Electronic Journal of Graph Theory and Applications **13** (1) (2025), 19–29

Electronic Journal of Graph Theory and Applications

On the rainbow connection numbers of line, middle, and total graphs of wheels

Lyra Yulianti*, Muhardiansyah, Admi Nazra

Department of Mathematics and Data Science, Faculty of Mathematics and Natural Sciences, Universitas Andalas
Kampus UNAND Limau Manis Padang 25163, Padang, Indonesia

lyra@sci.unand.ac.id, muhardiansyah97@gmail.com, nazra@sci.unand.ac.id

**corresponding author*

Abstract

An edge-colored graph G is called rainbow connected if any two vertices in G are connected by a path whose no two edges are colored the same. The rainbow connection of G, denoted by $rc(G)$, is the smallest number of colors needed such that G be a rainbow connected graph. Similarly defined, an edge-colored graph G is called strong rainbow connected if any two vertices in G are connected by a geodesic path whose no two of its edges are colored the same. The strong rainbow connection for G, denoted by $src(G)$, is the smallest number of colors needed such that G be a strong rainbow connected graph. This paper considers the determination of the rainbow connection and strong rainbow connection numbers of the line graph, the middle graph, and the total graph of a wheel W_n on $n+1$ vertices.

Keywords: rainbow connection number, wheel, line graph, middle graph, total graph
Mathematics Subject Classification: 05C12, 05C15, 05C40
DOI: 10.5614/ejgta.2025.13.1.2

1. Introduction

The concept of rainbow connection was introduced by Chartrand et al. [1]. For a nontrivial connected graph G and a positive integer k, let $c : E(G) \to \{1, 2, \cdots, k\}$ be an edge coloring of

Received: 8 January 2022, Revised: 17 April 2024, Accepted: 19 December 2024.

G, where the adjacent edges can be colored the same. A path in G is called a rainbow path if every edge is colored differently. A graph G is called a rainbow connected if for every two vertices x and y in G, there exists a rainbow (x, y)-path. In this case, the coloring c is a rainbow coloring. If c is a rainbow coloring with k colors, then c is a rainbow k-coloring. If k is the smallest number, then k is defined as the rainbow connection number $rc(G)$ of G. Clearly $diam(G) \leq rc(G)$, where $diam(G)$ is the diameter of G.

Let c be an edge coloring of a nontrivial graph G. For two vertices x and y of G, a (x, y)-rainbow geodesic in G is a (x, y)-rainbow path of length $d(x, y)$. The graph G is called a strongly rainbow connected if for every two vertices x and y in G, there exists a rainbow (x, y) geodesic. In this case, the coloring c is called a strong rainbow coloring of G. If c is a strong rainbow coloring with k colors, then c is defined as a rainbow k-coloring. The smallest k such that G has a strong rainbow k-coloring is defined as the strong rainbow connection number of G, denoted by $src(G)$. If G is a nontrivial connected graph with size m, then Chartrand et al. [1] stated that

$$diam(G) \leq rc(G) \leq src(G) \leq m.$$

Chartrand et al. [1] gave some characterizations of connected graphs G with $rc(G) = 1, 2$, and m. They also determined the rainbow connection number and strong rainbow connection for path, cycle, wheel, complete graph, complete bipartite graph, and complete multipartite graph. Other results discussing the rainbow connection numbers and strong rainbow connection numbers for various connected graphs were listed in an updated survey by Li and Sun [3]. Next, the rainbow connection and the strong rainbow connection numbers of the generalized triangle-ladder graph are determined by Yulianti et al. [9]. Moreover, in [10] Yulianti et al. determined the rainbow connection and the strong rainbow connection numbers of the amalgamation of the generalized triangle-ladder graphs, denoted by the triangle-net graph. In [5], Septyanto and Sugeng proved a new class of lower bounds of rainbow connection numbers, depending on the existence of sets with common neighbours. In [2], Fitriani et al. gave the sharp lower and upper bounds for the rainbow connection number of comb product between two connected graphs. Recent results were some new bounds for rainbow connection number of corona product of two graphs $G \circ H$, for two certain non-trivial graphs G and H in [6].

