



Motions of a connected subgraph representing a swarm of robots inside a graph of work stations

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Abstract

Imagine that a swarm of robots is given, these robots must communicate with each other, and they can do so if certain conditions are met. We say that the swarm is connected if there is at least one way to send a message between each pair of robots. A robot can move from a work station to another only if the connectivity of the swarm is preserved in order to perform some tasks. We model the problem via graph theory, we study connected subgraphs and how to motion them inside a connected graph preserving the connectivity. We determine completely the group of movements.

Keywords: edge-blocks, the Wilson group, motion planning, robots swarms, pebble motion

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1. Introduction

In this work we model the following problem. Imagine that a swarm of robots is given, these robots must communicate with each other, and they can do so if certain conditions are met. We say that the swarm is connected if there is at least one way to send a message between each pair of robots. A message between robots can be sent if either there is a direct communication between

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them or if there are intermediate robots which can send the message. Some work stations in a region are also given, the number of work stations are at least the number of robots. A robot can move from one of these work stations to another only if the connectivity of the swarm is preserved. The swarm of robots has one fixed initial position and, in order to perform some tasks, the robots move from one station to another as needed, always maintaining the swarm connected. After a while the swarm of robots returns to its initial position. In order to achieve this goal it is not necessary that each robot returns to its initial position, we only care about the position of the whole swarm, so as long as each one of the initial positions are occupied and the swarm is connected, we say that it has returned to its original position. Our intent in this paper is to study the different permutations that might appear once the swarm returns to its original position. In order to do so, we must also study the possible moves that the swarm can make, all moves must meet three conditions: 1. The connectivity of the swarm must be kept, 2. Only one robot is in each workstation at each time, 3. To avoid crashes, two robots are not allowed to swap positions.

We model the problem using a graph as follows. The work stations are represented by the vertices of a graph, two vertices are connected by an edge if their corresponding workstations allow a couple of robots, one in each workstation, to communicate with each other. Notice that the initial positions of the robots induce a unique subgraph of our workstations graph and that every time a robot moves this induced subgraph might change. Since we are interested only in the moves that assure the connectivity of the swarm, both the workstations graph and every induced subgraph must be connected. Under this model the subgraph of robots moves through the workstations graph and we ask how the permutations of the initial subgraph look like.

Related problems have been studied from a different perspective in the area of motion planning under the names of “robots swarm” and “pebble motion”, for example in [3] and [4].

A classical related problem is the well-known “15-puzzle” which was generalized to graphs by Wilson [10], who proved that for any nonseparable graph, except for one, its associate group is the symmetric group unless the graph is bipartite, for which it is the alternating group.

While Wilson considered just an empty workstation, the problem was generalized to k empty workstations in [7] where it was also given a polynomial time algorithm that decides reachability of a target configuration. In [2] optimal algorithms for specific graphs were explored. Colored versions were studied in [5] and [6]. In [8] it was proven that finding a shortest solution for the extended puzzle is NP-hard and therefore it is computationally intractable.

In the following section we define formally the problem. In Section 3, we prove that the set of possible movements is a group, and we define what we call the Wilson group (in honor of Richard M. Wilson). In Section 4, we characterize such group when there are no “empty workstations”. Finally, in Section 5, we characterize the group in the case when there is at least one “empty workstation”.

2. Definitions and basic results

In this section we introduce definitions, terminology and basic results. All the graphs considered in the paper are finite and simple.

Definition 2.1. Let G be a graph, R a k -set and f_t a function such that

$$f_t: V(G) \rightarrow R \cup \{\emptyset\}.$$

We denote $f_t^{-1}(R)$ by V_t , and we say f_t is an R -configuration over G if $f_t|_{V_t}$ is bijective, where $t \in \Delta$ and Δ is a set of natural numbers.

The elements of R are called *labels* and we use $R = [k]$ or a subset of $[k]$ for simplicity, where $[k] := \{1, 2, \dots, k\}$.

Definition 2.2. Let G be a graph and f_t a $[k]$ -configuration over G . We say f_t is a connected $[k]$ -configuration over G if the induced subgraph $G[V_t]$ of G is connected.

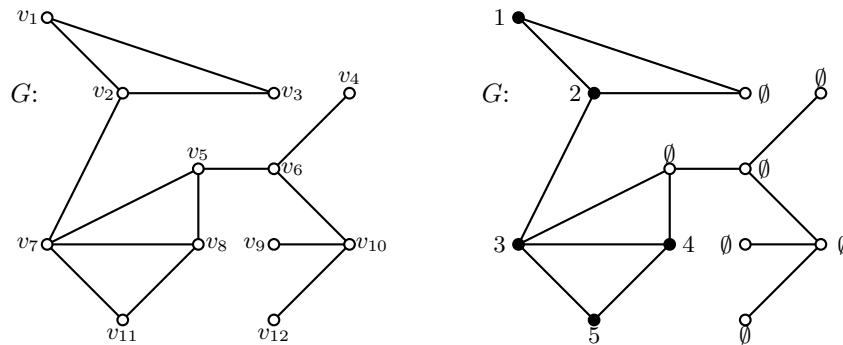


Figure 1. (Left) A graph G . (Right) The graph G labeled with a connected $[5]$ -configuration f_t .

If a vertex v is such that $f_t(v) = \emptyset$, we say that it is *empty*. The set of empty vertices is denoted by V_\emptyset . Figure 1 shows an example of a connected $[k]$ -configuration f_t over a graph G , for $k = 5$. We write

$$f_t = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} \\ 1 & 2 & \emptyset & \emptyset & \emptyset & \emptyset & 3 & 4 & \emptyset & \emptyset & 5 & \emptyset \end{pmatrix}.$$

Suppose that f_t and f_s are two connected $[k]$ -configurations over a graph G . If $V_t = V_s$, for $f_t(v_j) = i$ and $f_s(v_j) = i'$ we have that $\sigma(i) = i'$ where σ is a permutation of V_t .

Definition 2.3. Let f_t and f_s be two connected $[k]$ -configurations over a graph G . If $V_t = V_s$ we say f_t is similar to f_s and we denote it by $f_t \simeq f_s$.

It is not hard to see that the relation \simeq is an equivalence relation over the set of $[k]$ -configurations over G . The equivalence class of f_t is denoted by $[f_t]$. Therefore, a class $[f_t]$ is an unlabeled connected $[k]$ -configuration f_t .

2.1. Motioning connected subgraphs

In this subsection, we establish the rules to motion connected induced subgraphs preserving the connectivity.

Definition 2.4. Let f_t be a $[k]$ -connected configuration over a graph G . Let $w[f_t]$ be a function such that

$$w[f_t]: V(G) \rightarrow \{0, 1\}$$

for each $v \in V(G)$, $w[f_t](v) = 1$ if $f_t(v) \in [k]$ and $w[f_t](v) = 0$ otherwise.

The function $w[f_t]$ is called the weight function of f_t .

Clearly, if $f_t \simeq f_s$ then $w[f_t] = w[f_s]$. We recall that a cycle of a graph is denoted by (v_1, v_2, \dots, v_r) where $v_1 = v_r$. However, to keep our arguments as simple as possible, we choose to use $v_1 \neq v_r$ and then v_1 is adjacent to v_r .

