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Chromatically unique 6-bridge graph $\theta(a, a, a, b, b, c)$

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Abstract

For a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent if they share the same chromatic polynomial. A graph G is chromatically unique if for any graph chromatically equivalent to G is isomorphic to G. In this paper, the chromatically unique of a new family of 6-bridge graph $\theta(a, a, a, b, b, c)$ where $2 \leq a \leq b \leq c$ is investigated.

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1. Introduction

All graphs considered here are simple graphs. For such a graph G, let $P(G, \lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply $\chi-$equivalent), denoted by $G \sim H$, if $P(G, l) = P(H, l)$. A graph G is chromatically unique (or simply $\chi-$unique) if for any graph H such as $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G.

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The chromaticity of a graph G refers to questions about the chromatic equivalence class or chromatic uniqueness of G. For terminologies and notations which are not explained here, the reader is referred to [6, 20].

Let k be an integer with $k \geq 2$ and let $a_1, a_2, \ldots, a_k$ be positive integers with $a_i + a_j \geq 3$ for all i, j and $1 \leq i \leq j \leq k$. Let $\theta(a_1, a_2, \ldots, a_k)$ denote the graph obtained by connecting two distinct vertices with k independent (internally disjoint) paths of length $a_1, a_2, \ldots, a_k$, respectively. The graph $\theta(a_1, a_2, \ldots, a_k)$ is called a multi-bridge (more spesifically k-bridge) graph.

Given positive integers $a_1, a_2, \ldots, a_k$, where $k \geq 2$, what is the necessary and sufficient condition on $a_1, a_2, \ldots, a_k$ for $\theta(a_1, a_2, \ldots, a_k)$ to be chromatically unique? Many papers [4, 5, 14, 15] have been published on this problem, but it is still far from being completely solved.

For two non-empty graphs G and H, an *edge-gluing* of G and H is a graph obtained from G and H by identifying one edge of G with one edge of H. For example, the graph $K_4 - e$ (obtained from K_4 by deleting one edge) is an edge-gluing of K_3 and K_3. There are many edge-gluings of G and H. Let $g_e(G, H)$ denote the family of all edge-gluings of G and H. Zykov [25] showed that any member of $g_e(G, H)$ has chromatic polynomial

$$\frac{P(G, \lambda)P(H, \lambda)}{(\lambda(\lambda - 1))} \tag{1}$$

Thus any two members in $g_e(G, H)$ are χ-equivalent.

For any integer $k \geq 2$ and non-empty graphs $G_0, G_1, \cdots, G_k$, we can recursively define

$$g_e(G_0, G_1, \cdots, G_k) = \bigcup_{0 \leq i \leq k} g_e(G_i, G') \tag{2}$$

where $G' \in g_e(G_0, \cdots, G_{i-1}, G_{i+1}, \cdots, G_k)$.

Each graph in $g_e(G_0, G_1, \cdots, G_k)$ is also called an *edge-gluing* of $G_0, G_1, \cdots, G_k$. By (1), any two graphs in $g_e(G_0, G_1, \cdots, G_k)$ are χ-equivalent.

Let C_p denote the cycle of order p. It was shown independently in [19] and [21] that if G is χ-equivalent to a graph in $g_e(C_{i_0}, C_{i_1}, \cdots, C_{i_k})$, then $G \in g_e(C_{i_0}, C_{i_1}, \cdots, C_{i_k})$. In other words, this family is a χ-equivalence class.

A 2-bridge graph is simply a cycle graph, which is $\chi-$unique. Chao and Whitehead Jr. [2] showed that every 3-bridge graph $\theta(1, a_2, a_3)$ (or a theta graph) is $\chi-$unique. Loerinc [18] extended the above result to all 3-bridge graphs by showing that all 3-bridge graphs (or generalized θ-graph) are χ-unique. Assume therefore that $k \geq 4$. It is clear that if $a_i = 1$ for some i say $i = 1$, then $\theta(a_1, a_2, \cdots, a_k)$ is a member of $g_e(C_{a_2+1}, C_{a_3+1}, \cdots, C_{a_k+1})$ and thus $\theta(a_1, a_2, \cdots, a_k)$ is not χ-unique. Assume therefore that $a_i \geq 2$ for all i. For $k = 4$, Chen et al. [3] found that $\theta(a_1, a_2, a_3, a_4)$ may not be χ-unique.

Theorem 1.1. *(Chen et al. [3]) (a) Let a_1, a_2, a_3, a_4 be integers with $2 \leq a_1 \leq a_2 \leq a_3 \leq a_k$. Then $\theta(a_1, a_2, a_3, a_4)$ is χ-unique if and only if $(a_1, a_2, a_3, a_4) \neq (2, b, b+1, b+2)$ for any integer $b \geq 2$.*
(b) The χ-equivalence class of $\theta(2, b, b+1, b+2)$ is

$$\{\theta(2, b, b+1, b+2)\} \cup g_e\big(\theta(3, b, b+1), C_{b+2}\big).$$

Thus the problem of the chromaticity of $\theta(a_1, a_2, \cdots, a_k)$ has been completely settled for $k \leq 4$.

The results on the chromaticity of some families of 5-bridge graphs have been obtained by Bao and Chen [1], Li and Wei [17], Li [16], Khalaf [7], Khalaf and Peng [8], Khalaf et al. [13]. Ye [23, 24] proved that $\theta(2, 2, 2, 2, a, b)$ where $3 \leq a+1 \leq b$ and $\theta(2, 2, \ldots, 2, a, b)$ where $3 \leq a \leq b$ and $k \geq 5$ are χ-unique, respectively. Khalaf and Peng [9] also proved that $\theta(a, a, \ldots, a, b)$ for $a \leq b$ is χ-unique. The study on the chromaticity of 6-bridge graphs, $\theta(a_1, a_2, a_3, a_4, a_5, a_6)$ where $a_1, a_2, a_3, a_4, a_5, a_6$ assume exactly two distinct values and $\theta(3, 3, 3, 3, b, c)$ was done by Khalaf and Peng [10, 12]. Later on, Khalaf and Peng in [7, 11] solved the chromaticity of two types of 6-bridge graph $\theta(a_1, a_2, a_3, a_4, a_5, a_6)$ where $a_1, a_2, a_3, a_4, a_5, a_6$ assume exactly three distinct values, that is, the graphs $\theta(a, a, a, b, c, c)$ and $\theta(a, a, a, a, b, c)$, respectively. The aim of this paper is to investigate the chromaticity of another type of such graphs, that is, 6-bridge graphs $\theta(a, a, a, b, b, c)$ (see Figure 1).

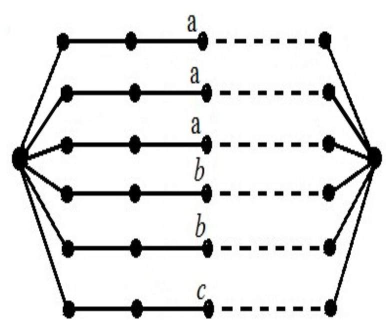

Figure 1. $\theta(a, a, a, b, b, c)$

2. Preliminary Results and Notations

In this section, we cite some results to be used in this paper. The following result is due to Xu et al. [22].

Lemma 2.1. *For $k \geq 4$, $\theta(a_1, a_2, \ldots, a_k)$ is χ-unique if $k-1 \leq a_1 \leq a_2 \leq \ldots \leq a_k$.*

Li and Wei [17] established that the 5-bridge graph $\theta(2, 2, 2, a, b)$ is χ-unique if and only if $(a, b) \neq (3, 4)$. Ye [23] extended the above result to any k-bridge graph $\theta(2, 2, \ldots, 2, a, b)$ with $b \geq a \geq 3$ and $k \geq 5$. For each positive integer h, the graph $G(h)$ is obtained from G by replacing each edge of G by a path of length h, respectively and is called the h-uniform subdivision of G. Xu et al. [21] showed that any h-uniform subdivision of θ_k denoted as $\theta_k(h)$, is χ-unique, as stated in the following theorem.

Lemma 2.2. *(Xu et al. [21]) For $k \geq 2$, the graph $\theta_k(h)$ is χ-unique.*

Dong et al. [5] proved the following result.

Lemma 2.3. *(Dong et al. [5]) If $2 \leq a_1 \leq a_2 \leq \ldots \leq a_k < a_1 + a_2$ where $k \geq 3$, then the graph $\theta(a_1, a_2, \ldots, a_k)$ is χ-unique.*

Let $k, a_1, a_2, \cdots, a_k \in N$, where N is natural number set and $G = \theta(a_1, a_2, \cdots, a_k)$. Then (see [4])

$$\begin{aligned} P(G,\lambda) &= \frac{1}{\lambda^{k-1}(\lambda-1)^{k-1}} \prod_{i=1}^{k} \big((\lambda-1)^{a_i+1} + (-1)^{a_i+1}(\lambda-1)\big) \\ &\quad + \frac{1}{\lambda^{k-1}} \prod_{i=1}^{k} \big((\lambda-1)^{a_i} + (-1)^{a_i}(\lambda-1)\big) \end{aligned}$$

Let $\lambda = 1 - x$, then

$$\begin{aligned} P(G, 1-x) &= \frac{(-1)^{a_1+a_2+\cdots+a_k+1}}{(1-x)^{k-1}} \big(x \prod_{i=1}^{k}(x^{a_i} - 1) - \prod_{i=1}^{k}(x^{a_i} - x)\big) \\ &= \frac{(-1)^{e(G)+1}}{(1-x)^{e(G)-v(G)+1}} \big(x \prod_{i=1}^{k}(x^{a_i} - 1) - \prod_{i=1}^{k}(x^{a_i} - x)\big) \end{aligned}$$

where $e(G) = \sum_{i=1}^{k} a_i$ and $v(G) = \sum_{i=1}^{k} a_i - k + 2$. Also define $Q(G, x)$ for any graph G and real number x as:

$$Q(G, x) = (-1)^{e(G)+1}(1-x)^{e(G)-v(G)+1} P(G, 1-x).$$

Then we have

Lemma 2.4. *(Dong et al. [5]) For any $k, a_1, a_2, \ldots, a_k \in N$,*

$$Q\big(\theta(a_1, a_2, \ldots, a_k), x\big) = x \prod_{i=1}^{k} \big(x^{a_i} - 1\big) - \prod_{i=1}^{k} \big(x^{a_i} - x\big).$$