Sun [7] investigated the rainbow connection numbers of the line graph, middle graph, and total graph of a connected triangle-free graph G. There were three (near) sharp upper bounds in terms of the number of vertex-disjoint cycles of the original graph G obtained in the paper. Next, Sy et al. [8] determined the rainbow connection number and strong rainbow connection number of fan and sunlet graphs. The rainbow connection numbers of the line graph, middle graph, and total graph of the sunlet graph were investigated by Rao and Murali [4]. The results in [4] were sharpened by Zhao et al. [11], by determining the exact values of $rc(G)$ and $src(G)$, where G are the line and the middle graphs of sunlet graph. This paper determines the rainbow connection and strong rainbow connection numbers of the line graph, the middle graph, and the total graph of wheel W_n on $n + 1$ vertices.

2. Preliminary Notes

The definitions of line, middle and total graphs are taken from [7].

Definition 2.1. *[7] Let G be an arbitrary graph with vertex set $V(G)$ and edge set $E(G)$. The line graph of a graph G, denoted by $L(G)$, is a graph whose vertex set is the edge set of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.*

Definition 2.2. *[7] Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of a graph G, denoted by $M(G)$, is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ if one of the following conditions holds:*

(1) x, y are in $E(G)$ and x, y adjacent in G,

(2) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

Definition 2.3. *[7] Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph of a graph G, denoted by $T(G)$, is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ if one of the following conditions holds:*

(1) x, y are in $V(G)$ and x is adjacent to y in G,

(2) x, y are in $E(G)$ and x, y are adjacent in G,

(3) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

3. Main Results

Let W_n be a wheel on $n+1$ vertices and $2n$ edges. The vertex set and the edge set of the line graph of a wheel, denoted by $L(W_n)$, are as follows.

$$\begin{aligned}
V(L(W_n)) &= \{v_i \mid 1 \leq i \leq n\} \cup \{w_i \mid 1 \leq i \leq n\}, \\
E(L(W_n)) &= \{v_i v_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{v_i w_i \mid 1 \leq i \leq n\} \\
&\quad \{w_i v_{i+1} \mid 1 \leq i \leq n, v_{n+1} = v_1\} \cup \{w_i w_{i+1} \mid 1 \leq i \leq n, w_{n+1} = w_1\}.
\end{aligned}$$

The vertex set and the edge set of the middle graph of wheel, denoted by $M(W_n)$, are as follows.

$$\begin{aligned}
V(M(W_n)) &= \{u_i \mid 0 \leq i \leq n\} \cup \{v_i \mid 1 \leq i \leq n\} \cup \{w_i \mid 1 \leq i \leq n\}, \\
E(M(W_n)) &= \{u_0 v_i \mid 1 \leq i \leq n\} \cup \{u_i v_i \mid 1 \leq i \leq n\} \\
&\quad \cup \ \{u_i w_i \mid 1 \leq i \leq n\} \cup \{w_{i-1} u_i \mid 1 \leq i \leq n, w_0 = w_n\} \\
&\quad \cup \ \{v_i v_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{v_i w_i \mid 1 \leq i \leq n\} \\
&\quad \cup \ \{w_{i-1} v_i \mid 1 \leq i \leq n, w_0 = w_n\} \cup \{w_{i-1} w_i \mid 1 \leq i \leq n, w_0 = w_n\}.
\end{aligned}$$

The vertex set and the edge set of the total graph of wheel, denoted by $T(W_n)$, are as follows.

$$\begin{aligned} V(T(W_n)) &= \{u_i \mid 0 \le i \le n\} \cup \{v_i \mid 1 \le i \le n\} \cup \{w_i \mid 1 \le i \le n\}, \\ E(T(W_n)) &= \{u_0u_i \mid 1 \le i \le n\} \cup \{u_iu_{i+1} \mid 1 \le i \le n, u_{n+1} = u_1\} \\ &\quad \cup \{u_0v_i \mid 1 \le i \le n\} \cup \{u_iv_i \mid 1 \le i \le n\} \\ &\quad \cup \{u_iw_i \mid 1 \le i \le n\} \cup \{w_iu_{i+1} \mid 1 \le i \le n, u_{n+1} = u_1\} \\ &\quad \cup \{v_iv_j \mid 1 \le i, j \le n, i \ne j\} \cup \{v_iw_i \mid 1 \le i \le n\} \\ &\quad \cup \{w_iv_{i+1} \mid 1 \le i \le n, v_{n+1} = v_1\} \cup \{w_iw_{i+1} \mid 1 \le i \le n, w_{n+1} = w_1\}. \end{aligned}$$