Definition 2.5. Let f_t be a connected $[k]$ -configuration over a graph G . An r -cycle p is a cycle $p = (v_1, v_2, \dots, v_r)$ such that $w[f_t](v_i) = 1$, for all $i \in [r]$. And an r -path p is a path $p = (v_1, v_2, \dots, v_r)$ such that $w[f_t](v_i) = 0$ if and only if $i = 1$, that is, only the vertex v_1 has weight 0.

Now, we associate a permutation to an r -cycle or path. In this paper, the product of permutations means composition of functions on the left. For a detailed introduction on permutations we refer to the book of Rotman [9].

Definition 2.6. Let f_t be a connected $[k]$ -configuration over a graph G and let p be an r -cycle or a path $p = (v_1, v_2, \dots, v_r)$. An elementary p -movement of V_t is a permutation σ_p such that

$$\sigma_p = (v_1 v_2 \dots v_r) = (v_1 v_2)(v_2 v_3) \dots (v_{r-1} v_r).$$

Hence, we can define configurations f_s arising from a given configuration f_t .

Definition 2.7. Let f_t be a connected $[k]$ -configuration over a graph G and σ_p an elementary p -movement of V_t . The $[k]$ -configuration $f_{t+1} = f_t \circ \sigma_p$ over G is an elementary configuration movement arising from f_t .

Note that if G is a tree such that $V(G) = V_t$ for a connected $[k]$ -configuration then there is no elementary p -movements. And in general, it is possible that $G[V_{t+1}]$ is a disconnected subgraph.

Definition 2.8. Let f_t be a connected $[k]$ -configuration over a graph G and σ_p an elementary p -movement of V_t . If f_{t+1} is an elementary configuration movement arising from f_t such that it is a connected $[k]$ -configuration then f_{t+1} is called valid as well as σ_p .

Figure 2 shows two valid elementary p -movements, namely, $\sigma_p = (v_5 v_8 v_7 v_2 v_1)$ and $\sigma_{p'} = (v_7 v_8 v_{11})$, therefore $f_{t+1} = f_t \circ \sigma_p$ and $f'_{t+1} = f_t \circ \sigma_{p'}$ where

$$f_{t+1} = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} \\ \emptyset & 1 & \emptyset & \emptyset & 4 & \emptyset & 2 & 3 & \emptyset & \emptyset & 5 & \emptyset \end{pmatrix} \text{ and}$$

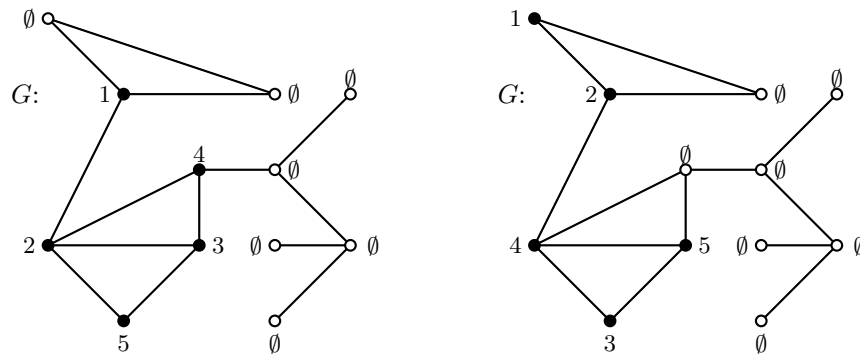


Figure 2. (Left) The graph G labeled with a connected $[5]$ -configuration f_{t+1} . (Right) The graph G labeled with a connected $[5]$ -configuration f'_{t+1} .

$$f'_{t+1} = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} \\ 1 & 2 & \emptyset & \emptyset & \emptyset & \emptyset & 4 & 5 & \emptyset & \emptyset & 3 & \emptyset \end{pmatrix}.$$

An example of an invalid elementary p -movement is $\sigma_{p''} = (v_3 v_2 v_7)$.

Since we obtain a connected $[k]$ -configuration f_{t+1} from a connected $[k]$ -configuration f_t via a valid elementary p -movement, we have the following proposition.

Proposition 2.1. *A σ_p is a valid elementary p -movement of V_t if and only if σ_p^{-1} is a valid elementary p -movement of V_{t+1} .*

Consider the set of empty vertices V_\emptyset . For any permutation σ in the symmetric group S_{V_\emptyset} of V_\emptyset we have that $f_t \circ \sigma = f_t$, therefore, σ is a valid elementary p -movement. Given a connected $[k]$ -configuration f_t , we denote the set of valid elementary p -movements of V_t as $\Gamma[f_t]$. Therefore $S_{V_\emptyset} \subseteq \Gamma[f_t]$.

Proposition 2.2. *If $f_t \simeq f_s$ then $\Gamma[f_t] = \Gamma[f_s]$ and $f_t \circ \sigma \simeq f_s \circ \sigma$ for any $\sigma \in \Gamma[f_t]$.*

By Proposition 2.1 we have the following.

Proposition 2.3. *$\sigma \in \Gamma[f_t]$ if and only if $\sigma^{-1} \in \Gamma[f_t \circ \sigma]$.*

Next, we define a valid sequence of connected configurations.

Definition 2.9. *Let $\{f_0, f_1, \dots, f_t\}$ be a set of connected $[k]$ -configuration over a graph G . We say that it is a valid $f_0 f_t$ -sequence if f_s is a valid $[k]$ -configuration arising from f_{s-1} for all $s \in [t]$.*

Figure 3 shows an example of a valid $f_0 f_2$ -sequence of a graph H for

$$f_0 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \emptyset & 1 & \emptyset & 2 & 3 & \emptyset & \emptyset \end{pmatrix}.$$