Lemma 2.5. *(Dong et al. [5]) For any graphs G and H,*
1. *If $H \sim G$, then $Q(H, x) = Q(G, x)$,*
2. *If $Q(H, x) = Q(G, x)$ and $v(H) = v(G)$, then $H \sim G$.*

Lemma 2.6. *(Dong et al. [5]) Suppose that $\theta(a_1, a_2, \ldots, a_k) \sim \theta(b_1, b_2, \ldots, b_k)$ where $k \geq 3$, $2 \leq a_1 \leq a_2 \leq \ldots \leq a_k$ and $2 \leq b_1 \leq b_2 \leq \ldots \leq b_k$, then $a_i = b_i$ for all $i = 1, 2, \ldots, k$.*

Lemma 2.7. *(Dong et al. [5]) Let $H \sim \theta(a_1, a_2, \ldots, a_k)$ where $k \geq 3$ and $a_i \geq 2$ for all i, then one of the following is true:*
(i) $H \cong \theta(a_1, a_2, \ldots, a_k)$,

(ii) $H \in g_e\big(\theta(b_1, b_2, \ldots, b_t), C_{b_{t+1}+1}, \ldots, C_{b_k+1}\big)$, *where* $3 \leq t \leq k-1$ *and* $b_i \geq 2$ *for all* $i = 1, 2, \ldots, k$.

Lemma 2.8. *(Dong et al. [5]) Let* $k, t, b_1, b_2, \ldots, b_k \in N$ *where* $3 \leq t \leq k-1$ *and* $b_i \geq 2$ *for all* $i = 1, 2, \ldots, k$. *If* $H \in g_e\big(\theta(b_1, b_2, \ldots, b_t), C_{b_{t+1}+1}, \ldots, C_{b_k+1}\big)$, *then*

$$Q\big(H, x\big) = x \prod_{i=1}^{k} \big(x^{b_i} - 1\big) - \prod_{i=1}^{t} \big(x^{b_i} - x\big) \prod_{i=t+1}^{k} \big(x^{b_i} - 1\big).$$

Lemma 2.9. *(Koh & Teo [14]) If* $G \sim H$, *then*

(i) $v(G) = v(H)$,

(ii) $e(G) = e(H)$,

(iii) $g(G) = g(H)$,

(iv) G *and* H *have the same number of shortest cycle.*

where $v(G)$, $v(H)$, $e(G)$, $e(H)$, $g(G)$ *and* $g(H)$ *denote the number of vertices, the number of edges and the girth of* G *and* H, *respectively.*

Lemma 2.10. *(Khalaf & Peng [11]) A 6-bridge graph* $\theta(a_1, a_2, \ldots, a_6)$ *is* χ*-unique if the positive integers* $a_1, a_2, \ldots, a_6$ *assume exactly two distinct values.*

3. Main Results

In this section, we present our main result on the chromaticity of 6-bridge graph $\theta(a, a, a, b, b, b, c)$.

Theorem 3.1. *The graph 6-bridge* $\theta(a, a, a, b, b, c)$, *where* $a \leq b \leq c$, *is* χ*-unique.*

Proof. Let G be a 6-bridge graph of the form $\theta(a, a, a, b, b, c)$ where $2 \leq a \leq b \leq c$. By Lemma 2.3, G is χ-unique if $c < 2a$. Suppose $c \geq 2a$ and $H \sim G$. We shall solve $Q(G) = Q(H)$ to get all solutions. Let the *lowest remaining power* and the *highest remaining power* be denoted by l.r.p. and h.r.p., respectively. By Lemma 2.9, $g(G) = g(H) = 2a$ and H has the same number of shortest cycles as G. Thus, we have

$$3a + 2b + c = b_1 + b_2 + b_3 + b_4 + b_5 + b_6 \tag{3}$$

By Lemmas 2.6 and 2.7, we have three cases to consider, (A) $H \in g_e\big(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1}\big)$ where $2 \leq b_1 \leq b_2 \leq b_3$ and $2 \leq b_4, b_5, b_6$ or (B) $H \in g_e\big(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1}\big)$ where $2 \leq b_1 \leq b_2 \leq b_3 \leq b_4$ and $2 \leq b_5, b_6$ or (C) $H \in g_e\big(\theta(b_1, b_2, b_3, b_4, b_5), C_{b_6+1}\big)$ where $2 \leq b_1 \leq b_2 \leq b_3 \leq b_4 \leq b_5$ and $2 \leq b_6$.

Case A $H \in g_e\big(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1}\big)$ where $2 \leq b_1 \leq b_2 \leq b_3$ and $2 \leq b_4, b_5, b_6$. As $G \cong \theta(a, a, a, b, b, c)$ and $H \in g_e\big(\theta(b_1, b_2, b_3), C_{b_4+1}, C_{b_5+1}, C_{b_6+1}\big)$, then by Lemma 2.8, we have

$$\begin{aligned}
Q(G) &= x(x^a-1)^3(x^b-1)^2(x^c-1)-(x^a-x)^3(x^b-x)^2(x^c-x),\\
Q(H) &= x(x^{b_1}-1)(x^{b_2}-1)(x^{b_3}-1)(x^{b_4}-1)(x^{b_5}-1)(x^{b_6}-1)-\\
&\quad (x^{b_1}-x)(x^{b_2}-x)(x^{b_3}-x)(x^{b_4}-1)(x^{b_5}-1)(x^{b_6}-1).
\end{aligned}$$

By equation 3, $Q(G)=Q(H)$ yields

$$\begin{aligned}
Q_1(G) &= 2x^{3a+b+1}+x^{3a+c+1}+x^{3a+3}+3x^{2a+2b+1}+6x^{2a+b+c+1}+\\
&\quad 6x^{2a+b+3}+3x^{2a+c+3}+3x^{2a+1}+3x^{a+2b+c+1}+3x^{a+2b+3}+\\
&\quad 6x^{a+b+c+3}+6x^{a+b+1}+3x^{a+c+1}+3x^{a+5}+x^{2b+c+3}+x^{2b+1}+\\
&\quad 2x^{b+c+1}+2x^{b+5}+x^{c+5}-\big(2x^{3a+b+2}+x^{3a+c+2}+x^{3a+1}+\\
&\quad 3x^{2a+2b+2}+6x^{2a+b+c+2}+6x^{2a+b+1}+3x^{2a+c+1}+3x^{2a+4}+\\
&\quad 3x^{a+2b+c+2}+3x^{a+2b+1}+6x^{a+b+c+1}+6x^{a+b+4}+3x^{a+c+4}+\\
&\quad 3x^{a+1}+x^{2b+c+1}+x^{2b+4}+2x^{b+c+4}+2x^{b+1}+x^{c+1}+x^{6}\big),\\
Q_1(H) &= x^{b_1+b_2+b_3+b_4+b_5}+x^{b_1+b_2+b_3+b_4+b_6}+x^{b_1+b_2+b_3+b_4+1}+\\
&\quad x^{b_1+b_2+b_3+b_5+b_6}+x^{b_1+b_2+b_3+b_5+1}+x^{b_1+b_2+b_3+b_6+1}+\\
&\quad x^{b_1+b_2+b_3}+x^{b_1+b_4+b_5+b_6+1}+x^{b_1+b_4+b_5+2}+x^{b_1+b_4+b_6+2}+\\
&\quad x^{b_1+b_4+1}+x^{b_1+b_5+b_6+2}+x^{b_1+b_5+1}+x^{b_1+b_6+1}+x^{b_1+2}+\\
&\quad x^{b_2+b_4+b_5+b_6+1}+x^{b_2+b_4+b_5+2}+x^{b_2+b_4+b_6+2}+x^{b_2+b_4+1}+\\
&\quad x^{b_2+b_5+b_6+2}+x^{b_2+b_5+1}+x^{b_2+b_6+1}+x^{b_2+2}+x^{b_3+b_4+b_5+b_6+1}+\\
&\quad x^{b_3+b_4+b_5+2}+x^{b_3+b_4+b_6+2}+x^{b_3+b_4+1}+x^{b_3+b_5+b_6+2}+\\
&\quad x^{b_3+b_5+1}+x^{b_3+b_6+1}+x^{b_3+2}+x^{b_4+b_5+b_6+3}+x^{b_4+b_5+1}+\\
&\quad x^{b_4+b_6+1}+x^{b_4+3}+x^{b_5+b_6+1}+x^{b_5+3}+x^{b_6+3}-\\
&\quad \big(x^{b_1+b_2+b_3+b_4+b_5+1}+x^{b_1+b_2+b_3+b_4+b_6+1}+x^{b_1+b_2+b_3+b_4}+\\
&\quad x^{b_1+b_2+b_3+b_5+b_6+1}+x^{b_1+b_2+b_3+b_5}+x^{b_1+b_2+b_3+b_6}+x^{b_1+b_2+b_3+1}+\\
&\quad x^{b_1+b_4+b_5+b_6+2}+x^{b_1+b_4+b_5+1}+x^{b_1+b_4+b_6+1}+x^{b_1+b_4+2}+\\
&\quad x^{b_1+b_5+b_6+2}+x^{b_1+b_5+2}+x^{b_1+b_6+2}+x^{b_1+1}+x^{b_2+b_4+b_5+b_6+2}+\\
&\quad x^{b_2+b_4+b_5+1}+x^{b_2+b_4+b_6+1}+x^{b_2+b_4+2}+x^{b_2+b_5+b_6+1}+\\
&\quad x^{b_2+b_5+2}+x^{b_2+b_6+2}+x^{b_2+1}+x^{b_3+b_4+b_5+b_6+2}+x^{b_3+b_4+b_5+1}+\\
&\quad x^{b_3+b_4+b_6+1}+x^{b_3+b_4+2}+x^{b_3+b_5+b_6+1}+x^{b_3+b_5+2}+x^{b_3+b_6+2}+\\
&\quad x^{b_3+1}+x^{b_4+b_5+b_6+1}+x^{b_4+b_5+3}+x^{b_4+b_6+3}+x^{b_4+1}+\\
&\quad x^{b_5+b_6+3}+x^{b_5+1}+x^{b_6+1}+x^{3}\big).
\end{aligned}$$

Compare the l.r.p. in $Q_1(G)$ and the l.r.p. in $Q_1(H)$. Thus, $a=2$. Therefore, $g(G)=g(H)=2a=4$ and both G and H has three cycles of length 4, respectively. Without loss of generality, we have four cases to consider, (1) $b_4=b_5=b_6=3$ or (2) $b_4=b_5=3, b_6\neq 3$ or (3) $b_4=3, b_5\neq 3, b_6\neq 3$ or (4) $b_4\neq 3, b_5\neq 3, b_6\neq 3$.