The following are the diameter of the line, middle, and total graphs of a wheel for $n \ge 3$.

$$\begin{aligned} diam(L(W_n)) &= \begin{cases} 2, & n = 3, 4, \\ 3, & n \ge 5. \end{cases} \\ diam(M(W_n)) &= \begin{cases} 2, & n = 3, \\ 3, & n \ge 4. \end{cases} \\ diam(T(W_n)) &= \begin{cases} 2, & n = 3, 4, \\ 3, & n \ge 5. \end{cases} \end{aligned}$$

Theorem 3.1 stated the rainbow and strong rainbow connection numbers of the line graph of a wheel W_n on $n + 1$ vertices for $n \ge 3$.

Theorem 3.1. *Let $G_1 = L(W_n)$ be the line graph of a wheel on $n + 1$ vertices for $n \ge 3$. Then*

$$rc(G_1) = src(G_1) = \begin{cases} 2, & n = 3, 4, \\ 3, & n \ge 5. \end{cases}$$

Proof. Consider the following cases.
Case 1. $n = 3$ or $n = 4$.
Because $diam(G_1) = 2$, then $rc(G_1) \ge 2$. Since $c_{11} : E(G_1) \to \{1, 2\}$ defined by:

$$c_{11}(e) = \begin{cases} 1, & e \in \{v_iv_j \mid 1 \le i, j \le n,\ i \ne j\} \\ & \cup \{w_jw_{j+1} \mid 1 \le j \le n - 1\} \text{ for odd } k \\ & \cup \{w_nw_1\} \text{ for odd } n, \\ 2, & \text{otherwise}, \end{cases} \tag{1}$$

is a rainbow strong coloring, it follows that $rc(G_1) = src(G_1) = 2$ for $n = 3$ or $n = 4$.

Case 2. $n = 5$.
Let $n = 5$. Because $diam(G_1) = 2$ then $rc(G_1) \ge 2$ for $n = 5$. Let c_{12} be a rainbow 2-coloring of G_1. Without loss of generalities, let $c_{12}(w_1w_2) = 1$. Then for $1 \le i \le 5$, there are some vertices w_i, w_{i+1}, w_{i+2} with $w_{n+1} = w_1$ and $w_{n+2} = w_2$ in G_1, such that there exists a (w_i, w_{i+2})-path with length 2 and therefore, $c_{12}(w_2w_3) = 2$. It follows that $c_{12}(w_3w_4) = 1$. Then, $c_{12}(w_4w_5) = 2$ and $c_{12}(w_5w_1) = 1$. Therefore, there is no rainbow (w_5, w_2)-path, which is a contradiction. Thus, $rc(G_1) \ge 3$.

Next, it will be shown that $rc(G_1) \leq 3$ for $n = 5$. Let $c_{13} : E(G_1) \to \{1, 2, 3\}$ be an edge-coloring, which is defined as follows.

$$c_{13}(e) = \begin{cases} 1, & e \in \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \cup \{w_nw_1\}, \\ 2, & e \in \{w_iv_{i+1} \mid 1 \leq i \leq n, v_{n+1} = v_1\} \cup \{w_iw_{i+1} \mid 1 \leq i \leq n, \text{for odd } i, \}, \\ 3, & \text{otherwise}, \end{cases} \quad (2)$$

For $1 \leq i, j \leq n$, $i \neq j$ and for every two vertices $a, b \in V(G_1)$ with $d(a, b) \geq 2$, it will be shown that there exists a rainbow (a, b)-path.

(i) If $a = w_i$, $b = w_j$, $i < j$ and $d(a, b) = 2$, then the rainbow (a, b)-path is: w_i, w_{i+1}, w_j.

(ii) If $a = w_i$, $b = w_j$, $i < j$ and $d(a, b) > 2$, then the rainbow (a, b)-path is: w_i, v_i, v_j, w_j.

