ K_2$.

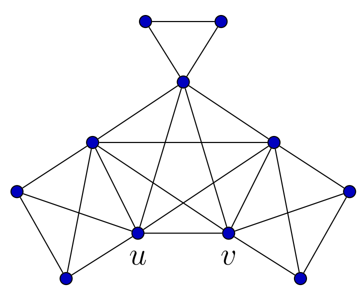


Figure 5: A member of $\mathcal{C}(K_3 \circ K_2)$.

For a given connected graph $H \in \mathbf{W}_2$, all the members of $\mathcal{C}(H)$ are in $\mathbf{W}_2$ by Theorem 3.1.

Proposition 4.6. *Let $H \in \mathbf{W}_2$ such that $S = V(H) - (I_1 \cup I_2)$ induces a clique in H for DMI sets I_1 and I_2. Then every member of the graph family $\mathcal{C}(H)$ is in $\mathbf{W}_2$.*

In the rest of the paper, we shall give our main result (Theorem 1.2) via a series of lemmas where we split the proof into three cases with respect to $\alpha(G)$.

Lemma 4.2. *Let $G \in \mathbf{W}_2$. Suppose that for DMI sets I_1 and I_2, the subgraph $G - (I_1 \cup I_2)$ is a clique. If $\alpha(G) = r \geq 4$, then G belongs to $\mathcal{C}(K_r \circ K_2)$.*

Proof. Let $\alpha(G) = r \geq 4$. Then $|S| = t \geq 3$ by Proposition 4.5, and we have $|S| \geq \alpha(G) \geq 4$ by Result 4.1. It follows from Theorem 1.1 that $n \leq 3|S| = 3t$.

Let $I_1 = \{x_1, x_2 \ldots, x_r\}$, $I_2 = \{y_1, y_2 \ldots, y_r\}$, and $S = \{u_1, u_2, \ldots, u_t\}$ with $t \geq r \geq 4$. By Theorem 2.1, we may assume $\{x_1y_1, x_2y_2, \ldots, x_ry_r\} \subset E(G)$. Notice that, for each $u_i \in S$, the graph $G - N_G[u_i]$ consists of K_2 components by Proposition 2.1 and Lemma 2.1, since $S \subset N_G[u_i]$.

We first show that $G[I_1 \cup I_2]$ is isomorphic to rK_2. Assume by contradiction that x_i is adjacent to y_j for some $i, j \in [r]$ with $i \neq j$. Recall that each vertex of $I_1 \cup I_2$ has a neighbour in S by Lemma 4.1. Moreover, for each vertex $s \in S$, the vertex s has a unique neighbour in I_ℓ for $\ell = 1, 2$ by Proposition 4.1. Consider a vertex $y_k \in I_2$ for $k \in [r]$ with $k \notin \{i, j\}$, there exists $u \in S \cap N_G(y_k)$. Since u has a unique neighbour in I_2, the vertex u has to be adjacent to x_i, since otherwise x_i and its two neighbours y_i, y_j would survive in $G - N_G[u]$, contradicting that $G - N_G[u]$ consists of K_2 components. Clearly, $G - N_G[u] = G[I_1 \cup I_2] - \{x_i, y_k\}$. We then deduce that $G[I_1 \cup I_2] - \{x_i, y_k\}$ is isomorphic to $(r-1)K_2$, and so $x_k y_i \in E(G)$. Let us next consider the vertex $y_j \in I_2$. By assumption, there exists $v \in S \cap N_G(y_j)$. Then, similarly as before, v has to be adjacent to x_k, since otherwise x_k and its two neighbours y_i, y_k would survive in $G - N_G[v]$, a contradiction with the fact that $G - N_G[v]$ consists of K_2 components. This again implies that $G[I_1 \cup I_2] - \{x_k, y_j\}$ is isomorphic to $(r-1)K_2$, and so $x_j y_k \in E(G)$. Finally, let us take the vertex $x_j \in I_1$, and we apply the same process as before. By assumption there exists $w \in S \cap N_G(x_j)$, and thus w has to be adjacent to y_i, since otherwise y_i and its two neighbours x_i, x_k would survive in $G - N_G[w]$, contradicting that $G - N_G[w]$ consists of K_2 components. This again implies that $G[I_1 \cup I_2] - \{x_j, y_i\}$ is isomorphic to $(r-1)K_2$. Since $r \geq 4$, there exists $x_\ell \in I_1$ for $\ell \in [r] \setminus \{i, j, k\}$, also we have $z \in S \cap N_G(x_\ell)$ by Lemma 4.1. It follows that z has to be adjacent to all $\{y_i, y_j, y_k\}$, since otherwise y_i (or y_j, y_k) and its two neighbours would survive in $G - N_G[z]$, contradicting that $G - N_G[z]$ consists of K_2 components. However, z can not have more than one neighbour in I_2 by Proposition 4.1, a contradiction. We therefore conclude that x_i is not adjacent to y_j. So, $G[I_1 \cup I_2]$ is isomorphic to rK_2.