Case 1 $b_4 = b_5 = b_6 = 3$. Note that there is $-3x^{a+1}$ in $Q_1(G)$. Hence, there are another two terms in $Q_1(H)$ that are equal to $-x^3$. Thus, we have $b_1 = b_2 = 2$ or $b_1 = b_3 = 2$ or $b_2 = b_3 = 2$.

Case 1.1 $b_1 = b_2 = 2$. Therefore, H has four cycles of length 4, a contradiction.

Case 1.2 $b_1 = b_3 = 2$. So $b_2 = 2$. Therefore, H has six cycles of length 4, a contradiction.

Case 1.3 $b_2 = b_3 = 2$. So $b_1 = 2$. Therefore, H has six cycles of length 4, a contradiction.

Case 2 $b_4 = b_5 = 3, b_6 \neq 3$. Since the girth of H is 4, then $b_6 \geq 4$. Given that H has three cycles of length 4, then $b_1 + b_2 = 4$. So $b_1 = b_2 = 2$. It follows from equation 3 that $2b + c = b_3 + b_6 + 4$. We obtain the following after simplification.

$$\begin{aligned}
Q_2(G) &= x^{2b+c+3} + 12x^{b+c+5} + 2x^{b+c+1} + 6x^{2b+5} + x^{2b+1} + 8x^{b+7} + \\
&\quad 6x^{b+3} + 4x^{c+7} + 3x^{c+3} + x^9 + 2x^7 + 3x^5 - \big(x^{2b+c+1} + \\
&\quad 6x^{b+c+6} + 2x^{b+c+4} + 6x^{b+c+3} + 3x^{2b+6} + x^{2b+4} + 3x^{2b+3} + \\
&\quad 2x^{b+8} + 6x^{b+6} + 4x^{b+5} + 2x^{b+1} + x^{c+8} + 3x^{c+6} + 2x^{c+5} + \\
&\quad x^{c+1} + 3x^8 + x^6\big), \\
Q_2(H) &= 3x^{b_3+b_6+5} + x^{b_3+b_6+1} + x^{b_3+10} + 3x^{b_3+8} + 3x^{b_3+4} + x^{b_3+2} + \\
&\quad 3x^{b_6+9} + 3x^{b_6+7} + 3x^{b_6+3} + 2x^{10} + 6x^6 - \big(3x^{b_3+b_6+4} + \\
&\quad x^{b_3+b_6+2} + x^{b_3+11} + 3x^{b_3+7} + 3x^{b_3+5} + x^{b_3+1} + 2x^{b_6+10} + \\
&\quad 6x^{b_6+6} + x^{b_6+1} + 3x^9 + 3x^7\big).
\end{aligned}$$

Consider the l.r.p. in $Q_2(G)$ and the l.r.p. in $Q_2(H)$. We have $b = c = 4$. Therefore, $G \cong \theta(2,2,2,4,4,4)$. By Lemma 2.10, G is χ-unique.

Case 3 $b_4 = 3, b_5 \neq 3, b_6 \neq 3$. Since the girth of H is 4, then $b_5 \geq 4$ and $b_6 \geq 4$. Given that H has three cycles of length 4, then $b_1 + b_2 = 4$ and $b_1 + b_3 = 4$. So $b_1 = b_2 = b_3 = 2$. Now, H has four cycles of length 4, a contradiction.

Case 4 $b_4 \neq 3, b_5 \neq 3, b_6 \neq 3$. Since the girth of H is 4, then $b_4 \geq 4$, $b_5 \geq 4$ and $b_6 \geq 4$. Given that H has three cycles of length 4, then $b_1 + b_2 = 4$, $b_1 + b_3 = 4$ and $b_2 + b_3 = 4$. Thus, $b_1 = b_2 = b_3 = 2$. It follows from equation 3 that $2b + c = b_4 + b_5 + b_6$. We obtain the following after simplification.

$$\begin{aligned}
Q_3(G) &= 12x^{b+c+5} + 2x^{b+c+1} + 6x^{2b+5} + x^{2b+1} + 8x^{b+7} + 6x^{b+3} + \\
&\quad 4x^{c+7} + 3x^{c+3} + x^9 + 2x^7 + 3x^5 - \big(x^{2b+c+1} + 3x^{2b+6} + \\
&\quad x^{2b+4} + 3x^{2b+3} + 6x^{b+c+6} + 2x^{b+c+4} + 6x^{b+c+3} + 2x^{b+8} + \\
&\quad 6x^{b+6} + 4x^{b+5} + 2x^{b+1} + x^{c+8} + 3x^{c+6} + 2x^{c+5} + x^{c+1} + \\
&\quad 3x^8 + x^6\big),
\end{aligned}$$

$$\begin{aligned} Q_3(H) &= x^{b_4+b_5+6} + 3x^{b_4+b_5+4} + x^{b_4+b_5+1} + x^{b_4+b_6+6} + 3x^{b_4+b_6+4} + \\ &\quad x^{b_4+b_6+1} + x^{b_4+7} + 4x^{b_4+3} + x^{b_5+b_6+6} + 3x^{b_5+b_6+4} + \\ &\quad x^{b_5+b_6+1} + x^{b_5+7} + 4x^{b_5+3} + x^{b_6+7} + 4x^{b_6+3} + x^6 + 3x^4 - \\ &\quad \big(x^{b_4+b_5+b_6+1} + x^{b_4+b_5+7} + 4x^{b_4+b_5+3} + x^{b_4+b_6+7} + 4x^{b_4+b_6+3} + \\ &\quad x^{b_4+6} + 3x^{b_4+4} + x^{b_4+1} + x^{b_5+b_6+7} + 4x^{b_5+b_6+3} + x^{b_5+6} + \\ &\quad 3x^{b_5+4} + x^{b_5+1} + x^{b_6+6} + 3x^{b_6+4} + x^{b_6+1} + x^7 + x^3\big). \end{aligned}$$

Compare the l.r.p. in $Q_3(G)$ and the l.r.p. in $Q_3(H)$. We have $b = 2$ or $c = 2$.
If $b = 2$, then $G \cong \theta(2, 2, 2, 2, 2, c)$. By Lemma 2.10, G is χ-unique.
If $c = 2$, then $G \cong \theta(2, 2, 2, 2, 2, 2)$. By Lemma 2.2, G is χ-unique.

<u>Case B</u> $H \in g_e\big(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1}\big)$ where $2 \le b_1 \le b_2 \le b_3 \le b_4$ and $2 \le b_5, b_6$. As $G \cong \theta(a, a, a, b, b, c)$ and $H \in g_e\big(\theta(b_1, b_2, b_3, b_4), C_{b_5+1}, C_{b_6+1}\big)$, then

$$\begin{aligned} Q_4(G) &= x\big(x^a - 1\big)^3\big(x^b - 1\big)^2\big(x^c - 1\big) - \big(x^a - x\big)^3\big(x^b - x\big)^2\big(x^c - x\big), \\ Q_4(H) &= x\big(x^{b_1} - 1\big)\big(x^{b_2} - 1\big)\big(x^{b_3} - 1\big)\big(x^{b_4} - 1\big)\big(x^{b_5} - 1\big)\big(x^{b_6} - 1\big) - \\ &\quad \big(x^{b_1} - x\big)\big(x^{b_2} - x\big)\big(x^{b_3} - x\big)\big(x^{b_4} - x\big)\big(x^{b_5} - 1\big)\big(x^{b_6} - 1\big). \end{aligned}$$