(iii) If $a = v_i$ and $b = w_j$, then the rainbow (a, b)-path is: v_i, v_j, w_j.

From (i) - (iii), it can be shown that those rainbow (a, b)-paths are also geodesic. Therefore, c_{13} (eq. 2) is a strong rainbow 3-coloring. Thus, we have that $rc(G_1) = src(G_1) = 3$ for $n = 5$.

Case 3. $n > 5$.

Since $diam(G_1) = 3$ for $n > 5$, then $rc(G_1) \geq 3$ for $n > 5$. Using c_{13} as the strong rainbow 3-coloring of G_1, we have that $rc(G_1) = src(G_1) = 3$ for $n > 5$. □

In Theorem 3.2 we determined the rainbow and strong rainbow connection numbers of the middle graph of wheel W_n on $n + 1$ vertices for $n \geq 3$.

Theorem 3.2. *If $n \geq 3$ and $G_2 = M(W_n)$ is the middle graph of a wheel W_n on $n + 1$ vertices, then*

$$rc(G_2) = \begin{cases} 2, & n = 3, \\ 3, & 4 \leq n \leq 9, \\ 4, & n \geq 10, \end{cases} \quad \textit{and } src(G_2) = \begin{cases} 2, & n = 3, \\ 3, & 4 \leq n \leq 9, \\ \lceil n/3 \rceil, & n \geq 10. \end{cases}$$

Proof. Consider the following cases.

Case 1. $n = 3$.

(a) Since $diam(G_2) = 2$, then $rc(G_2) \geq 2$. Next, it will be shown that $rc(G_2) \leq 2$. We construct a rainbow 2-coloring $c_{21} : E(G_2) \to \{1, 2\}$, which is defined as

$$c_{21}(e) = \begin{cases} 1, & e \in \{u_0v_i \mid 1 \leq i \leq n\} \cup \{u_iw_i \mid 1 \leq i \leq n\} \\ & \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\}, \\ 2, & \text{otherwise}. \end{cases} \quad (3)$$

Therefore, $rc(G_2) = 2$ for $n = 3$.

(b) Next, since $rc(G_2) = 2$ for $n = 3$, it is clear that $src(G_2) \geq 2$. Moreover, because c_{21} (eq. 3) is a strong rainbow 2-coloring, it follows that $src(G_2) = 2$ for $n = 3$.

Case 2. $4 \leq n \leq 9$.

(a) Since $diam(G_2) = 3$ for $n \geq 4$, then $rc(G_2) \geq 3$. Let $c_{22} : E(G_2) \rightarrow \{1, 2, 3\}$ be an edge coloring defined as follows.

$$c_{22}(e) = \begin{cases} 1, & e \in \{u_0v_s \mid 4 \leq s \leq n\} \cup \{u_pv_p \mid 1 \leq p \leq 3\} \\ & \cup\{u_iw_i \mid 1 \leq i \leq n\} \cup \{v_qv_r \mid 4 \leq q, r \leq n, q \neq r)\} \\ & \cup\{v_pw_{p-1} \mid 1 \leq p \leq 3, w_0 = w_n\}, \\ 2, & e \in \{u_zv_z, w_zv_z, w_{z-1}v_z \mid 4 \leq z \leq 6\} \\ & \cup\{w_iw_{i+1} \mid 1 \leq i \leq n, w_{n+1} = w_1\} \\ & \cup\{v_bv_c \mid 1 \leq b \leq 3, 7 \leq c \leq n\}, \\ 3, & \text{otherwise}. \end{cases} \tag{4}$$

For every two vertices $a, b \in V(G_2)$ with $d(a, b) \geq 2$, we will show that there exists a rainbow (a, b)-path as follows.

(i) If $a = u_i$ and $b = u_j$, then the rainbow (a, b)-path is u_i, v_i, v_j, u_j or u_i, w_i, w_{j-1}, w_j, where $w_0 = w_n$.

(ii) If $a = u_i$ and $b = v_j$, then the rainbow (a, b)-path is u_i, v_i, v_j.

(iii) If $a = u_i$ and $b = w_j$, then the rainbow (a, b)-path is u_i, w_i, w_j or $u_i, v_i, v_j + 1, w_j$, where $v_{n+1} = v_1$.