Observe that if a vertex $u \in S$ is adjacent to x_i, y_j with $i \neq j$, then the edge $x_j y_i$ must appear in G since $G - N_G[u]$ consists of K_2 components. However, this is not possible because $G[I_1 \cup I_2]$ is isomorphic to rK_2 by above claim. We therefore infer that each vertex of S is adjacent to only both endpoints of an edge $x_i y_i$ in $G[I_1 \cup I_2]$ for $i \in [r]$. It follows that there exists $S' \subset S$ with $|S'| = r$ such that $G[I_1 \cup I_2 \cup S']$ is isomorphic to $K_r \circ K_2$. On the other hand, if S has more than r vertices, then some vertices of S have the same neighbours in $I_1 \cup I_2$, since each vertex of S is adjacent to only both endpoints of an edge $x_i y_i$ in $G[I_1 \cup I_2]$ for $i \in [r]$. Let $S_1, S_2 \ldots, S_k$ be subsets of S such that each S_i consists of the vertices of S having the same neighbours in $I_1 \cup I_2$. Obviously, each S_i consists of c-twin vertices, and we have $S_i \cap S_j = \emptyset$ for $i, j \in [k]$. It then follows that the sets $S_1, S_2 \ldots, S_k$ correspond to a partition of S. Hence, G belongs to $\mathcal{C}(K_r \circ K_2)$. □

Corollary 4.3. *Let $G \in \mathbf{W}_2$. Suppose that for DMI sets I_1 and I_2, the set $S = V(G) - (I_1 \cup I_2)$ induces a clique of size t in G. If $\alpha(G) = r \geq 4$ and $n = 3|S|$, then $t = r$ and $G = K_r \circ K_2$.*

Lemma 4.3. *Let $G \in \mathbf{W}_2$. Suppose that for DMI sets I_1 and I_2, the subgraph $G - (I_1 \cup I_2)$ is a clique. If $\alpha(G) = 3$, then G is in either $\mathcal{C}(G_5)$ or $\mathcal{C}(G_6)$ or $\mathcal{C}(K_3 \circ K_2)$ (see Figures 3 and 6).*

Proof. Let I_1, I_2 be two DMI sets in G, and let $S = V(G) - (I_1 \cup I_2)$ induce a clique of size t in G. Suppose $\alpha(G) = 3$. Then G has $|S| + 6$ vertices. Let $I_1 = \{x_1, x_2, x_3\}$, $I_2 = \{y_1, y_2, y_3\}$.

We may assume $\{x_1y_1, x_2y_2, x_3y_3\} \subset E(G)$ by Theorem 2.1. Observe that, by Proposition 4.1, for each vertex $s \in S$, the vertex s has a unique neighbour in I_ℓ for $\ell = 1, 2$. It follows from Proposition 4.2 that S has at least two vertices.

We first assume that every vertex in $I_1 \cup I_2$ has a neighbour in S. Then $|S| \geq 3$, because for each vertex $s \in S$, the vertex s has a unique neighbour in I_ℓ for $\ell = 1, 2$. If $G[I_1 \cup I_2]$ is isomorphic to $3K_2$, then G belongs to $\mathcal{C}(K_3 \circ K_2)$ as we deduce in the proof of Lemma 4.2. Else, x_i is adjacent to y_j for some $i, j \in \{1, 2, 3\}$ with $i \neq j$. Again, following from the proof of Lemma 4.2, there exists $u, v, w \in S$ such that $I_1 \cup I_2 \cup \{u, v, w\}$ induces the graph G_6 (see Figure 6). If S has more than 3 vertices, then some vertices of S have to have the same neighbours in $I_1 \cup I_2$. Let $S_1, S_2 \ldots, S_k$ be subsets of S such that each S_i consists of c-twin vertices of S. It follows that the sets $S_1, S_2 \ldots, S_k$ corresponds to a partition of S. Hence, G belongs to $\mathcal{C}(G_6)$.