By equation 3, $Q_4(G) = Q_4(H)$ yields

$$\begin{aligned} Q_5(G) &= 2x^{3a+b+1} + x^{3a+c+1} + x^{3a+3} + 3x^{2a+2b+1} + 6x^{2a+b+c+1} + \\ &\quad 6x^{2a+b+3} + 3x^{2a+c+3} + 3x^{2a+1} + 3x^{a+2b+c+1} + 3x^{a+2b+3} + \\ &\quad 6x^{a+b+c+3} + 6x^{a+b+1} + 3x^{a+c+1} + 3x^{a+5} + x^{2b+c+3} + \\ &\quad x^{2b+1} + 2x^{b+c+1} + 2x^{b+5} + x^{c+5} - \big(2x^{3a+b+2} + x^{3a+c+2} + \\ &\quad x^{3a+1} + 3x^{2a+2b+2} + 6x^{2a+b+c+2} + 6x^{2a+b+1} + 3x^{2a+c+1} + \\ &\quad 3x^{2a+4} + 3x^{a+2b+c+2} + 3x^{a+2b+1} + 6x^{a+b+c+1} + 6x^{a+b+4} + \\ &\quad 3x^{a+c+4} + 3x^{a+1} + x^{2b+c+1} + x^{2b+4} + 2x^{b+c+4} + 2x^{b+1} + \\ &\quad x^{c+1} + x^6\big), \end{aligned}$$

$$\begin{aligned} Q_5(H) &= x^{b_1+b_2+b_3+b_4+b_5}+x^{b_1+b_2+b_3+b_4+b_6}+x^{b_1+b_2+b_3+b_4+1}+\\ &x^{b_1+b_2+b_5+b_6+1}+x^{b_1+b_2+b_5+2}+x^{b_1+b_2+b_6+2}+x^{b_1+b_2+1}+\\ &x^{b_1+b_3+b_5+b_6+1}+x^{b_1+b_3+b_5+2}+x^{b_1+b_3+b_6+2}+x^{b_1+b_3+1}+\\ &x^{b_1+b_4+b_5+b_6+1}+x^{b_1+b_4+b_5+2}+x^{b_1+b_4+b_6+2}+x^{b_1+b_4+1}+\\ &x^{b_1+b_5+b_6+3}+x^{b_1+b_5+1}+x^{b_1+b_6+1}+x^{b_1+3}+x^{b_2+b_3+b_5+b_6+1}+\\ &x^{b_2+b_3+b_5+2}+x^{b_2+b_3+b_6+2}+x^{b_2+b_3+1}+x^{b_2+b_4+b_5+b_6+1}+\\ &x^{b_2+b_4+b_5+2}+x^{b_2+b_4+b_6+2}+x^{b_2+b_4+1}+x^{b_2+b_5+b_6+3}+\\ &x^{b_2+b_5+1}+x^{b_2+b_6+1}+x^{b_2+3}+x^{b_3+b_4+b_5+b_6+1}+x^{b_3+b_4+b_5+2}+\\ &x^{b_3+b_4+b_6+2}+x^{b_3+b_4+1}+x^{b_3+b_5+b_6+3}+x^{b_3+b_5+1}+x^{b_3+b_6+1}+\\ &x^{b_3+3}+x^{b_4+b_5+b_6+3}+x^{b_4+b_5+1}+x^{b_4+b_6+1}+x^{b_4+3}+x^{b_5+b_6+1}+\\ &x^{b_5+4}+x^{b_6+4}-\big(x^{b_1+b_2+b_3+b_4+b_5+1}+x^{b_1+b_2+b_3+b_4+b_6+1}+\\ &x^{b_1+b_2+b_3+b_4}+x^{b_1+b_2+b_5+b_6+2}+x^{b_1+b_2+b_5+1}+x^{b_1+b_2+b_6+1}+\\ &x^{b_1+b_2+2}+x^{b_1+b_3+b_5+b_6+2}+x^{b_1+b_3+b_5+1}+x^{b_1+b_3+b_6+1}+\\ &x^{b_1+b_3+2}+x^{b_1+b_4+b_5+b_6+2}+x^{b_1+b_4+b_5+1}+x^{b_1+b_4+b_6+1}+\\ &x^{b_1+b_4+2}+x^{b_1+b_5+b_6+1}+x^{b_1+b_5+3}+x^{b_1+b_6+3}+x^{b_1+1}+\\ &x^{b_2+b_3+b_5+b_6+2}+x^{b_2+b_3+b_5+1}+x^{b_2+b_3+b_6+1}+x^{b_2+b_3+2}+\\ &x^{b_2+b_4+b_5+b_6+2}+x^{b_2+b_4+b_5+1}+x^{b_2+b_4+b_6+1}+x^{b_2+b_4+2}+\\ &x^{b_2+b_5+b_6+1}+x^{b_2+b_5+3}+x^{b_2+b_6+3}+x^{b_2+1}+x^{b_3+b_4+b_5+b_6+2}+\\ &x^{b_3+b_4+b_5+1}+x^{b_3+b_4+b_6+1}+x^{b_3+b_4+2}+x^{b_3+b_5+b_6+1}+\\ &x^{b_3+b_5+3}+x^{b_3+b_6+3}+x^{b_3+1}+x^{b_4+b_5+b_6+1}+x^{b_4+b_5+3}+\\ &x^{b_4+b_6+3}+x^{b_4+1}+x^{b_5+b_6+4}+x^{b_5+1}+x^{b_6+1}+x^{4}\big). \end{aligned}$$

Since $2 \le a \le b \le c$, by comparing the l.r.p. in $Q_5(G)$ and the l.r.p. in $Q_5(H)$, we have $a = 2$ or $a = 3$.

Case 1 $a = 2$. Then $g(G) = g(H) = 2a = 4$. There are three cycles of length 4 in G, and H has the same number as well. Without loss of generality, we have the following cases to consider.

Case 1.1 $b_5 = b_6 = 3$. Since H has three cycles of length 4, then $b_1+b_2 = 4$. So $b_1 = b_2 = 2$. Note that there is $-3x^{a+1}$ in $Q_5(G)$. Hence, there is one more term in $Q_5(H)$ that equal to $-x^3$. Thus, $b_3 = 2$ or $b_4 = 2$.

If $b_3 = 2$, then H has five cycles of length 4, a contradiction.

If $b_4 = 2$, then $b_3 = 2$. So H has eight cycles of length 4, a contradiction.

Case 1.2 $b_5 = 3, b_6 \neq 3$. Since H has girth 4, then $b_6 \ge 4$. Given that H has three cycles of length 4, then $b_1 + b_2 = 4$ and $b_1 + b_3 = 4$. So $b_1 = b_2 = b_3 = 2$. Now H has four cycles of length 4, a contradiction.

Case 1.3 $b_5 \neq 3, b_6 \neq 3$. Since H has girth 4, then $b_5 \ge 4$ and $b_6 \ge 4$. Given that H has three cycles of length 4, then $b_1 + b_2 = 4$, $b_1 + b_3 = 4$ and $\big(b_1 + b_4 = 4$ or $b_2 + b_3 = 4\big)$. Therefore, we have two cases to consider.

Case 1.3.1 $b_1 + b_4 = 4$. So $b_1 = b_4 = 2$. Thus, $b_1 = b_2 = b_3 = b_4 = 2$. Hence H has six cycles of length 4, a contradiction.

<u>Case 1.3.2</u> $b_2 + b_3 = 4$. So $b_2 = b_3 = 2$. Thus, $b_1 = b_2 = b_3 = 2$. It follows from equation 3 that $2b + c = b_4 + b_5 + b_6$. Then we obtain the following after simplification.

$$\begin{aligned}
Q_6(G) &= 6x^{2b+5} + x^{2b+1} + 12x^{b+c+5} + 2x^{b+c+1} + 8x^{b+7} + 6x^{b+3} + \\
&\quad 4x^{c+7} + 3x^{c+3} + x^9 + 2x^7 - \big(x^{2b+c+1} + 3x^{2b+6} + x^{2b+4} + \\
&\quad 3x^{2b+3} + 6x^{b+c+6} + 2x^{b+c+4} + 6x^{b+c+3} + 2x^{b+8} + x^{b+6} + \\
&\quad 4x^{b+5} + 2x^{b+1} + x^{c+8} + 3x^{c+6} + 2x^{c+5} + x^{c+1} + 3x^8\big), \\
Q_6(H) &= x^{b_4+b_5+6} + 3x^{b_4+b_5+4} + x^{b_4+b_5+1} + x^{b_4+b_6+6} + 3x^{b_4+b_6+4} + \\
&\quad x^{b_4+b_6+1} + x^{b_4+7} + 4x^{b_4+3} + 6x^{b_5+b_6+5} + x^{b_5+b_6+1} + 3x^{b_5+6} + \\
&\quad x^{b_5+4} + 3x^{b_5+3} + 3x^{b_6+6} + x^{b_6+4} + 3x^{b_6+3} + 3x^5 - \\
&\quad \big(x^{b_4+b_5+b_6+1} + x^{b_4+b_5+7} + 4x^{b_4+b_5+3} + x^{b_4+b_6+7} + 4x^{b_4+b_6+3} + \\
&\quad x^{b_4+6} + 3x^{b_4+4} + x^{b_4+1} + 3x^{b_5+b_6+6} + x^{b_5+b_6+4} + 3x^{b_5+b_6+3} + \\
&\quad 3x^{b_5+6} + 6x^{b_5+5} + x^{b_5+1} + x^{b_6+5} + x^{b_6+1} + 2x^6 + x^4\big).
\end{aligned}$$

Comparing the l.r.p. in $Q_6(G)$ and the l.r.p. in $Q_6(H)$, we have $b = 3$ or $c = 3$. If $c = 3$, then $b = 2$ or $b = 3$. By Lemma 2.10, G is χ-unique for both cases. So $b = 3$. Note that there is $2x^{b+1}$ in $Q_6(G)$, thus, $b_4 = 3$. Simplifying $Q_6(G)$ and $Q_6(H)$, we obtain the following.

$$\begin{aligned}
Q_7(G) &= 5x^{c+8} + x^{c+7} + x^{c+4} + 3x^{c+3} + 4x^{11} + 6x^{10} + 3x^7 + \\
&\quad 4x^6 - \big(3x^{c+9} + 6x^{c+6} + 2x^{c+5} + x^{c+1} + 3x^{12} + 7x^9 + \\
&\quad 7x^8\big), \\
Q_7(H) &= x^{b_5+9} + 3x^{b_5+7} + 2x^{b_5+4} + 3x^{b_5+3} + x^{b_6+9} + 3x^{b_6+7} + \\
&\quad 2x^{b_6+4} + 3x^{b_6+3} + 3x^5 - \big(2x^{b_5+b_6+4} + x^{b_5+10} + x^{b_5+6} + \\
&\quad 6x^{b_5+5} + x^{b_5+1} + x^{b_6+10} + x^{b_6+6} + 6x^{b_6+5} + x^{b_6+1} + 3x^7\big).
\end{aligned}$$

Consider the term $3x^5$ in $Q_7(H)$. Thus, $b_5 = 4$ and $b_6 = 4$. Therefore $c = 5$. However, we obtain $Q_7(G) \neq Q_7(H)$, a contradiction.