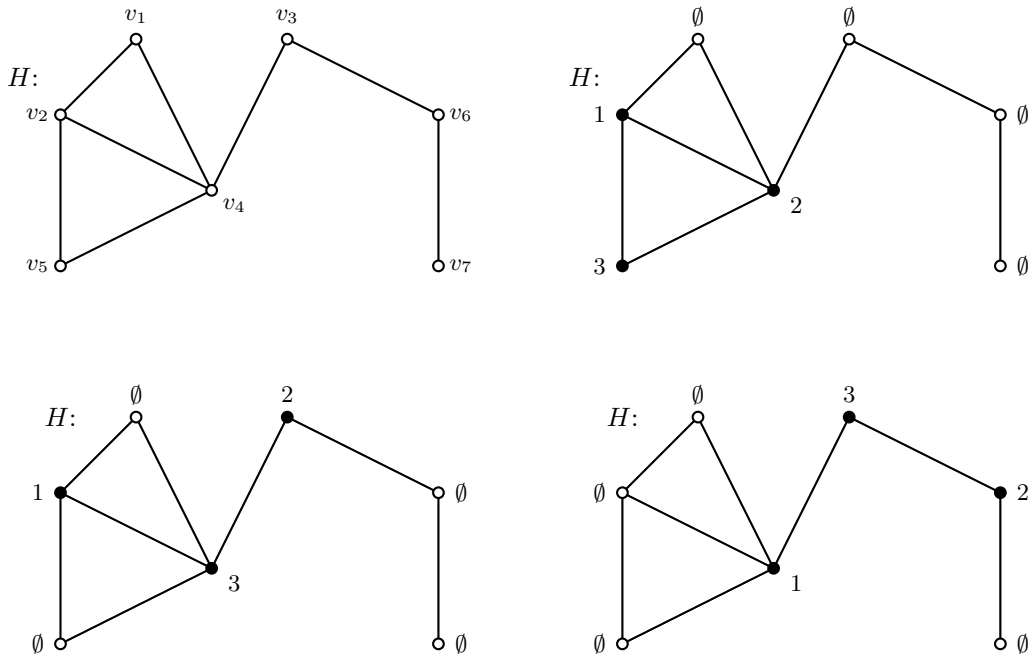


Figure 3. (Left-top) A graph H . (Right-top) The graph H labeled with a connected $[3]$ -configuration f_0 . (Left-bottom) The graph H labeled with a connected $[3]$ -configuration f_1 . (Right-bottom) The graph H labeled with a connected $[3]$ -configuration f_2 .

Taking a 3-path $p_1 = (v_3, v_4, v_5)$ and the permutation $\sigma_{p_1} = (v_3 v_4 v_5)$ we get $f_1 = f_0 \circ \sigma_{p_1}$ obtaining $f_1 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \emptyset & 1 & 2 & 3 & \emptyset & \emptyset & \emptyset \end{pmatrix}$. Then, taking a 4-path $p_1 = (v_6, v_3, v_4, v_2)$ and the permutation $\sigma_{p_2} = (v_6 v_3 v_4 v_2)$ we get $f_2 = f_1 \circ \sigma_{p_2}$ obtaining $f_2 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\ \emptyset & \emptyset & 3 & 1 & \emptyset & 2 & \emptyset \end{pmatrix}$.

Therefore, $f_2 = f_0 \circ \sigma_{p_1} \circ \sigma_{p_2}$, i.e., $f_2 = f_0 \circ \sigma$ where $\sigma = \sigma_{p_1} \circ \sigma_{p_2}$. In general, if $\{f_0, f_1, \dots, f_t\}$ is a set of $[k]$ -configuration over a graph G , hence $f_i = f_{i-1} \circ \sigma_{p_i}$ for some $\sigma_{p_i} \in \Gamma[f_{i-1}]$ with $i \in [t]$ and then $f_t = f_0 \circ \sigma$ for $\sigma = \sigma_{p_1} \circ \sigma_{p_2} \circ \dots \circ \sigma_{p_t}$.

Definition 2.10. Let f_0 and f_t be two connected $[k]$ -configurations over a graph G . If there exists a valid $f_0 f_t$ -sequence $\{f_0, f_1, \dots, f_t\}$ for which $f_i = f_{i-1} \circ \sigma_{p_i}$ for some $\sigma_{p_i} \in \Gamma[f_{i-1}]$ with $i \in [t]$, the permutation $\sigma = \sigma_{p_1} \circ \sigma_{p_2} \circ \dots \circ \sigma_{p_t}$ is called a valid movement of V_0 having that $f_t = f_0 \circ \sigma$.

In a natural way we have the following two propositions.

Proposition 2.4. A permutation σ is a valid movement of V_0 if and only if σ^{-1} is a valid movement of V_t .

Proposition 2.5. If $f_t \simeq f_s$ then $f_t \circ \sigma \simeq f_s \circ \sigma$ for any valid movement σ of V_t .

To end this section, we have the following theorem about the classes $[f_t]$.

Theorem 2.1. *Let $[f_t]$ and $[f_s]$ be two unlabeled connected $[k]$ -configurations over a graph G . Then there exists a valid $f_t f_{t+r}$ -sequence $\{f_t, f_{t+1}, \dots, f_{t+r}\}$ for which $f_{t+r} \simeq f_s$.*

Proof. Let T_t and T_s spanning trees be of $G[V_t]$ and $G[V_s]$, respectively. And let $P = (x_0, x_1, \dots, x_r)$ be a $T_t T_s$ -geodesic. If the length r of P is positive, then $w[f_t(x_1)] = 0$. Take a leaf y of T_t contained in $V_t \setminus \{x_0\}$ and let $p_1 = (x_1, x_0, \dots, y', y)$ a path containing the $x_0 y$ -path of T_t . Therefore, σ_{p_1} is a valid elementary p_1 -movement of V_t . The tree T_{t+1} with vertex set $V_{t+1} = V_t \cup \{x_1\} \setminus \{y\}$ and edge set $E(T_t) \cup \{x_1 x_0\} \setminus \{yy'\}$ is a spanning tree of $G[V_{t+1}]$ with $P' = (x_1, \dots, x_r)$ a $T_{t+1} T_s$ -geodesic shorter than P .

Now, we can assume that $r = 0$. Consider a maximal component T of $G[V_t \cap V_s]$ and take spanning trees T_t and T_s of $G[V_t]$ and $G[V_s]$, respectively, such that T is a subgraph of them. If $T_t = T_s$ then $V_t = V_s$ and then $f_t \simeq f_s$. We can assume that there exist a leaf y of T_t contained in $V_t \setminus V(T)$ and then there exist a vertex x in $V_s \setminus V(T)$ such that xy'' is an edge of T_s and y'' is a vertex of T . Let $q_1 = (x, y'', \dots, y', y)$ the path of $T_t \cup T_s$. Therefore, σ_{q_1} is a valid elementary q_1 -movement of V_t . The tree T_{t+1} with vertex set $V_{t+1} = V_t \cup \{x\} \setminus \{y\}$ and edge set $E(T_t) \cup \{xy''\} \setminus \{yy'\}$ is a spanning tree of $G[V_{t+1}]$ and the maximal component T' of $G[V_{t+1} \cap V_s]$ has order largest than T . Since the graph is finite, the result follows. \square

Corollary 2.1. *Let $[f_t]$ and $[f_s]$ be two unlabeled connected $[k]$ -configurations over a graph G . Then there exists a valid movement σ from V_t to V_s .*

3. The Wilson group

Given two connected $[k]$ -configurations over a graph G , by Theorem 2.1 and Corollary 2.1, we know that we can move the first one to the second one via connected subgraphs. In this section, we prove a similar result but considering the case when the labels are sorted.

Firstly, we define the following interesting set Φ regarding to valid movements.

Definition 3.1. *Let f_t be a connected $[k]$ -configuration over a graph G . The Wilson set $\Phi[f_t]$ is the set of valid movements of V_t such that $f_t = f_t \circ \sigma$ for all $\sigma \in \Phi[f_t]$.*

Clearly, the symmetric group S_{V_\emptyset} of V_\emptyset is a subset of $\Phi[f_t]$. The following proposition establishes that the Wilson set is independent to the labels of f_t .