(iv) If $a = w_i$ and $b = w_j$ then the rainbow (a, b)-path is w_i, w_{j-1}, u_j, w_j or w_i, u_i, u_j, w_j.

Therefore, $rc(G_2) = 3$ for $4 \leq n \leq 9$.

(b) Since $rc(G_2) = 3$ for $4 \leq n \leq 9$, then $src(G_2) \geq 3$. To show that $src(G_2) \leq 3$, we provide a strong rainbow 3-coloring defined by $c_{23} : E(G_2) \rightarrow \{1, 2, \cdots, \lceil\frac{n}{3}\rceil\}$ as follows.

(i) If $n \mod 3 \neq 1$.

$$c_{23}(e) = \begin{cases} \lceil\frac{i}{3}\rceil, & e \in \{w_{i-1}v_i \mid 1 \leq i \leq n, w_0 = w_n\} \\ & \cup\{u_iv_i, v_iw_i \mid 1 \leq i \leq n\}, \\ f(i), & e \in \{w_{i-1}u_i, w_{i-1}w_i \mid i \leq i \leq n, w_0 = w_n\} \\ & \cup\{u_iw_i \mid 1 \leq i \leq n\}, \\ k, & e \in \{v_iv_j \mid 1 \leq i, j \leq n, i < j\} \cup \{u_0v_i \mid 1 \leq i \leq n\}. \end{cases} \tag{5}$$

(ii) If $n \mod 3 = 1$.

$$c_{24}(e) = \begin{cases} \lceil\frac{i}{3}\rceil, & e \in \{w_{i-1}v_i \mid 1 \leq i \leq n, w_0 = w_n\} \\ & \cup\{u_iv_i, v_iw_i \mid 1 \leq i \leq n\}, \\ f(j), & e \in \{w_{j-1}u_j, w_{j-1}w_j \mid 1 \leq j \leq n-1, w_0 = w_n\} \\ & \cup\{u_jw_j \mid 1 \leq j \leq n-1\}, \\ k, & e \in \{v_iv_j \mid 1 \leq i, j \leq n, i < j\} \cup \{u_0v_i \mid 1 \leq i \leq n\}, \\ 2, & e \in \{w_{n-1}u_n, u_nw_n, w_{n-1}w_n\}, \end{cases} \tag{6}$$

where

$$f(i) = \begin{cases} i, & i = 1, 2, \\ i \mod 3, & i \mod 3 \neq 0, \\ 3, & i \mod 3 = 0, \end{cases}$$

and k is a number which is assigned to $e = v_iv_j$, where $k \neq c(v_iv_j) \neq c(v_ju_j)$. Therefore, $src(G_2) = 3$ for $4 \leq n \leq 9$.

Case 3. $n \geq 10$.

(a) Because $diam(G_2) = 3$ for $n \geq 10$ then $rc(G_2) \geq 3$. Assume to the contrary that $rc(G_2) = 3$. Let c_{25} be a rainbow 3-coloring of G_2. Let H be a subgraph of G_2 with vertex set and edge set as follows.

$$\begin{aligned} V(H) &= \{u_i, w_i \mid 1 \leq i \leq n\}, \\ E(H) &= \{u_iw_i, w_{i-1}u_i, w_{i-1}w_i \mid 1 \leq i \leq n, w_0 = w_n\}. \end{aligned}$$

Let $V' = \{v_i \mid 1 \leq i \leq n\}$ and $E' = \{u_iv_i \mid 1 \leq i \leq n\}$. Thus, there exist vertices $x, y \in \{u_i \mid 1 \leq i \leq n\} \subset V(H)$ where $d_H(x, y) > 3$, and vertices $x', y' \in V'$ such that $c_{25}(xx') = c_{25}(yy')$. Because x, x', y', y is the only (x, y)- path which has $d(x, y) = 3$ in G_2, it follows that there is no rainbow (x, y)-path in G_2, which is a contradiction. Thus $rc(G_2) \geq 4$.