<u>Case 2</u> $a = 3$. Therefore, $g(G) = g(H) = 2a = 6$. There are three cycles of length 6 in G and H, respectively. Without loss of generality, we have three cases to consider, that are $b_5 = b_6 = 5$ or $b_5 = 5, b_6 \neq 5$ or $b_5 \neq 5, b_6 \neq 5$.

<u>Case 2.1</u> $b_5 = b_6 = 5$. Therefore, $b_1 + b_2 = 6$. Thus, we have $b_1 = 2, b_2 = 4$ or $b_1 = b_2 = 3$.

<u>Case 2.1.1</u> $b_1 = 2, b_2 = 4$. It follows from equation 3 that $2b + c = b_3 + b_4 + 7$. Since $3 \leq b \leq c$ and $4 \leq b_3 \leq b_4$, by cancelling the equal terms, there is $-x^3$ in $Q_5(H)$ but not in $Q_5(G)$, a contradiction.

<u>Case 2.1.2</u> $b_1 = b_2 = 3$. From equation 3, we obtain

$$2b + c = b_3 + b_4 + 7 \tag{4}$$

We obtain the following after simplification.

$$\begin{aligned}
Q_8(G) &= x^{2b+c+3}+6x^{b+c+7}+6x^{b+c+6}+2x^{b+c+1}+3x^{2b+7}+3x^{2b+6}+\\
&\quad x^{2b+1}+2x^{b+10}+6x^{b+9}+2x^{b+5}+6x^{b+4}+x^{c+10}+3x^{c+9}+\\
&\quad x^{c+5}+3x^{c+4}+x^{12}+3x^{8}+2x^{7}-\big(x^{2b+c+1}+6x^{b+c+8}+\\
&\quad 8x^{b+c+4}+3x^{2b+8}+4x^{2b+4}+2x^{b+11}+12x^{b+7}+2x^{b+1}+\\
&\quad x^{c+11}+6x^{c+7}+x^{c+1}+4x^{10}+x^{6}\big),\\
Q_8(H) &= 3x^{b_3+b_4+7}+x^{b_3+b_4+1}+2x^{b_3+14}+x^{b_3+13}+4x^{b_3+10}+2x^{b_3+6}+\\
&\quad 2x^{b_3+4}+x^{b_3+3}+2x^{b_4+14}+x^{b_4+13}+4x^{b_4+10}+2x^{b_4+6}+\\
&\quad 2x^{b_4+4}+x^{b_4+3}+x^{17}+2x^{16}+2x^{13}+6x^{9}-\big(3x^{b_3+b_4+6}+\\
&\quad x^{b_3+b_4+2}+2x^{b_3+15}+x^{b_3+11}+4x^{b_3+9}+2x^{b_3+8}+2x^{b_3+5}+\\
&\quad x^{b_3+1}+2x^{b_4+15}+x^{b_4+11}+4x^{b_4+9}+2x^{b_4+8}+2x^{b_4+5}+\\
&\quad x^{b_4+1}+x^{18}+3x^{14}+2x^{12}+3x^{11}+x^{8}\big).
\end{aligned}$$

Comparing the l.r.p. in $Q_8(G)$ and the l.r.p. in $Q_8(H)$, we have $b_3=5$ or $b_4=5$.

Case 2.1.2.1 $b_3=5$. Then we obtain the following after simplification.

$$\begin{aligned}
Q_9(G) &= x^{2b+c+3}+6x^{b+c+7}+6x^{b+c+6}+2x^{b+c+1}+3x^{2b+7}+3x^{2b+6}+\\
&\quad x^{2b+1}+2x^{b+10}+6x^{b+9}+2x^{b+5}+6x^{b+4}+x^{c+10}+3x^{c+9}+\\
&\quad x^{c+5}+3x^{c+4}+x^{12}+3x^{8}+2x^{7}-\big(x^{2b+c+1}+6x^{b+c+8}+\\
&\quad 8x^{b+c+4}+3x^{2b+8}+4x^{2b+4}+2x^{b+11}+12x^{b+7}+2x^{b+1}+\\
&\quad x^{c+11}+6x^{c+7}+x^{c+1}+2x^{10}\big),\\
Q_9(H) &= 2x^{b_4+14}+x^{b_4+13}+3x^{b_4+12}+4x^{b_4+10}+3x^{b_4+6}+2x^{b_4+4}+\\
&\quad x^{b_4+3}+2x^{19}+x^{17}+x^{16}+4x^{15}+8x^{9}-\big(2x^{b_4+15}+4x^{b_4+11}+\\
&\quad 4x^{b_4+9}+2x^{b_4+8}+x^{b_4+7}+2x^{b_4+5}+x^{b_4+1}+2x^{20}+7x^{14}+\\
&\quad 2x^{12}+x^{11}\big).
\end{aligned}$$

Consider the term $2x^7$ in $Q_9(G)$. We have $b=6$ or $c=6$.

If $b=6$, then $c=b_4$. However, $Q_9(G)\neq Q_9(H)$, a contradiction.

If $c=6$, then we obtain $b=6$ and $b_4=6$. Therefore, $G\cong\theta(3,3,3,6,6,6)$. By Lemma 2.10, G is χ-unique.

Case 2.1.2.2 $b_4=5$. Then, we have $b_3=3$ or $b_3=4$ or $b_3=5$.

Case 2.1.2.2(a) If $b_3=3$, then H has five cycles of length 6, a contradiction.

Case 2.1.2.2(b) If $b_3=4$, then there is $-x^5$ in $Q_8(H)$. Hence, $b=4$ or $c=4$.

If $b=4$, by equation 4 we have $c=8$. But $Q_8(G)\neq Q_8(H)$, a contradiction.

If $c=4$, by equation 4 we have $b=6$. But $3\le b\le 4$, a contradiction.

Case 2.1.2.2(c) If $b_3=5$, then it follows from equation 4 that $2b+c=17$. We obtain the following after simplification.

$$\begin{aligned} Q_{10}(G) &= x^{2b+c+3} + 6x^{b+c+7} + 6x^{b+c+6} + 2x^{b+c+1} + 3x^{2b+7} + 3x^{2b+6} + \\ & x^{2b+1} + 2x^{b+10} + 6x^{b+9} + 2x^{b+5} + 6x^{b+4} + x^{c+10} + 3x^{c+9} + \\ & x^{c+5} + 3x^{c+4} + x^{12} + 2x^8 + 2x^7 - \big(x^{2b+c+1} + 6x^{b+c+8} + \\ & 8x^{b+c+4} + 3x^{2b+8} + 4x^{2b+4} + 2x^{b+11} + 12x^{b+7} + 2x^{b+1} + \\ & x^{c+11} + 6x^{c+7} + x^{c+1}\big), \\ Q_{10}(H) &= 4x^{19} + x^{18} + 4x^{17} + 8x^{15} + 2x^{11} + 10x^9 - \big(4x^{20} + 3x^{16} + \\ & 11x^{14} + 2x^{13} + 3x^{12} + x^6\big). \end{aligned}$$

Compare the l.r.p. in $Q_{10}(G)$ and the l.r.p. in $Q_{10}(H)$. We have $b = 5$ or $c = 5$. Since the coefficient of $-x^{b+1}$ is 2, then $c = 5$. If $c = 5$, we have $b = 6$. But $3 \leq b \leq 5$, a contradiction.

Case 2.2 $b_5 = 5, b_6 \neq 5$. Therefore, $b_1 + b_2 = 6$ and $b_1 + b_3 = 6$. Thus, $b_2 = b_3$. Hence, we have $b_1 = 2, b_2 = b_3 = 4$ or $b_1 = b_2 = b_3 = 3$.

Case 2.2.1 $b_1 = 2, b_2 = b_3 = 4$. It follows from equation 3 that $2b + c = b_4 + b_6 + 6$. However, cancelling the equal terms we obtain $Q_5(G) \neq Q_5(H)$, a contradiction.

Case 2.2.2 $b_1 = b_2 = b_3 = 3$. Then H has four cycles of length 6, a contradiction.

Case 2.3 $b_5 \neq 5, b_6 \neq 5$. Note that the girth of H is 6, thus $b_5 \geq 6$ and $b_6 \geq 6$. Since H has three cycles of length 6, therefore $b_1 + b_2 = 6$, $b_1 + b_3 = 6$ and $\big(b_1 + b_4 = 6$ or $b_2 + b_3 = 6\big)$.

Case 2.3.1 $b_1 + b_4 = 6$. Considering $b_1 + b_2 = 6$ and $b_1 + b_3$=6, then $b_2 = b_3 = b_4$. Hence, we have $b_1 = 2, b_2 = b_3 = b_4 = 4$ or $b_1 = b_2 = b_3 = b_4 = 3$.

Case 2.3.1.1 $b_1 = 2, b_2 = b_3 = b_4 = 4$. It follows from equation 3 that $2b + c = b_5 + b_6 + 5$. Cancelling the equal terms we obtain $Q_5(G) \neq Q_5(H)$, a contradiction.

Case 2.3.1.2 $b_1 = b_2 = b_3 = b_4 = 3$. Then H has six cycles of length 6, a contradiction.