Proposition 3.1. *If $f_t \simeq f_s$ then $\Phi[f_t] = \Phi[f_s]$. In particular, if $\sigma \in \Phi[f_t]$ then $\Phi[f_t] = \Phi[f_t \circ \sigma]$.*

Proof. Let $\sigma \in \Phi[f_t]$. Since σ is a valid movement of V_t , by Proposition 2.5, we have $f_s \simeq f_t \simeq f_t \circ \sigma \simeq f_s \circ \sigma$, i.e., $\sigma \in \Phi[f_s]$ and then $\Phi[f_t] \subseteq \Phi[f_s]$. Analogously, $\Phi[f_s] \subseteq \Phi[f_t]$ and then $\Phi[f_t] = \Phi[f_s]$. Now, in particular, if $f_s = f_t \circ \sigma$ for some $\sigma \in \Phi[f_t]$ then $\Phi[f_t] = \Phi[f_t \circ \sigma]$. \square

Proposition 3.2. *Let σ be a valid movement of V_t . If $\phi \in \Phi[f_t \circ \sigma]$ then $\sigma \circ \phi \circ \sigma^{-1} \in \Phi[f_t]$.*

Proof. Since σ is a valid movement of V_t , by Proposition 2.4, σ^{-1} is a valid movement of $\sigma(V_t)$. On the other hand, since $\phi \in \Phi[f_t \circ \sigma]$ then $f_t \circ \sigma \simeq f_t \circ \sigma \circ \phi$. By Proposition 2.5, σ^{-1} is a valid movement of $\sigma \circ \phi$ and then $f_t \simeq f_t \circ \sigma \circ \phi \circ \sigma^{-1}$. Because $\sigma \circ \phi \circ \sigma^{-1}$ is a valid movement of V_t we have that $\sigma \circ \phi \circ \sigma^{-1} \in \Phi[f_t]$. \square

Next, we prove that the Wilson set is, in fact, a group.

Theorem 3.1. *The pair $(\Phi[f_t], \circ)$ is a group.*

Proof. Let $\sigma_1, \sigma_2 \in \Phi[f_t]$, then $f_t \simeq f_t \circ \sigma_1$ and $\Phi[f_t] = \Phi[f_t \circ \sigma_1]$ hence $\sigma_2 \in \Phi[f_t \circ \sigma_1]$ and $f_t \circ \sigma_1 \simeq f_t \circ \sigma_1 \circ \sigma_2$. Then $f_t \simeq f_t \circ \sigma_1 \circ \sigma_2$ and $\sigma_1 \circ \sigma_2 \in \Phi[f_t]$. Therefore, $\Phi[f_t]$ is closed under the operation \circ .

Now, it is clear that $f_t \simeq f_t \circ (1)$ and then the identity function is an element of $\Phi[f_t]$. Also it is clear the operation \circ is associative.

Finally, let $\sigma \in \Phi[f_t]$. By Proposition 2.4, σ^{-1} is a valid movement of V_t and by Proposition 2.5, we have that $\sigma^{-1} \in \Phi[f_t]$. \square

Theorem 3.2. *Let f_0 and f_t be connected $[k]$ -configurations and σ a valid movement of V_0 with $f_s = f_t \circ \sigma$ and $f_t \simeq f_s$. Then there exists a valid $f_0 f_t$ -sequence if and only if $f_t = f_s \circ \phi$ for some $\phi \in \Phi[f_t]$.*

Proof. Assume that there exists a valid $f_0 f_t$ -sequence and σ_1 is a valid movement from V_0 to V_t . By Proposition 2.4, σ_1^{-1} is a valid movement from V_t to V_0 , i.e., $f_0 = f_t \circ \sigma_1^{-1}$. Since σ is a valid movement from V_s to V_t then σ^{-1} is a valid movement from V_t to V_s , i.e., $f_s = f_t \circ \sigma^{-1}$, then $f_t \circ \sigma^{-1} \simeq f_s \circ \sigma^{-1} = f_0$. Therefore, $f_0 = f_t \circ \sigma_1^{-1} \simeq f_t \circ \sigma^{-1}$ and by Proposition 2.5, $f_t \simeq f_t \circ \sigma^{-1} \circ \sigma_1$ and for $\phi = \sigma^{-1} \circ \sigma_1$ we have that $f_t = f_s \circ \phi$ with $\phi \in \Phi[f_t]$.

Now, we verify the converse, let $f_t = f_s \circ \phi$ be for some $\phi \in \Phi[f_t]$, then ϕ is a valid movement from V_s to V_t and σ is a valid movement from V_0 to V_s , therefore $\sigma \circ \phi$ is a valid movement from V_0 to V_t and the theorem follows. \square

The existence of the valid movement σ is guaranteed by Theorem 2.1 and Corollary 2.1, hence, in order to verify the existences of a valid $f_0 f_t$ -sequence, we only need to find some $\phi \in \Phi[f_t]$ for which $f_t = f_s \circ \phi$.

3.1. Some Wilson groups

Therefore, we need to know the structure of the Wilson group of a given subgraph. We begin with some particular configurations.

Theorem 3.3. *Let f_t be a connected $[n]$ -configuration over a graph G of order n .*

1. *If G is a cycle, then $\Phi[f_t]$ is the cyclic group \mathbb{Z}_n .*
2. *If G is a tree, then $\Phi[f_t]$ is the trivial group $\{(1)_V\}$.*

Proof. First, since V_\emptyset is empty, then only the elementary valid movements are the cycles, for instance $\sigma = (v_1 v_2 \dots v_n)$ and then $\Phi[f_t] = \langle \sigma \rangle$ which is \mathbb{Z}_n . Second, there is no elementary valid movements different to the identity permutation, and the result follows. \square

Theorem 3.4. *Let f_t be a connected $[k]$ -configuration over a graph G of order $n > k$. If G is a cycle or a path then $\Phi[f_t]$ is $\{(1)_{V_t}\} \times S_{V_\emptyset}$.*

Proof. The valid movements are given by k -paths only into the two opposite directions, namely σ_{p_1} and σ_{p_2} , see Figure 4. Since the labels of V_t are invariant under the valid movements, we have that $\Phi[f_t] = \{(1)_{V_t}\} \times S_{V_\emptyset}$ since any permutation of V_\emptyset also leaves invariant the labels of V_t . \square

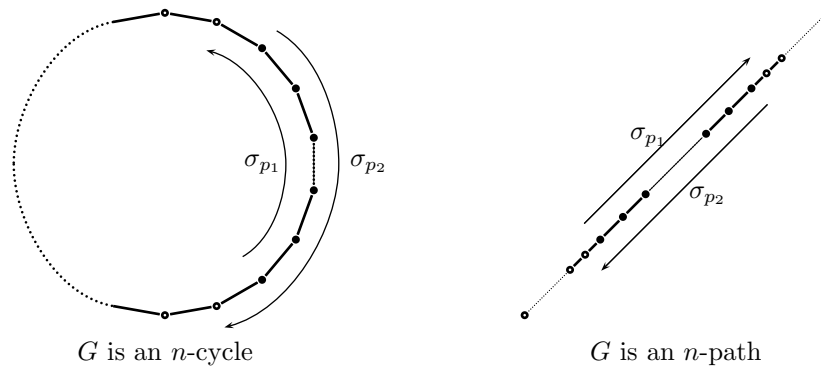


Figure 4. The two possible direction of a k -path, for $k < n$, into a n -cycle or a n -path.