Next, let $c_{26} : E(G_2) \to \{1, 2, 3, 4\}$ be an edge coloring of G_2 as follows.

$$c_{26}(e) = \begin{cases} 1, & e \in \{u_0v_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\}, \\ 2, & e \in \{u_iv_i \mid 1 \leq i \leq n, \text{ for odd } i\} \cup \{v_iw_{i-1} \mid 1 \leq i \leq n, w_0 = w_n\}, \\ 3, & e \in \{u_yv_y \mid 2 \leq y \leq n, \text{ for even } i\} \cup \{v_iw_i \mid 1 \leq i \leq n\}, \\ 4, & \text{otherwise.} \end{cases} \quad (7)$$

Under this coloring, we will show that for $a, b \in V(G_2)$ with $d(a, b) \geq 2$, there exists a rainbow (a, b)-path as follows.

(i) If $a = u_i$ and $b = u_j$ for $1 \leq i, j \leq n$, $i \neq j$, then the rainbow (a, b)-path is:

1. $a = u_i, v_i, v_j, w_j, u_j = b$, if i and j are both odd,
2. $a = u_i, v_i, v_j, w_{j-1}, v_j = b$, if i and j are both even,
3. $a = u_i, v_i, v_j, u_j = b$, if i is odd and j is even or i is even and j is odd.

(ii) u_i, v_i, v_j, w_j or $a = u_i, v_i, v_{j+1}, w_j$, where $v_{n+1} = v_1$, if $a = u_i$ and $b = w_j$.

(iii) w_i, v_{i+1}, v_j, w_j, if $a = w_i$ and $b = w_j$.

Therefore, $rc(G_2) = 4$ for $n \geq 10$.

(b) Because $n \geq 10$, then there is an integer z such that $3z - 2 \leq n \leq 3z$. Let G_2 consists of an n-cycle $C_n : u_1, u_2, \cdots, u_n, u_1$ and $V' = \{v_i \mid 1 \leq i \leq n\}$. First, it will be shown that $src(G) \geq z$. Assume to the contrary that $src(G) \leq z - 1$. Let c be a strong rainbow

$(z-1)$-coloring of G. Since $deg(v) = n+2 > 3(z-1)$ for $v \in V'$ in G, then there exists $V^* \subseteq V(C_n)$ such that $|V^*| = 4$ and all edges $\{uv \mid u \in V^*, v \in V\prime,\ uv \in E(G_2)\}$ are assigned the same. Thus, there exist at least two vertices $x, y \in V^*$ such that $d(x,y) \geq 3$ in C_n and $d(x,y) = 3$ in G_2. Let $E' = \{u_iv_i \mid 1 \leq i \leq n\} \subseteq E(G)$. Since x, x', y', y $(xx', yy' \in E')$ is the only $x-y$ geodesic in G_2, it follows that there is no rainbow (x,y)-geodesic in G_2, which is a contradiction. Thus, $src(G_2) \geq z$.

Next, to show that $src(G) \leq z$, we provide a strong rainbow z-coloring which is defined by c_{23} and c_{24} (eq. (5) and (6)). Therefore, $src(G_) = \lceil n/3 \rceil$ for $n \geq 10$.

□

In Theorem 3.3 we determined the rainbow and strong rainbow connection numbers of the total graph of wheel W_n on $n+1$ vertices for $n \geq 3$.

Theorem 3.3. *If $n \geq 3$ and $G_3 = T(W_n)$ is the total graph of a wheel W_n on $n+1$ vertices, then*

$$rc(G_3) = \begin{cases} 2, & n = 3, 4, \\ 3, & n \geq 5, \end{cases} \text{, and } src(G_3) = \begin{cases} 2, & n = 3,4, \\ 3, & 5 \leq n \leq 9, \\ \lceil n/3 \rceil, & n \geq 10. \end{cases}$$

Proof. Consider the following cases.

Case 1. $n = 3$ or $n = 4$.

(a) Since $diam(G_3) = 2$ for $n = 3, 4$, then $rc(G_3) \geq 2$. Next, it will be shown that $rc(G_3) \leq 2$. Since $c_{31} : E(G_3) \to \{1, 2\}$ defined by

$$c_{31}(e) = \begin{cases} 1, & e \in \{u_0u_i, u_0v_i, u_iw_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i,j \leq n, i \neq j\} \\ & \cup \{u_iu_{i+1}, w_iw_{i+1} \mid 1 \leq i \leq n, \text{for odd } i, u_{n+1} = u_1, w_{n+1} = w_1\} \\ 2, & \text{otherwise} \end{cases} \quad (8)$$

is a rainbow 2-coloring, it follows that $rc(G_3) = 2$ for $n = 3$ or $n = 4$.