Case 2.3.2 $b_2 + b_3 = 6$. Considering $b_1 + b_2 = 6$ and $b_1 + b_3 = 6$, we have $b_1 = b_2 = b_3 = 3$. It follows from equation 3 that $2b + c = b_4 + b_5 + b_6$. We obtain the following after simplification.

$$\begin{aligned} Q_{11}(G) &= 6x^{b+c+7} + 6x^{b+c+6} + 2x^{b+c+1} + 3x^{2b+7} + 3x^{2b+6} + x^{2b+1} + \\ & 2x^{b+10} + 6x^{b+9} + 2x^{b+5} + 6x^{b+4} + x^{c+10} + 3x^{c+9} + x^{c+5} + \\ & 3x^{c+4} + x^{12} + 3x^8 - \big(6x^{b+c+8} + 8x^{b+c+4} + 3x^{2b+8} + 4x^{2b+4} + \\ & 2x^{b+11} + 12x^{b+7} + 2x^{b+1} + x^{c+11} + 6x^{c+7} + x^{c+1} + 4x^{10} + x^6\big), \\ Q_{11}(H) &= x^{b_4+b_5+9} + 3x^{b_4+b_5+5} + x^{b_4+b_5+1} + x^{b_4+b_6+9} + 3x^{b_4+b_6+5} + \\ & x^{b_4+b_6+1} + x^{b_4+10} + 3x^{b_4+4} + x^{b_4+3} + 3x^{b_5+b_6+7} + 3x^{b_5+b_6+6} + \\ & x^{b_5+b_6+1} + 3x^{b_5+8} + 4x^{b_5+4} + 3x^{b_6+8} + 4x^{b_6+4} + 3x^6 - \\ & \big(x^{b_4+b_5+10} + 3x^{b_4+b_5+4} + x^{b_4+b_5+3} + x^{b_4+b_6+10} + 3x^{b_4+b_6+4} + \\ & x^{b_4+b_6+3} + x^{b_4+9} + 3x^{b_4+5} + x^{b_4+1} + 3x^{b_5+b_6+8} + 4x^{b_5+b_6+4} + \\ & 3x^{b_5+7} + 3x^{b_5+6} + x^{b_5+1} + 3x^{b_6+7} + 3x^{b_6+6} + x^{b_6+1} + 3x^8 + x^4\big). \end{aligned}$$

Comparing the l.r.p. in $Q_{11}(G)$ and the l.r.p. in $Q_{11}(H)$, we have $b = 3$ or $c = 3$. If $b = 3$, then $G \cong \theta(3, 3, 3, 3, 3, c)$. By Lemma 2.10, G is χ-unique. If $c = 3$, then $G \cong \theta(3, 3, 3, 3, 3, 3)$. By Lemma 2.2, G is χ-unique.

Case C $H \in g_e\big(\theta(b_1,b_2,b_3,b_4,b_5), C_{b_6+1}\big)$ where $2 \le b_1 \le b_2 \le b_3 \le b_4 \le b_5$ and $2 \le b_6$. As $G \cong \theta(a,a,a,b,b,c)$ and $H \in g_e\big(\theta(b_1,b_2,b_3,b_4,b_5), C_{b_6+1}\big)$, then

$$\begin{aligned}
Q_{12}(G) &= x\big(x^a-1\big)^3\big(x^b-1\big)^2\big(x^c-1\big) - \big(x^a-x\big)^3\big(x^b-x\big)^2\big(x^c-x\big),\\
Q_{12}(H) &= x\big(x^{b_1}-1\big)\big(x^{b_2}-1\big)\big(x^{b_3}-1\big)\big(x^{b_4}-1\big)\big(x^{b_5}-1\big)\big(x^{b_6}-1\big) -\\
&\quad \big(x^{b_1}-x\big)\big(x^{b_2}-x\big)\big(x^{b_3}-x\big)\big(x^{b_4}-x\big)\big(x^{b_5}-x\big)\big(x^{b_6}-1\big).
\end{aligned}$$

By equation 3, $Q_{12}(G) = Q_{12}(H)$ yields,

$$\begin{aligned}
Q_{13}(G) &= 2x^{3a+b+1} + x^{3a+c+1} + x^{3a+3} + 3x^{2a+2b+1} + 6x^{2a+b+c+1} + 6x^{2a+b+3} +\\
&\quad 3x^{2a+c+3} + 3x^{2a+1} + 3x^{a+2b+c+1} + 3x^{a+2b+3} + 6x^{a+b+c+3} + 6x^{a+b+1} +\\
&\quad 3x^{a+c+1} + 3x^{a+5} + x^{2b+c+3} + x^{2b+1} + 2x^{b+c+1} + 2x^{b+5} + x^{c+5} -\\
&\quad \big(2x^{3a+b+2} + x^{3a+c+2} + x^{3a+1} + 3x^{2a+2b+2} + 6x^{2a+b+c+2} + 6x^{2a+b+1} +\\
&\quad 3x^{2a+c+1} + 3x^{2a+4} + 3x^{a+2b+c+2} + 3x^{a+2b+1} + 6x^{a+b+c+1} + 6x^{a+b+4} +\\
&\quad 3x^{a+c+4} + 3x^{a+1} + x^{2b+c+1} + x^{2b+4} + 2x^{b+c+4} + 2x^{b+1} + x^{c+1} + x^6\big),
\end{aligned}$$

$$\begin{aligned}
Q_{13}(H) &= x^{b_1+b_2+b_3+b_4+b_5} + x^{b_1+b_2+b_3+b_6+1} + x^{b_1+b_2+b_3+2} + x^{b_1+b_2+b_4+b_6+1} +\\
&\quad x^{b_1+b_2+b_4+2} + x^{b_1+b_2+b_5+b_6+1} + x^{b_1+b_2+b_5+2} + x^{b_1+b_2+b_6+3} + x^{b_1+b_2+1} +\\
&\quad x^{b_1+b_3+b_4+b_6+1} + x^{b_1+b_3+b_4+2} + x^{b_1+b_3+b_5+b_6+1} + x^{b_1+b_3+b_5+2} +\\
&\quad x^{b_1+b_3+b_6+3} + x^{b_1+b_3+1} + x^{b_1+b_4+b_5+b_6+1} + x^{b_1+b_4+b_5+2} + x^{b_1+b_4+b_6+3} +\\
&\quad x^{b_1+b_4+1} + x^{b_1+b_5+b_6+3} + x^{b_1+b_5+1} + x^{b_1+b_6+1} + x^{b_1+4} + x^{b_2+b_3+b_4+b_6+1} +\\
&\quad x^{b_2+b_3+b_4+2} + x^{b_2+b_3+b_5+b_6+1} + x^{b_2+b_3+b_5+2} + x^{b_2+b_3+b_6+3} + x^{b_2+b_3+1} +\\
&\quad x^{b_2+b_4+b_5+b_6+1} + x^{b_2+b_4+b_5+2} + x^{b_2+b_4+b_6+3} + x^{b_2+b_4+1} + x^{b_2+b_5+b_6+3} +\\
&\quad x^{b_2+b_5+1} + x^{b_2+b_6+1} + x^{b_2+4} + x^{b_3+b_4+b_5+b_6+1} + x^{b_3+b_4+b_5+2} + x^{b_3+b_4+b_6+3} +\\
&\quad x^{b_3+b_4+1} + x^{b_3+b_5+b_6+3} + x^{b_3+b_5+1} + x^{b_3+b_6+1} + x^{b_3+4} + x^{b_4+b_5+b_6+3} +\\
&\quad x^{b_4+b_5+1} + x^{b_4+b_6+1} + x^{b_4+4} + x^{b_5+b_6+1} + x^{b_5+4} + x^{b_6+5} - \big(x^{b_1+b_2+b_3+b_4+b_5+1} +\\
&\quad x^{b_1+b_2+b_3+b_6+2} + x^{b_1+b_2+b_3+1} + x^{b_1+b_2+b_4+b_6+2} + x^{b_1+b_2+b_4+1} + x^{b_1+b_2+b_5+b_6+2} +\\
&\quad x^{b_1+b_2+b_5+1} + x^{b_1+b_2+b_6+1} + x^{b_1+b_2+3} + x^{b_1+b_3+b_4+b_6+1} + x^{b_1+b_3+b_4+1} +\\
&\quad x^{b_1+b_3+b_5+b_6+2} + x^{b_1+b_3+b_5+1} + x^{b_1+b_3+b_6+1} + x^{b_1+b_3+3} + x^{b_1+b_4+b_5+b_6+2} +\\
&\quad x^{b_1+b_4+b_5+1} + x^{b_1+b_4+b_6+1} + x^{b_1+b_4+3} + x^{b_1+b_5+b_6+1} + x^{b_1+b_5+3} +\\
&\quad x^{b_1+b_6+4} + x^{b_1+1} + x^{b_2+b_3+b_4+b_6+2} + x^{b_2+b_3+b_4+1} + x^{b_2+b_3+b_5+b_6+2} +\\
&\quad x^{b_2+b_3+b_5+1} + x^{b_2+b_3+b_6+1} + x^{b_2+b_3+3} + x^{b_2+b_4+b_5+b_6+2} + x^{b_2+b_4+b_5+1} +\\
&\quad x^{b_2+b_4+b_6+1} + x^{b_2+b_4+3} + x^{b_2+b_5+b_6+1} + x^{b_2+b_5+3} + x^{b_2+b_6+4} + x^{b_2+1} +\\
&\quad x^{b_3+b_4+b_5+b_6+2} + x^{b_3+b_4+b_5+1} + x^{b_3+b_4+b_6+1} + x^{b_3+b_4+3} +\\
&\quad x^{b_3+b_5+b_6+1} + x^{b_3+b_5+3} + x^{b_3+b_6+4} + x^{b_3+1} + x^{b_4+b_5+b_6+1} +\\
&\quad x^{b_4+b_5+3} + x^{b_4+b_6+4} + x^{b_4+1} + x^{b_5+b_6+4} + x^{b_5+1} + x^{b_6+1} + x^5\big).
\end{aligned}$$

Consider the l.r.p. in $Q_{13}(G)$ that is $a+1$ and the l.r.p. in $Q_{13}(H)$, that is 5. Since $a=4$ and $a\geq 2$, we have three cases to consider, (1) $a=2$ or (2) $a=3$ or (3) $a=4$.