4. Saturated configurations

In this section, we study the configurations without empty vertices, that is, each vertex has weight 1.

Definition 4.1. Let G be a connected graph of order n . A connected $[n]$ -configuration is called saturated.

Theorem 3.3 states a result concerning to saturated configurations, namely, when G is a cycle or a tree. Note that, the elementary movements are only given by cycles, that is, a vertex can be moved if it is in a cycle.

Corollary 4.1. Let f_t be a connected $[n]$ -configuration over a unicyclic connected graph G of order n for which its cycle has order k . Then $\Phi[f_t]$ is the cyclic group \mathbb{Z}_k .

The following definitions are about edge-connectivity and edge-blocks.

Definition 4.2. A non-empty bridgeless connected subgraph \mathbf{B} of G is called an edge-block of G if \mathbf{B} is maximal.

Note that the Wilson group induces a (left) group action φ into the set of vertices $\varphi: \Phi[f_t] \times V(G) \rightarrow V(G)$ where $\varphi(\sigma, v) = \sigma(v)$, therefore we have Theorem 4.1.

Theorem 4.1. If f_t is a saturated configuration over G and $v \in V(\mathbf{B})$ with \mathbf{B} an edge-block of G , then the orbit $\Phi[f_t]v$ of v is $V(\mathbf{B})$.

Proof. First, note that an edge-block could be separable if it contains cut-vertices. Now, let u be a vertex of $V(\mathbf{B})$. By Menger's Theorem, there exist two edge disjoint uv -paths such that they internally share only cut-vertices v_1, \dots, v_{r-1} . The union of this two paths is a union of cycles p_1, \dots, p_r where $v = v_0$ is a vertex of p_1 and $u = v_r$ is a vertex of p_r . The permutation $\sigma = \sigma_{p_r}^{a_r} \circ \dots \circ \sigma_{p_1}^{a_1}$ for which $\sigma_{p_i}^{a_i}(v_{i-1}) = v_i$ for $i \in \{1, \dots, r\}$ maps v to u , therefore $u \in \Phi[f_t]v$ and $V(\mathbf{B}) \subseteq \Phi[f_t]v$. Finally, because a vertex x out of $V(\mathbf{B})$ is connected to $V(\mathbf{B})$ via a bridge, it is not possible to move x into $V(\mathbf{B})$. Therefore $V(\mathbf{B}) = \Phi[f_t]v$. \square

In consequence, the Wilson group of a saturated configuration of a graph in the product of the Wilson groups arising from each edge-block. Hence, we analyze the edge-blocks to know which is the Wilson group of a saturated graph.

We recall that S_X denotes the symmetric group over X , while A_X denotes the alternating group over X .

Lemma 4.1. *Let $G = (V, E)$ be a graph which is a cycle C with a chord with some subdivisions. And let f_t be a saturated configuration of G . Then $\Phi[f_t] = S_V$.*

Proof. The cycle C can be interpreted as the union of the cycles $C' = (v_1 \dots v_r w_m \dots w_1)$ and $C'' = (w_1 \dots w_m u_s \dots u_1)$ where the chord with its subdivisions is the path $T = (w_1 \dots w_m)$ and the cycle C is $(u_1 \dots u_s w_m v_r \dots v_1 w_1)$, see Figure 5.

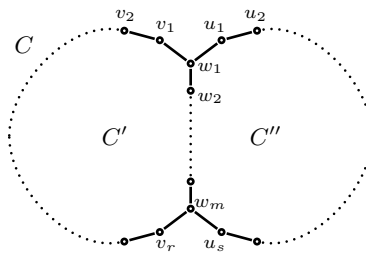


Figure 5. Cycles of Lemma 4.1.

Since for the permutations $\sigma_{C'}$ and $\sigma_{C''}$ is turning clockwise while for the permutations σ_C is turning counterclockwise, we have that

$$\sigma_C \circ \sigma_{C''} \circ \sigma_{C'} = (v_1 w_1).$$

We call σ' to this transposition. Now, we show $(vu) \in \Phi[f_t]$ for any $v, u \in V(G)$ with $v \neq u$. Without lossing of generality, $v = w_1$ and $u \in V(C'')$. For some i , $\sigma_{C''}^i$ we have $\sigma_{C''}^i(u) = v$. Suppose $u \notin \{w_2, \dots, w_m\}$, then the permutation $\sigma = \sigma_{C''}^i \circ \sigma_{C'}^{-1}$ is such that $\sigma(v) = u$ with $v \sim u$. If $u \in \{w_2, \dots, w_m\}$, then $\sigma = \sigma_{C''}^{i+1} \circ \sigma_{C'}^{-1}$ is such that $\sigma(v) = u$ such that $v \sim u$. Hence, $(vu) = \sigma^{-1} \circ \sigma' \circ \sigma$ and $(vu) \in \Phi[f_t]$ and then $\Phi[f_t] = S_V$. \square

A permutation is an element of A_X if and only if it is a product of an even number of transpositions in X . Since every 3-cycle (ijk) is the product of two transpositions $(ij)(ik)$ and the product of two transpositions $(ij)(kl)$ is the product of two 3-cycles $(ikj)(kjl)$ then the alternating group is generated by 3-cycles.

Lemma 4.2. *Let $G = (V, E)$ be a graph which is the union of two cycles C and C' with exactly a cut vertex. And let f_t be a saturated configuration of G . Then $\Phi[f_t] = S_V$ if C or C' is even, otherwise $\Phi[f_t] = A_V$.*

Proof. First, we prove $A_V \subseteq \Phi[f_t]$, that is (wvu) is an element of $\Phi[f_t]$ for any $w, v, u \in V$. We can assume $w = w_1$ since, for some j , $\sigma = \sigma_C^j$ or $\sigma = \sigma_{C'}^j$ is such that $\sigma(w) = w_1$ and then $(wvu) = \sigma^{-1} \circ (w_1vu) \circ \sigma$.

Second, let $C = (w_1 v_1 \dots v_r)$ be and $C' = (w_1 u_1 \dots u_s)$, see Figure 6. Therefore $\sigma_{C'}^{-1} \circ \sigma_C^{-1} \circ \sigma_{C'} \circ \sigma_C = (w_1 v_1 u_1)$. We call σ' to this 3-cycle.

Then, we divide the proof into three cases:

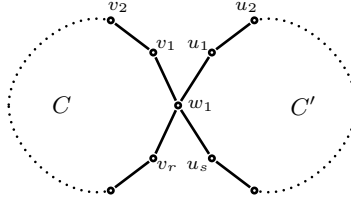


Figure 6. Cycles of Lemma 4.2.