Now, assume that there exist $x_i \in I_1$ for $i \in \{1, 2, 3\}$ such that x_i has no neighbour in S. Then $y_i \in N_G(S)$ by Proposition 4.2, and it follows from Corollary 2.1 and Proposition 4.4 that x_i has only two neighbours y_i, y_j for an index $j \in \{1, 2, 3\} \setminus \{i\}$. We therefore deduce that every vertex in S is adjacent to either y_i or y_j, since otherwise x_i and its two neighbours y_i, y_j would survive in $G - N_G[u]$ for some $u \in S$, however, $G - N_G[u]$ must consist of K_2 components by Proposition 2.1 and Lemma 2.1, a contradiction. Moreover, no vertex of S is adjacent to $I_2 - \{y_i, y_j\}$ by Proposition 4.1. Then, there exists $y_\ell \in I_2$ for $\ell \in \{1, 2, 3\} \setminus \{i, j\}$ such that y_ℓ has no neighbour in S due to $\alpha(G) = 3$. It then follows from Proposition 4.2 that $x_\ell \in N_G(S)$, say $x_\ell \in N_G(u)$ for a vertex $u \in S$. Recall that u is adjacent to either y_i or y_j. We note that if u is adjacent to y_i, then y_j and its both neighbours x_i, x_j would survive in $G - N_G[u]$, contradicting that $G - N_G[u]$ consists of K_2 components. Therefore, u is adjacent to only y_j in I_2. This also implies that x_j is adjacent to y_ℓ since $G - N_G[u]$ consists of K_2 components. On the other hand, there must be another vertex $v \in S - u$ such that $v \in N_G(y_i) \cap S$ since $x_i \notin N_G(S)$. The vertex v must be adjacent to x_j, since otherwise x_j and its two neighbours y_j, y_ℓ would survive in $G - N_G[v]$, a contradiction. Consequently, $G[I_1 \cup I_2]$ contains the edges $x_iy_i, x_jy_j, x_\ell y_\ell, x_iy_j, x_jy_\ell$, and we will show that the graph $G[I_1 \cup I_2]$ has no more edges. For simplicity, we assume that $i = 1$, $j = 2$ and $\ell = 3$. Since $G - N_G[u]$ consists of K_2 components, we can say $x_1y_3, x_2y_1 \notin E(G)$. By the same reason, $x_3y_2 \notin E(G)$ since $G - N_G[v]$ consists of K_2 components. Similarly, $x_3y_1 \notin E(G)$, since otherwise $N_G(x_1)$ would be dominated by $\{x_2, x_3\}$, contradicting that x_1 is a shedding vertex. Hence, $G[I_1 \cup I_2]$ consists of only the edges $x_1y_1, x_2y_2, x_3y_3, x_1y_2, x_2y_3$. In addition, u (resp. v) has only neighbours x_3, y_2 (resp. x_2, y_1) in $I_1 \cup I_2$. Observe that $I_1 \cup I_2 \cup \{u, v\}$ induces the subgraph G_5 in G (see Figure 3). Moreover, if S has more than two vertices, then every vertex in $S - \{u, v\}$ must be c-twin with one of u, v. Hence, we conclude that G is in $\mathcal{C}(G_5)$. □

Lemma 4.4. *Let $G \in \mathbf{W}_2$. Suppose that $G - (I_1 \cup I_2)$ is a clique for DMI sets I_1 and I_2. If $\alpha(G) = 2$, then G belongs to one of the graph classes $\mathcal{C}(C_5), \mathcal{C}(G_2), \mathcal{C}(G_3), \mathcal{C}(G_4)$, $\mathcal{C}(G_7), \mathcal{C}(G_8), \mathcal{C}(G_9)$, and $\mathcal{C}(K_2 \circ K_2)$ (see Figures 3 and 6).*

Proof. Let $\alpha(G) = 2$. By Corollary 2.2, $G = C_5$ when $|S| = 1$. We may therefore assume $|S| \geq 2$. Let $I_1 = \{x_1, x_2\}$, $I_2 = \{y_1, y_2\}$, and $S = \{u_1, u_2, \ldots, u_t\}$ for $t \geq 2$. By Theorem 2.1,