Case 1 $a=2$. Therefore, $g(G)=g(H)=2a=4$ and H has three cycles of length 4. Hence, we have $b_6=3$ or $b_6\neq 3$.

Case 1.1 $b_6=3$. Then, we have $b_1+b_2=4$ or $b_1+b_3=4$. Thus, $b_1=b_2=b_3=2$. However, H has four cycles of length 4, a contradiction.

Case 1.2 $b_6\neq 3$. Note that $g(H)=4$, then $b_6\geq 4$. Then, we have $b_1+b_2=4$, $b_1+b_3=4$ and $\big(b_1+b_4=4$ or $b_2+b_3=4\big)$. So, we have two cases to consider.

Case 1.2.1 $b_1+b_4=4$. Since $b_1+b_2=4$ and $b_1+b_3=4$, then we know that $b_1=b_2=b_3=b_4=2$. Now H has six cycles of length 4, a contradiction.

Case 1.2.2 $b_2+b_3=4$. Since $b_1+b_2=4$ and $b_1+b_3=4$, then $b_1=b_2=b_3=2$. From equation 3, we obtain

$$2b+c=b_4+b_5+b_6 \tag{5}$$

Then, we obtain the following after simplification.

$$\begin{aligned}
Q_{14}(G) &= 6x^{2b+5}+x^{2b+1}+12x^{b+c+5}+2x^{b+c+1}+8x^{b+7}+6x^{b+3}+4x^{c+7}+\\
&\quad 3x^{c+3}+x^{9}+2x^{7}+x^{5}-\big(x^{2b+c+1}+3x^{2b+6}+x^{2b+4}+3x^{2b+3}+\\
&\quad 6x^{b+c+6}+2x^{b+c+4}+6x^{b+c+3}+2x^{b+8}+6x^{b+6}+4x^{b+5}+\\
&\quad 2x^{b+1}+x^{c+8}+3x^{c+6}+2x^{c+5}+x^{c+1}+3x^{8}+x^{6}\big),\\
Q_{14}(H) &= x^{b_4+b_5+6}+3x^{b_4+b_5+4}+x^{b_4+b_5+1}+6x^{b_4+b_6+5}+x^{b_4+b_6+1}+\\
&\quad 3x^{b_4+6}+x^{b_4+4}+3x^{b_4+3}+6x^{b_5+b_6+5}+x^{b_5+b_6+1}+3x^{b_5+6}+\\
&\quad x^{b_5+4}+3x^{b_5+3}+4x^{b_6+7}+3x^{b_6+3}+x^{8}+3x^{6}-\big(x^{b_4+b_5+b_6+1}+\\
&\quad x^{b_4+b_5+7}+4x^{b_4+b_5+3}+3x^{b_4+b_6+6}+x^{b_4+b_6+4}+3x^{b_4+b_6+3}+\\
&\quad 6x^{b_4+5}+x^{b_4+1}+3x^{b_5+b_6+6}+x^{b_5+b_6+4}+3x^{b_5+b_6+3}+6x^{b_5+5}+\\
&\quad x^{b_5+1}+x^{b_6+8}+3x^{b_6+6}+2x^{b_6+5}+x^{b_6+1}+4x^{7}\big).
\end{aligned}$$

Consider the l.r.p. in $Q_{14}(G)$. We have $b=4$ or $c=4$.

Case 1.2.2.1 $b=4$. Since there is $-2x^{b+1}$ in $Q_{14}(G)$, we have $b_4=4$ or $b_5=4$ or $b_6=4$.

If $b_4=4$, then it follows from equation 5 that $c+4=b_5+b_6$. However, $Q_{14}(G)\neq Q_{14}(H)$, a contradiction.

If $b_5=4$, then it follows from equation 5 that $c+4=b_4+b_6$. However, $Q_{14}(G)\neq Q_{14}(H)$, a contradiction.

If $b_6=4$, then it follows from equation 5 that $c+4=b_4+b_5$. However, $Q_{14}(G)\neq Q_{14}(H)$, a contradiction.

Case 1.2.2.2 $c=4$. We obtain the following after simplification.

$$\begin{aligned}
Q_{15}(G) &= 5x^{2b+5}+x^{2b+1}+12x^{b+9}+2x^{b+7}+6x^{b+3}+4x^{11}+5x^{7}-\\
&\quad \big(3x^{2b+6}+x^{2b+4}+3x^{2b+3}+6x^{b+10}+4x^{b+8}+6x^{b+6}+\\
&\quad 2x^{b+5}+2x^{b+1}+x^{12}+3x^{10}+x^{9}+3x^{8}\big),\\
Q_{15}(H) &= x^{b_4+b_5+6}+3x^{b_4+b_5+4}+x^{b_4+b_5+1}+6x^{b_4+b_6+5}+x^{b_4+b_6+1}+\\
&\quad 3x^{b_4+6}+x^{b_4+4}+3x^{b_4+3}+6x^{b_5+b_6+5}+x^{b_5+b_6+1}+3x^{b_5+6}+\\
&\quad x^{b_5+4}+3x^{b_5+3}+4x^{b_6+7}+3x^{b_6+3}+x^{8}+4x^{6}-\\
&\quad \big(x^{b_4+b_5+b_6+1}+x^{b_4+b_5+7}+4x^{b_4+b_5+3}+3x^{b_4+b_6+6}+x^{b_4+b_6+4}+\\
&\quad 3x^{b_4+b_6+3}+6x^{b_4+5}+x^{b_4+1}+3x^{b_5+b_6+5}+x^{b_5+b_6+4}+\\
&\quad 3x^{b_5+b_6+3}+6x^{b_5+5}+x^{b_5+1}+x^{b_6+8}+3x^{b_6+6}+2x^{b_6+5}+\\
&\quad x^{b_6+1}+4x^{7}\big).
\end{aligned}$$

Compare the l.r.p. in $Q_{15}(G)$ and the l.r.p. in $Q_{15}(H)$. Then we have $b=3$. Simplifying $Q_{15}(G)$ and $Q_{15}(H)$, we obtain the following.

$$\begin{aligned}
Q_{16}(G) &= 8x^{12}+5x^{11}+6x^{7}+6x^{6}-\big(6x^{13}+2x^{10}+10x^{9}+5x^{8}+2x^{4}\big),\\
Q_{16}(H) &= x^{b_4+b_5+6}+3x^{b_4+b_5+4}+x^{b_4+b_5+1}+6x^{b_4+b_6+5}+x^{b_4+b_6+1}+\\
&\quad 3x^{b_4+6}+x^{b_4+4}+3x^{b_4+3}+6x^{b_5+b_6+5}+x^{b_5+b_6+1}+3x^{b_5+6}+\\
&\quad x^{b_5+4}+3x^{b_5+3}+4x^{b_6+7}+3x^{b_6+3}+x^{8}+4x^{6}-\big(x^{b_4+b_5+b_6+1}+\\
&\quad x^{b_4+b_5+7}+4x^{b_4+b_5+3}+3x^{b_4+b_6+6}+x^{b_4+b_6+4}+3x^{b_4+b_6+3}+\\
&\quad 6x^{b_4+5}+x^{b_4+1}+3x^{b_5+b_6+6}+x^{b_5+b_6+4}+3x^{b_5+b_6+3}+6x^{b_5+5}+\\
&\quad x^{b_5+1}+x^{b_6+8}+3x^{b_6+6}+2x^{b_6+5}+x^{b_6+1}+4x^{7}\big).
\end{aligned}$$

Considering the term $-2x^4$ in $Q_{16}(G)$, we have $b_4=3$ and $b_5=3$. It follows from equation 5 that $b_6=4$. However, we obtain $Q_{16}(G)\neq Q_{16}(H)$, a contradiction.

Case 2 $a=3$. So $g(G)=g(H)=2a=6$. Both G and H has three cycles of length 6, respectively. Then, we have $b_6=5$ or $b_6\neq 5$.

Case 2.1 $b_6=5$. Therefore, $b_1+b_2=6$ and $b_1+b_3=6$. So $b_2=b_3$. Thus, we have $b_1=2, b_2=b_3=4$ or $b_1=b_2=b_3=3$.

Case 2.1.1 $b_1=2, b_2=b_3=4$. It follows from equation 3 that $2b+c=b_4+b_5+4$. However, we obtain $Q_{13}(G)\neq Q_{13}(H)$, a contradiction.

Case 2.1.2 $b_1=b_2=b_3=3$. Then H has four cycles of length 6, a contradiction.

Case 2.2 $b_6\neq 5$. Since H has three cycles of length 6, then we have $b_1+b_2=6$, $b_1+b_3=6$ and $\big(b_1+b_4=6$ or $b_2+b_3=6\big)$.

Case 2.2.1 $b_1+b_4=6$. Note that $b_1+b_2=6$ and $b_1+b_3=6$. Therefore, $b_2=b_3=b_4$. Hence, we have $b_1=2, b_2=b_3=b_4=4$ or $b_1=b_2=b_3=b_4=3$.

Case 2.2.1.1 $b_1=2, b_2=b_3=b_4=4$. It follows from equation 3 that $2b+c=b_5+b_6+5$. However, after simplification, we obtain $Q_{13}(G)\neq Q_{13}(H)$, a contradiction.

Case 2.2.1.2 $b_1 = b_2 = b_3 = b_4 = 3$. Therefore, H has six cycles of length 6, a contradiction.