1. If $v = v_1$ and $u \in V(C')$, for some j , $\sigma = \sigma_{C'}^j$ is such that $\sigma(u) = w_1$ then $(w_1 v u) = \sigma^{-1} \circ \sigma_C^{-1} \circ \sigma \circ \sigma_C$.
2. If $v = v_1$ and $u \in V(C)$, for some j , $\sigma = \sigma_C^j$ is such that $\sigma(u) = w_1$ then $(w_1 v u) = \sigma^{-1} \circ \sigma_{C'} \circ \sigma' \circ \sigma_{C'}^{-1} \circ \sigma$.
3. Similar to (1.), but $v \in V(C) \setminus \{v_1\}$. If $u = u_1$ then the case is analogous to (1.) by symmetry. If $u \neq u_1$ then $\sigma = \sigma_{C'}^{-1} \circ \sigma^j \circ \sigma_{C'}$, for $\sigma^j(v) = v_1$ and some j , gets a configuration similar to the case (1.)

Now, if C and C' are odd cycles, then σ_C and $\sigma_{C'}$ are even permutations, therefore $A_V = \Phi[f_t]$. On the other hand, if C or C' is even, then it is an odd permutations, therefore $S_V = \Phi[f_t]$. \square

In order to prove our main results, we define the following concept.

Definition 4.3. An edge-block of a graph is called weak if it is a cycle or if every two cycles sharing vertices have exactly a vertex in common.

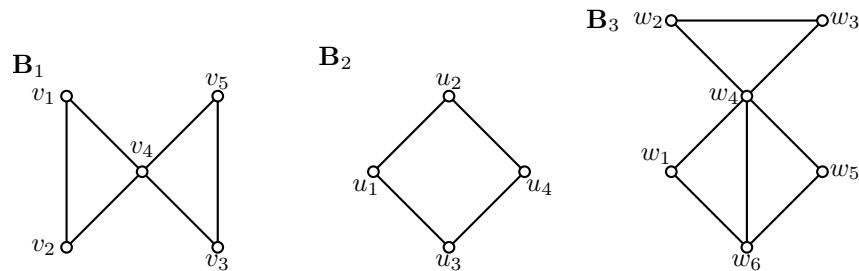


Figure 7. The edge-blocks \mathbf{B}_1 and \mathbf{B}_2 are weak but \mathbf{B}_3 is not.

Lemma 4.3. Let \mathbf{B} be an edge-block that is not a cycle and f_t be a saturated configuration. If \mathbf{B} is weak such that every cycle is odd, then $\Phi[f_t] = A_V$ otherwise $\Phi[f_t] = S_V$.

Proof. Let \mathbf{B} be as defined above and suppose that it is weak and every cycle is odd. We prove that $(uvw) \in \Phi[f_t]$ for any $u, v, w \in V(\mathbf{B})$ as follows: since u, v, w are in the same orbit and there are (at least) two cycles C and C' with exactly a cut vertex in common. We can send there all of them via a permutation σ , by Lemma 4.2, we can get this 3-cycle there and then, via σ^{-1} we get the desire 3-cycle. Because the parity of the odd cycles, $\Phi[f_t] = A_V$.

On the other hand, if \mathbf{B} contains an even cycle $\Phi[f_t] = S_V$ or if it is not weak, we can assume that C and C' have in common a path with more than a vertex therefore any transposition (uv) can be done via a permutation σ sending u and v to the cycles C and C' , by Lemma 4.1, we can get this 2-cycle there and finally, via σ^{-1} we get the desire transposition getting that $\Phi[f_t] = S_V$. \square

Now, we can describe the Wilson group of a saturated configuration.

Theorem 4.2. *Let f_t be a saturated configuration of a graph G , then*

$$\Phi[f_t] = \prod_{i=1}^r \Gamma_i$$

where Γ_i is a cyclic group, an alternating group or a symmetric group.

Proof. Let \mathbf{B}_i be the set of edge-blocks G , for $i \in \{1, \dots, r\}$. By Theorem 4.1, the non-trivial orbits are $V(\mathbf{B}_i)$ for each $i \in \{1, \dots, r\}$. Hence, for the set $V(\mathbf{B}_i)$, its Wilson group is cyclic, by Theorem 3.3, or it is an alternating group or a symmetric group by Lemma 4.3 and the result follows. \square

5. No-saturated configurations

In this section, we only study the Wilson group for no-saturated configurations. The main difference between saturated and no-saturated configurations is the existence of valid movements given by paths. Theorem 3.4 is an example of this fact.

To begin with, we analyze the behavior of the bipartite complete graph $K_{1,3}$, also called as the 3-star graph, which is a relevant graph in no-saturated configurations.

Let G be a graph containing at least a 3-star subgraph and f_t a connected configuration over G such that

$$f_t = \begin{pmatrix} \dots & v & v_1 & u_1 & w_1 & \dots \\ \dots & 1 & 2 & 3 & 4 & \dots \end{pmatrix}$$

where v is a vertex of degree at least 3 and v_1, u_1 and w_1 are adjacent to v . Figure 8 shows the sequence of movements to generate the transposition (vv_1) supposing some empty vertices. Before to verify the details to produce the movements to generate such transposition, we give the following definition.

Definition 5.1. *Let G be a graph and w an edge of G . A vertex w is in the direction of u with respect to v if there exists a w -path containing u .*

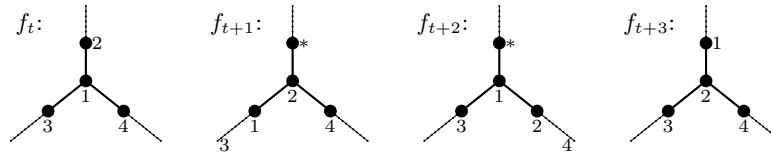


Figure 8. Movements over a 3-star.

Therefore, the set of vertices of the component of $G - v$ containing u are the vertices in the direction of u with respect to v . If v is not a cut-vertex, each vertex is in the direction of u with respect to v , for every u in $N(v)$. The set of empty vertices in the direction of u with respect to v is denoted by $B_v[f_t](u)$ and its cardinality by $b_v[f_t](u)$ (or simply by $B_v(u)$ and $b_v(u)$, respectively, when f_t is understood), see Figure 9.

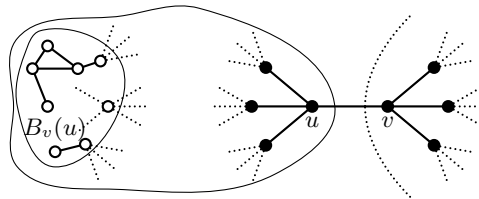


Figure 9. A set $B_v(u)$.

Proposition 5.1. *Let G be a connected graph. If $G[B_v(u)]$ is a tree, then there exists a spanning tree of G containing $G[B_v(u)]$.*

Lemma 5.1. *Let f_t be a connected $[k]$ -configuration over a graph G and (w, v, u) a path such that $u, v \in V_t$. If $b_v(w) > 0$ then there exists a cycle or a path p containing (w, v, u) for which σ_p is in $\Gamma[f_t]$.*

Proof. Since $b_v(w) > 0$, there exists an empty vertex in the direction of w with respect to v . Therefore, there is an r -path $p_1 = (w_r, \dots, w_1 = w, v)$ with w_r an empty vertex.

On one hand, if $u = w_i$ for some $i \in [r - 1]$ then for the cycle $p = (w_1 = w, v, u = w_i, w_{i-1}, \dots, w_2)$, σ_p is in $\Gamma[f_t]$.