(b) Since $rc(G_3) = 2$ for $n = 3$ or $n = 4$, then $src(G_3) \geq 2$. Since c_{31} (eq. 8) is a strong rainbow 2-coloring, it follows that $src(G_3) = 2$ for $n = 3$ or $n = 4$.

Case 2. $n \geq 5$.

(a) Since $diam(G_3) = 3$, then $rc(G_3) \geq 3$. To show that $rc(G) \leq 3$, let $c_{32} : E(G_3) \to \{1, 2, 3\}$ as follows.

$$c_{32}(e) = \begin{cases} 1, & e \in \{u_0u_i \mid 1 \leq i \leq n, \text{for odd } i\} \cup \{u_0v_i \mid 1 \leq i \leq n\} \\ & \cup \{v_iv_j \mid 1 \leq i,j \leq n, i \neq j\} \\ 2, & e \in \{u_0u_i \mid 2 \leq i \leq n, \text{for even } i\} \cup \{v_iu_i \mid 1 \leq i \leq n, \text{for odd } i\} \\ & \cup \{v_iw_{i-1} \mid 1 \leq i \leq n, w_0 = w_n\}, \\ 3, & \text{otherwise.} \end{cases} \quad (9)$$

The rainbow $(a,b)-path$ for every two vertices $a, b \in V(G_3)$ under this coloring are as follows.

(i) If $a = u_i$ and $b = u_j$, then the rainbow (a, b)-path is:
   * u_i, v_0, v_{j-1}, u_j if i, j are both odd or i, j are both even.
   * u_i, u_0, u_j if i is odd and j is even, or i is even and j is odd.

(ii) If $a = v_i$ and $b = u_j$, then the rainbow (a, b)-path is v_i, v_j, u_j.

(iii) If $a = u_i$ and $b = w_j$, then the rainbow (a, b)-path is u_i, v_i, v_j, w_j or u_i, v_i, v_{j+1}, w_j.

(iv) If $a = v_i$ and $b = w_j$, then the rainbow (a, b)-path is v_i, v_j, w_j.

(v) If $a = w_i$ and $b = w_j$, then the rainbow (a, b)-path is w_i, v_{i+1}, v_j, w_j.

Therefore, $rc(G) = 3$ for $n \geq 5$.

(b) Since $rc(G_3) = 3$ for $n \geq 5$ then $src(G_3) \geq 3$. Consider the following cases.

(i) $5 \leq n \leq 9$. Claim that $src(G_3) \leq 3$ for $5 \leq n \leq 9$, by defining strong rainbow 3-colorings $c_{33} : E(G_3) \to \{1, 2, \cdots, \lceil \frac{n}{3} \rceil\}$ and $c_{34} : E(G_3) \to \{1, 2, \cdots, \lceil \frac{n}{3} \rceil\}$ as follows.

   * If n is even then

$$c_{33}(e) = \begin{cases} \lceil \frac{i}{3} \rceil, & e \in \{u_0u_i \mid 1 \leq i \leq n\}, \\ 1, & e \in \{u_0v_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \\ & \cup \{u_tu_{t+1}, u_tw_t, w_tu_{t+1}, w_tw_{t+1} \mid 1 \leq t \leq n-1, \\ & \text{for odd } t\} \\ 2, & e \in \{u_iv_i \mid 1 \leq i \leq n\} \\ & \cup \{v_iw_{i-1} \mid 1 \leq i \leq n, i \neq j, w_0 = w_1\}, \\ & \cup \{u_ku_{k+1}, w_kw_{k+1}, w_ku_{k+1}, u_kw_k \mid 2 \leq i \leq n, \\ & \text{for even } k, u_{n+1} = u_1, w_{n+1} = w_1\} \\ 3, & \text{otherwise.} \end{cases} \quad (10)$$