Case 2.2.2 $b_2 + b_3 = 6$. Note that $b_1 + b_2 = 6$ and $b_1 + b_3 = 6$. Therefore $b_1 = b_2 = b_3 = 3$. From equation 3, we have

$$2b + c = b_4 + b_5 + b_6 \tag{6}$$

We obtain the following after simplification.

$$\begin{aligned}
Q_{17}(G) &= 3x^{2b+7} + 3x^{2b+6} + x^{2b+1} + 6x^{b+c+7} + 6x^{b+c+6} + 2x^{b+c+1} + \\
&\quad 2x^{b+10} + 6x^{b+9} + 2x^{b+5} + 6x^{b+4} + x^{c+10} + 3x^{c+9} + x^{c+5} + \\
&\quad 3x^{c+4} + x^{12} + 3x^{8} - \big(3x^{2b+8} + 4x^{2b+4} + 6x^{b+c+8} + \\
&\quad 8x^{b+c+4} + 2x^{b+11} + 12x^{b+7} + 2x^{b+1} + x^{c+11} + 6x^{c+7} + \\
&\quad x^{c+1} + 3x^{10} + x^{6}\big), \\
Q_{17}(H) &= x^{b_4+b_5+9} + 3x^{b_4+b_5+5} + x^{b_4+b_5+1} + 3x^{b_4+b_6+7} + 3x^{b_4+b_6+6} + \\
&\quad x^{b_4+b_6+1} + 3x^{b_4+8} + 4x^{b_4+4} + 3x^{b_5+b_6+7} + 3x^{b_5+b_6+6} + \\
&\quad x^{b_5+b_6+1} + 3x^{b_5+8} + 4x^{b_5+4} + x^{b_6+10} + 3x^{b_6+9} + x^{b_6+5} + \\
&\quad 3x^{b_6+4} + x^{11} + 3x^{7} - \big(x^{b_4+b_5+10} + 3x^{b_4+b_5+4} + x^{b_4+b_5+3} + \\
&\quad 3x^{b_4+b_6+8} + 4x^{b_4+b_6+4} + 3x^{b_4+7} + 3x^{b_4+6} + x^{b_4+1} + \\
&\quad 3x^{b_5+b_6+8} + 4x^{b_5+b_6+4} + 3x^{b_5+7} + 3x^{b_5+6} + x^{b_5+1} + x^{b_6+11} + \\
&\quad 6x^{b_6+7} + x^{b_6+1} + 3x^{9} + x^{5}\big).
\end{aligned}$$

Compare the l.r.p. in $Q_{17}(G)$ and the l.r.p. in $Q_{17}(H)$. We have $b = 4$ or $c = 4$.

Case 2.2.2.1 $b = 4$. There is one term in $Q_{17}(H)$ that equal to $-x^5$. Since $b_6 \geq 6$, we have $b_4 = 4$ or $b_5 = 4$.

If $b_4 = 4$, then it follows from equation 6 that $c + 4 = b_5 + b_6$. Cancelling the equal terms, we obtain $b_5 = 5$ and $b_6 = 6$. So $c = 7$. But, $Q_{17}(G) \neq Q_{17}(H)$, a contradiction.

If $b_5 = 4$, then it follows from equation 6 that $c + 4 = b_4 + b_6$. Since $b_6 \geq 6$, by cancelling the equal terms in $Q_{17}(G)$ and $Q_{17}(H)$, we obtain $b_4 = 5$. But $3 \leq b_4 \leq 4$, a contradiction.

Case 2.2.2.2 $c = 4$. Therefore, $b = 3$ or $b = 4$. If $b = 3$, then $G \cong \theta(3,3,3,3,3,4)$. By Lemma 2.10, G is χ-unique. If $b = 4$, then $G \cong \theta(3,3,3,4,4,4)$. Similarly, by Lemma 2.10, G is χ-unique.

Case 3 $a = 4$. Therefore, $g(G) = g(H) = 2a = 8$ and both G and H has three cycles of length 8, respectively. Then, we have to consider for $b_6 = 7$ or $b_6 \neq 7$.

Case 3.1 $b_6 = 7$. Therefore, $b_1 + b_2 = 8$ and $b_1 + b_3 = 8$. So, we know that $b_2 = b_3$. Hence, we have $b_1 = 2, b_2 = b_3 = 6$ or $b_1 = 3, b_2 = b_3 = 5$ or $b_1 = b_2 = b_3 = 4$.

Case 3.1.1 $b_1 = 2, b_2 = b_3 = 6$. It follows from equation 3 that $2b + c = b_4 + b_5 + 9$. Since $4 \leq b \leq c$, after simplification, we obtain $-x^3$ is in $Q_{13}(H)$ but not in $Q_{13}(G)$, a contradiction.

Case 3.1.2 $b_1 = 3, b_2 = b_3 = 5$. It follows from equation 3 that $2b + c = b_4 + b_5 + 8$. Similar to Case 3.1.1, we obtain $Q_{13}(G) \neq Q_{13}(H)$, a contradiction.

Case 3.1.3 $b_1 = b_2 = b_3 = 4$. But H has four cycles of length 8, a contradiction.

Case 3.2 $b_6 \neq 7$. Since the girth of H is 8, then $b_6 \geq 8$. So $b_1 + b_2 = 8$, $b_1 + b_3 = 8$ and $\big(b_1 + b_4 = 8$ or $b_2 + b_3 = 8\big)$. Hence, we have two cases to consider.

Case 3.2.1 $b_1 + b_4 = 8$. Since $b_1 + b_2 = 8$ and $b_1 + b_3 = 8$, we know that $b_2 = b_3 = b_4$. So we have $b_1 = 2, b_2 = b_3 = b_4 = 6$ or $b_1 = 3, b_2 = b_3 = b_4 = 5$ or $b_1 = b_2 = b_3 = b_4 = 4$.

Case 3.2.1.1 $b_1 = 2, b_2 = b_3 = b_4 = 6$. It follows from equation 3 that $2b + c = b_5 + b_6 + 8$. Similar to the above cases, we obtain $Q_{13}(G) \neq Q_{13}(H)$, a contradiction.

Case 3.2.1.2 $b_1 = 3, b_2 = b_3 = b_4 = 5$. It follows from equation 3 that $2b + c = b_5 + b_6 + 6$. Similar to the above cases, we obtain $Q_{13}(G) \neq Q_{13}(H)$, a contradiction.

Case 3.2.1.3 $b_1 = b_2 = b_3 = b_4 = 4$. But H has six cycles of length 8, a contradiction.

Case 3.2.2 $b_2 + b_3 = 8$. Since $b_1 + b_2 = 8$ and $b_1 + b_3 = 8$, we know that $b_1 = b_2 = b_3$. Therefore, $b_1 = b_2 = b_3 = 4$. It follows from equation 3 that $2b + c = b_4 + b_5 + b_6$. We obtain the following after simplification.

$$
\begin{aligned}
Q_{18}(G) \;=\; & 3x^{2b+9} + 3x^{2b+7} + x^{2b+1} + 6x^{b+c+9} + 6x^{b+c+7} + 2x^{b+c+1} + \\
& 2x^{b+13} + 6x^{b+11} + 8x^{b+5} + x^{c+13} + 3x^{c+11} + 4x^{c+5} + x^{15} + \\
& 3x^9 - \big(3x^{2b+10} + 3x^{2b+5} + x^{2b+4} + 6x^{b+c+10} + 6x^{b+c+5} + \\
& 2x^{b+c+4} + 2x^{b+14} + 6x^{b+9} + 6x^{b+8} + 2x^{b+1} + x^{c+14} + 3x^{c+9} + \\
& 3x^{c+8} + x^{c+1} + 3x^{12} + x^6\big), \\
Q_{18}(H) \;=\; & x^{b_4+b_5+12} + 3x^{b_4+b_5+6} + x^{b_4+b_5+1} + 3x^{b_4+b_6+9} + 3x^{b_4+b_6+7} + \\
& x^{b_4+b_6+1} + 3x^{b_4+10} + 3x^{b_4+5} + x^{b_4+4} + 3x^{b_5+b_6+9} + 3x^{b_5+b_6+7} + \\
& x^{b_5+b_6+1} + 3x^{b_5+10} + 3x^{b_5+5} + x^{b_5+4} + x^{b_6+13} + 3x^{b_6+11} + \\
& 4x^{b_6+5} + x^{14} + 3x^8 - \big(x^{b_4+b_5+13} + 3x^{b_4+b_5+5} + x^{b_4+b_5+3} + \\
& 3x^{b_4+b_6+10} + 3x^{b_4+b_6+5} + x^{b_4+b_6+4} + 3x^{b_4+9} + 3x^{b_4+7} + x^{b_4+1} + \\
& 3x^{b_5+b_6+10} + 3x^{b_5+b_6+5} + x^{b_5+b_6+4} + 3x^{b_5+9} + 3x^{b_5+7} + x^{b_5+1} + \\
& x^{b_6+14} + 3x^{b_6+9} + 3x^{b_6+8} + x^{b_6+1} + 3x^{11} + x^5\big).
\end{aligned}
$$

Compare the l.r.p. in $Q_{18}(G)$ and the l.r.p. in $Q_{18}(H)$. We have $b = 4$ or $c = 4$.
If $b = 4$, then $G \cong \theta(4, 4, 4, 4, 4, c)$. By Lemma 2.10, G is χ-unique.
If $c = 4$, then $G \cong \theta(4, 4, 4, 4, 4, 4)$. By Lemma 2.2, G is χ-unique.
This completes the proof of Theorem 3.1. □

4. Conclusion

It is natural to ask the following question: for which choices $(a_1, a_2, \cdots, a_6)$ where $a_1 \leq a_2 \leq \cdots \leq a_6$, the graph $\theta(a_1, a_2, \cdots, a_6)$ is χ-unique? In general, this problem still remains open. In the next paper, the chromaticity of another type of the graph $\theta(a_1, a_2, \cdots, a_6)$ where $a_1, a_2, \cdots, a_6$ assume exactly three distinct values will be given.

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