On the other hand, if $u \neq w_i$ for all $i \in [r - 1]$ and let T a spanning tree of $G[V_t]$ containing the path $p = (w_{r-1}, \dots, w_1 = w, v, u = u_0, \dots, u_s)$ where u_s is a leaf of T , and then σ_p is in $\Gamma[f_t]$, see Figure 10.

□

Theorem 5.1. *Let $K_{1,3}$ be a star subgraph of G with partition $(\{v\}, \{u_1, v_1, w_1\})$ such that $b_v(u_1), b_v(w_1) > 0$. If $v, v_1 \in V_t$ and vv_1 is a bridge, then $(vv_1) \in \Phi[f_t]$.*

Proof. Let f_t be a connected $[k]$ -configuration over G . For the paths (u_1, v, v_1) and (w_1, v, v_1) , by Lemma 5.1, there exist the permutations

$$\sigma_1 = (u_s \dots u_1 v v_1 \dots v_r) \text{ and } \sigma'_1 = (w_m \dots w_1 v v_1 \dots v_r).$$

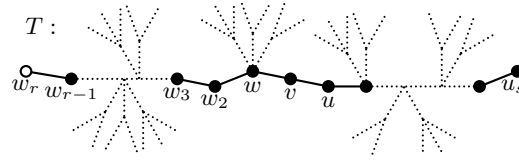


Figure 10. The spanning tree in the proof of Lemma 5.1.

Since vv_1 is a bridge, we can assume that the corresponding paths of the permutations are sharing the leaf v_r .

If $u_s = w_m$ then the vertex v is in a cycle $(v, u_1, \dots, u_i = w_j, w_1)$, see Figure 11. Let σ_2 and σ_3 be the permutations

$$\sigma_2 = (vu_1 \dots u_i \dots w_1) \text{ and } \sigma_3 = (v_r \dots v_1vw_1 \dots u_i \dots u_s).$$

Then $\sigma = \sigma_3 \circ \sigma_2 \circ \sigma_1 = (vv_1) \in \Phi[f_t]$.

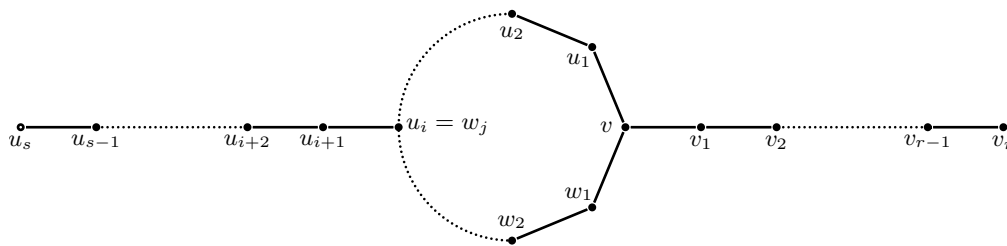


Figure 11. Two paths sharing both leaves in the proof of Theorem 5.1.

If $u_s \neq w_m$, consider the following permutations, see Figure 12.

$$\sigma_5 = (w_m \dots w_1vu_1 \dots u_s) \text{ and } \sigma_6 = (v_r \dots v_1vw_1 \dots w_m) \text{ and } \sigma_7 = (u_s w_s).$$

Therefore $\sigma = \sigma_7 \circ \sigma_6 \circ \sigma_5 \circ \sigma_1 \in \Phi[f_t]$ because $\sigma_7 \in \Phi[f_t]$. □

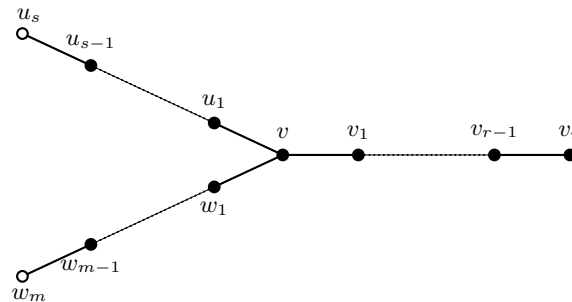


Figure 12. Two paths sharing only a leaf in the proof of Theorem 5.1.

In order to analyze the Wilson group and use the previous ideas, we give the following definition.

Definition 5.2. Let $x, y \in V_t$ be and v a vertex of degree 3 or more. We say that v is an exchange-vertex for the pair $\{x, y\}$ if there is a vertex v_1 adjacent to v and a valid permutation σ such that $\sigma(v) = x$, $\sigma(v_1) = y$ and $f_t \circ \sigma$ satisfy the hypothesis of Theorem 5.1.

Moreover, if $x \sim y$ we say v is an exchange-vertex for the edge xy .

Note that if v is an exchange-vertex of $\{x, y\}$ then $(xy) \in \Phi[f_t]$ because Proposition 3.2, indeed, if $(vv_1) \in \Phi[f_t \circ \sigma]$ then $(xy) = \sigma \circ (vv_1) \circ \sigma^{-1} \in \Phi[f_t]$.

Lemma 5.2. If there exists an exchange-vertex for the pair $\{x, y\}$ then $(xy) \in \Phi[f_t]$.

Theorem 5.2. If (u_1, u_2, \dots, u_m) is a path such that for each edge there is an exchange vertex, then $(u_1 u_m) \in \Phi[f_t]$.

Proof. Since each transposition $(u_i u_{i+1}) \in \Phi[f_t]$ with $i \in [m - 1]$ and therefore

$$(u_1 u_2) \circ (u_2 u_3) \circ \dots \circ (u_{m-1} u_m) \circ (u_{m-1} u_{m-2}) \circ \dots \circ (u_2 u_1) = (u_1 u_m) \in \Phi[f_t].$$

□

Now, we analyze the edge-blocks of a graph G when it is not a cycle and f_t is a no-saturated configuration.

Lemma 5.3. If \mathbf{B} is an edge-block of $G[V_t]$ for which every vertex has weight 1, then for each edge $xy \in E(\mathbf{B})$ there exists an exchange-vertex for xy .

Proof. Since f_t is a no-saturated configuration there is a vertex $v_1 \notin V(\mathbf{B})$ adjacent to some vertex $v \in V(\mathbf{B})$ such that $b_v(v_1) > 0$. Since \mathbf{B} is a bridgeless subgraph, let u, w adjacent to v in \mathbf{B} living in a cycle there. Note that v has degree at least 3.

Let xy be an edge of \mathbf{B} . Without loss of generality, we can assume that $(xu) \circ (yv) \in \Phi[f_y]$ by Theorem 3.3 and Lemma 4.3. Let $f_s = f_t \circ (xu) \circ (yv)$.

Note that the path (v_1, v, u) satisfies the hypothesis of Lemma 5.1. Hence, there exists a cycle or a path p containing (v_1, v, u) for which $\sigma_p \in \Gamma[f_s]$ with $\sigma_p(v_1) = v$ and $\sigma_p(v) = u$. Moreover, if vv_1 is not a bridge, we can delete edges adjacent to v but not in $E(\mathbf{B})$, $b_v[f_s \circ \sigma_p](w) > 0$ and $b_v[f_s \circ \sigma_p](u) > 0$. By Theorem 5.1, the vertex v is an exchange-vertex for xy .