   * If n is odd then

$$c_{34}(e) = \begin{cases} \lceil \frac{i}{3} \rceil, & e \in \{u_0u_i \mid 1 \leq i \leq n\}, \\ 1, & e \in \{u_0v_i \mid 1 \leq i \leq n\} \cup \{v_iv_j \mid 1 \leq i, j \leq n, i \neq j\} \\ & \cup \{u_su_{s+1}, u_sw_s, w_su_{s+1}, w_sw_{s+1} \mid 1 \leq s \leq n-2, \\ & \text{for odd } s\} \\ 2, & e \in \{u_iv_i \mid 1 \leq i \leq n\} \\ & \cup \{v_iw_{i-1} \mid 1 \leq i \leq n, i \neq j, w_0 = w_1\}, \\ & \cup \{u_lu_{l+1}, w_lw_{l+1}, w_lu_{l+1}, u_lw_l \mid 2 \leq l \leq n-1, \\ & \text{for even } l, u_{n+1} = u_1, w_{n+1} = w_1\} \\ 3, & \text{otherwise.} \end{cases} \quad (11)$$

(ii) $n \geq 10$.
For the lower bound of $src(G_3)$, the proof is similar with Theorem 3.2, Case 3.2.3 (b). For the upper bound, we provide c_{33} (eq. (10) and c_{34} (11)) as the strong rainbow k-coloring of G_3 for $n \geq 10$.

□

4. Conclusion

In this paper, we have determined the rainbow connection numbers and strong rainbow connection numbers of the line, middle, and total graphs of wheel W_n on $n + 1$ vertices.

Acknowledgement

The author thanks the reviewers for the valuable comments that were useful for improving our paper.

References

[1] G. Chartrand, G.L. Johns, K.A. McKeon, and P. Zhang, Rainbow connection in graph. *Mathematica Bohemica* **15** (2006), 85–89, DOI: 10.21136/mb.2008.133947.

[2] D. Fitriani, A.N.M. Salman, and Z.Y. Awanis, Rainbow connection number of comb product of graphs, *Electron. J. Graph Theory Appl.* **10** (2) (2022), 461–473, DOI: 10.5614/ejgta.2022.10.2.9.

[3] X. Li and Y. Sun, An updated survey on rainbow connection of graphs – A dynamics survey, *Theory and Application of Graphs* **0** Issue 1 (2017), Article 3. DOI: 10.20429/tag.2017.000103.

[4] K.S. Rao and R. Murali, Rainbow connection number of sunlet graph and its line, middle and total graphs, *International Journal of Mathematics and its Applications* **3** Issue 4A (2015), 105–113.

[5] F. Septyanto, Rainbow connection number of corona product of graphs, *Electron. J. Graph Theory Appl.* **12** (2) (2024), 363–378, DOI: 10.5614/ejgta.2024.12.2.14.

[6] F. Septyanto and K.A. Sugeng, Color code techniques in rainbow connection, *Electron. J. Graph Theory Appl.* **6** (2) (2018), 347–361, DOI: 0.5614/ejgta.2018.6.2.14.

[7] Y. Sun, Rainbow connection number of line graphs, middle graphs and total graphs, *International Journal of Applied Mathematics and Statistics* **42** (12) (2013), 361–369.

[8] S. Sy, G.H. Medika, and L. Yulianti, The rainbow connection number of fan and sun, *Applied Mathematical Sciences* **7** (2013), 3155–3160, DOI: http://dx.doi.org/10.12988/ams.2013.13275.

[9] L. Yulianti, N. Narwen, S. Fitrianda, and K. Al Azizu, On the rainbow connection number and strong rainbow connection number of generalized triangle ladder graph, *Proceeding of the* 2^{nd} *International Conference on Science and Technology* (2020), 86–90.

[10] L. Yulianti, A. Nazra, M. Muhardiansyah, and N. Narwen, On the rainbow connection number of triangle-net graphs, *IOP Conf Series: Journal of Physics: Conf Series* **1836** (2021) 012004, DOI: 10.1088/1742-6596/1836/1/012004.

[11] Y. Zhao, S. Li, and S. Liu, 2018, (Strong) rainbow connection number of line, middle and total graph of sunlet graph, *The 2018 IEEE International Conference on Progress in Informatics and Computing* (PIC) (2018), 175–179, DOI: 10.1109/PIC.2018.8706299.