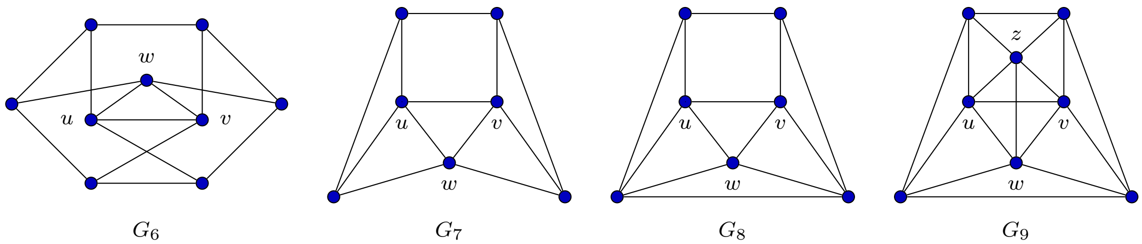


Figure 6: The graphs G_6, G_7, G_8, and G_9.

we assume $\{x_1y_1, x_2y_2\} \subset E(G)$. Notice that for each $u_i \in S$, the graph $G - N_G[u_i]$ consists of K_2 components by Proposition 2.1 and Lemma 2.1, since $G - N_G[u_i] \in \mathbf{W}_2$ and $S \subset N_G[u_i]$. Also, by Proposition 4.1, for each vertex $s \in S$, the vertex s has a unique neighbour in I_ℓ for $\ell = 1, 2$.

First, we suppose that there exists a vertex of $I_1 \cup I_2$ having no neighbour in S. Without loss of generality, we assume that $x_1 \in I_1$ has no neighbour in S. Then $y_1 \in N_G(S)$ by Proposition 4.2. Also, $x_2 \in N_G(S)$ by Proposition 4.1. Since x_1 has exactly two neighbours in I_2 by Corollary 2.1 and Proposition 4.4, we may assume without loss of generality that $y_2 \in N_G(x_1)$, and so $N_G(x_1) = \{y_1, y_2\}$. Notice that $x_2y_1 \notin E(G)$, since otherwise $N_G(x_1)$ would be dominated by $\{x_2\}$, a contradiction as x_1 is a shedding vertex. It follows that $G[I_1 \cup I_2]$ is isomorphic to a P_4 whose middle vertices are x_1, y_2. On the other hand, since $x_2 \in N_G(S)$, we have two cases: $y_2 \notin N(S)$ or $y_2 \in N(S)$. If y_2 has no neighbour in S, then y_1 has a neighbour in S. It follows from Proposition 4.4, every vertex in S is adjacent to both x_2 and y_1 in $I_1 \cup I_2$. This means that every pair of vertices in S is twin. Hence, G belongs to $\mathcal{C}(C_5)$. We now suppose that $y_2 \in N(S)$. Since for each vertex $s \in S$, the vertex s has a unique neighbour in I_ℓ for $\ell = 1, 2$, every vertex in S is adjacent to x_2 and y_1 (or y_2) where we recall that $y_1 \in N_G(S)$. It follows that there exists $u, v \in S$ such that $N_G(u) \cap (I_1 \cup I_2) = \{x_2, y_2\}$ and $N_G(v) \cap (I_1 \cup I_2) = \{x_2, y_1\}$. Obviously, the set $I_1 \cup I_2 \cup \{u, v\}$ induces the subgraph G_3 in the graph G (see Figure 3). Moreover, if S has more than 2 vertices, then every vertex in $S - \{u, v\}$ would be a c-twin with u or v. Hence, G belongs to $\mathcal{C}(G_3)$.

Let us next assume that every vertex in $I_1 \cup I_2$ has a neighbour in S. Observe that if a vertex $u \in S$ is adjacent to both x_i and y_j with $i \neq j$, then the edge x_jy_i must appear in G since $G - N_G[u]$ consists of K_2 components. This means that that any vertex of S is adjacent to only both endpoints of either x_1y_1 or x_2y_2 when $G[I_1 \cup I_2]$ induces $2K_2$. Then, by Proposition 4.2, G belongs to $\mathcal{C}(K_2 \circ K_2)$ when $G[I_1 \cup I_2]$ induces $2K_2$. Hence, we further suppose that $G[I_1 \cup I_2] \ncong 2K_2$. Without loss of generality, assume $x_1y_2 \in E(G)$. We then observe that $G[I_1 \cup I_2]$ is isomorphic to either P_4 or C_4.

Suppose first that $G[I_1 \cup I_2]$ induces P_4. Then any vertex $u \in S$ cannot be adjacent to both x_1 and y_2 in $I_1 \cup I_2$, since otherwise $G - N_G[u]$ would consists of two isolated vertices x_2, y_1 due to $G[I_1 \cup I_2] \cong P_4$, a contradiction. This implies that if $u \in S$ is a neighbour of x_1 (resp. y_2) in G, then u is adjacent to y_1 (resp. x_2). It then follows from Proposition 4.2 that there exist $u, v \in S$ with $u \neq v$ such that $x_1, y_1 \in N_G(u)$ and $x_2, y_2 \in N_G(v)$. Observe that $G[x_1, x_2, y_1, y_2, u, v]$ is isomorphic to the graph G_2 (see Figure 3). If y_1 and x_2 have no common neighbour in S, then G belongs to $\mathcal{C}(G_2)$. Otherwise, y_1 and x_2 have a common neighbour w in S, then $G[x_1, x_2, y_1, y_2, u, v, w]$ is isomorphic to the graph G_7 (see Figure 6). Similarly, if S has some twin vertices in respect to u, v, w, then G belongs to $\mathcal{C}(G_7)$.

Finally, we suppose that $G[I_1 \cup I_2]$ is isomorphic to C_4. Recall that for each vertex $s \in S$, the vertex s has a unique neighbour in I_ℓ for $\ell = 1, 2$, also every vertex in $I_1 \cup I_2$ has a neighbour in S. Then, we deduce that there exist $u, v \in S$ such that $\{u, v\}$ dominates all x_1, x_2, y_1, y_2 in the graph G. Since $I_1 \cup I_2$ induces C_4 in G, we may then assume without loss of generality that $x_1, y_1 \in N_G(u)$ and $x_2, y_2 \in N_G(v)$. Obviously, $G[x_1, x_2, y_1, y_2, u, v]$ is isomorphic to the graph G_4 (see Figure 3). Therefore, G belongs to $\mathcal{C}(G_4)$ when $S = \{u, v\}$ or every vertex in $S - \{u, v\}$ is a c-twin with one of u and v. Now, we suppose that there exists $w \in S - \{u, v\}$ such that w is not a c-twin with u and v. Then w is adjacent to x_1, y_2 (or x_2, y_1), assume without loss of generality that $x_1, y_2 \in N_G(w)$. In such a case, $G[x_1, x_2, y_1, y_2, u, v, w]$ is isomorphic to the graph G_8 (see Figure 6). Therefore, G belongs to $\mathcal{C}(G_8)$ when $S = \{u, v, w\}$ or each vertex of $S - \{u, v, w\}$ is a c-twin with one of u, v, w. At last, we suppose that there exists $z \in S - \{u, v, w\}$ such that z is not a c-twin with u, v and w, then the only possibility is that $x_2, y_1 \in N_G(z)$. It follows that $G[x_1, x_2, y_1, y_2, u, v, w, z]$ is isomorphic to the graph G_9 (see Figure 6). Also, if $|S| \geq 5$, then some vertices of S must form a c-twin with one of u, v, w, z. Hence, G belongs to $\mathcal{C}(G_9)$. □

Notice that any connected graph with independence number 1 is a complete graph. Since all complete graphs having at least two vertices are in $\mathbf{W}_2$, we say that any graph in $\mathbf{W}_2$ with independence number 1 belongs to $\mathcal{C}(K_2)$.

By combining Lemmas 4.2, 4.3, 4.4 and Proposition 4.6, we get the promised characterization of $\mathbf{W}_2$ graphs for which $G - (I_1 \cup I_2)$ is a clique for DMI sets I_1 and I_2.

Theorem 1.2. *A connected graph G is in $\mathbf{W}_2$ such that the removal of two DMI sets from G leaves a clique if and only if G belongs to one of the graph classes $\mathcal{C}(G_2), \mathcal{C}(G_3), \ldots, \mathcal{C}(G_9)$, $\mathcal{C}(K_2), \mathcal{C}(C_5)$ and $\mathcal{C}(K_t \circ K_2)$ for $t \geq 2$.*

Acknowledgement

This research was supported by TUBITAK (The Scientific and Technological Research Council of Turkey) under the project number 121F018. The author thanks the anonymous referee for their careful reading and invaluable comments.

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