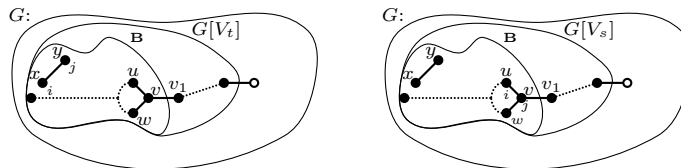


Figure 13. The vertex v is an exchange-vertex of xy .

□

The previous result can be established for vertex-blocks instead of edge-blocks because if \mathbf{B} is a vertex-block of $G[V_t]$ for which every vertex has weight 1, consider an edge $vv_1 \in E(\mathbf{B})$. If vv_1 is not a bridge, vv_1 is in an edge-block and there exists an exchange-vertex for vv_1 . And if vv_1 is a bridge following the hypothesis of Theorem 5.1, then v is an exchange-vertex for vv_1 .

Theorem 5.3. *Let xy be an edge for which its vertices have weight 1. If xy is not a bridge and G is not a cycle, then there exists an exchange-vertex for xy .*

Proof. If xy belongs to an edge-block of $G[V_t]$, by Lemma 5.3, the result is done. Now assume to the contrary. By hypothesis, xy belongs to a cycle $C = (u_1 = x, u_2 = y, \dots, u_s)$ of G and u_i has degree at least 3 for some $i \in [s]$.

Via a valid movement σ , we can move the edge xy to the edge $u_i u_{i-1}$, see Figure 14, and by Theorem 5.1, $u_i = \sigma(y)$ (or $u_i = \sigma(x)$) is an exchange-vertex of $\sigma(x)\sigma(y)$, then u_i is an exchange-vertex of xy .

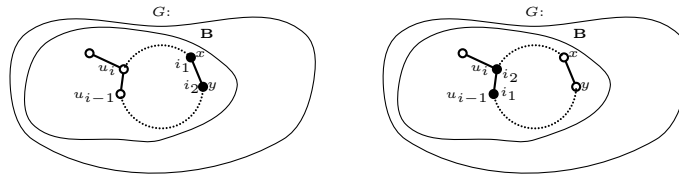


Figure 14. The vertex v is an exchange-vertex of xy .

□

Theorem 5.4. *If u and v are two different vertices of an edge-block \mathbf{B} of a graph G , both of weight 1 and G is not a cycle, then $(uv) \in \Phi[f_t]$*

Proof. Since there exist a path P contained in \mathbf{B} for which every vertex is in $G[V_t]$, by Lemma 5.3 and Theorem 5.3 the result follows. □

Corollary 5.1. *If G is an edge-block, G is not a cycle and f_t is a no-saturated configuration, then $\Phi[f_t] = S_{V_t}$.*

Before to analyzing the bridges of a graph with no-saturated configurations, consider the set of vertices C_v for which v is an exchange-vertex, i.e.,

$$C_v = \{x, y \in V_t : v \text{ is an exchange-vertex for the pair } \{x, y\}\}.$$

In order to see the relation between C_v and the orbits of $\Phi[f_t]$ we have the following definition and results.

Corollary 5.2. *Let \mathbf{B} be an edge-block of G . If v is an exchange-vertex for some edge of \mathbf{B} , then v is an exchange-vertex for each edge of \mathbf{B} .*

Since the set of bridges and edge-blocks induces a partition into the set of edges, we can define the following graph.

Definition 5.3. Given a graph G , the graph $G_{\mathbf{B}}$ obtained from G by contracting each edge-block into a vertex is called the edge-block graph.

Proposition 5.2. The edge-block graph $G_{\mathbf{B}}$ of a connected graph G is a tree.

We denote a vertex of $G_{\mathbf{B}}$ as $[v]$ where v is any vertex of the corresponding edge-block of G , i.e., $[v]$ is the equivalence class of vertices of an edge-block of G . If the equivalence class is trivial, we use v instead of $[v]$.

For example, if G is a unicyclic graph, the cycle of G is denoted by $[v]$ in $G_{\mathbf{B}}$ but the remainder vertices are denoted u instead of $[u]$ in the edge-block graph of G .

Proposition 5.3. The edge $[u][v]$ of $G_{\mathbf{B}}$ is a bridge if and only if u_1v_1 is a bridge of G for some $u_1 \in [u]$ and $v_1 \in [v]$.

Definition 5.4. Let $G_{\mathbf{B}}$ be the edge-block graph of G and f_t a $[k]$ -connected configuration. We define a weight function $\omega[f_t]$ such that

$$\omega[f_t] : V(G_{\mathbf{B}}) \rightarrow \mathbb{N}$$

and $\omega[f_t]([v])$ is the number of empty vertices in the corresponding edge-block of v in G .

For example, if f_t is a saturated connected configuration over G , the weighted function of $G_{\mathbf{B}}$ is zero for any of its vertices.

Let $[u][v]$ be an edge of $G_{\mathbf{B}}$ and u_1v_1 the bridge such that $u_1 \in [u]$ and $v_1 \in [v]$. Recall that the set of empty vertices in the direction of u_1 with respect to v_1 is $b_{v_1}(u_1)$. We denote by $\beta_{[v]}([u])$ to the sum of $\omega[f_t]([x])$ for all $[x]$ in the component of $G_{\mathbf{B}} - [v]$ containing $[u]$.

Lemma 5.4. Let $[u][v]$ be an edge of $G_{\mathbf{B}}$ where $[v]$ is not a trivial equivalence class. If $\beta_{[u]}([v]) > 0$ then v_1 is an exchange-vertex of u_1v_1 where u_1v_1 is the corresponding bridge for $[u][v]$.

Proof. Suppose $u_1, v_1 \in V_t$. Since $[v]$ is not a trivial equivalence class, v_1 is a vertex of a cycle C . Let v_2 be another vertex of C . If $b_{v_1}(v_2) > 0$ then there exists a valid movement $\sigma \in \Gamma[f_t]$ such that $\sigma(v_2) = v_1$ and $\sigma(v_1) = u_1$. Due to the fact that v_1 is an exchange-vertex of v_1v_2 , v_1 is an exchange-vertex of $\sigma(v_1)\sigma(v_2) = u_1v_1$, see Figure 15 (left).

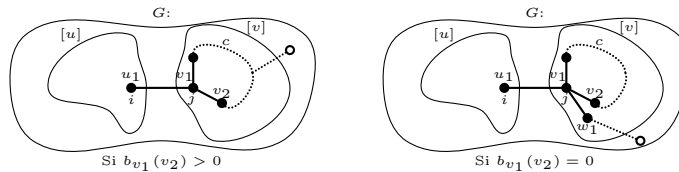


Figure 15. If $[v]$ is not a trivial equivalence class, then v_1 is an exchange-vertex of u_1v_1 .

If $b_{v_1}(v_2) = 0$, then C is saturated and there is a valid movement σ_1 such that $\sigma_1(v_2) = v_1$. Since u_1v_1 is a bridge and $\beta_{[u]}([v]) > 0$ there exists a vertex $w_1 \notin [v]$ adjacent to v_1 such that $b_{v_1}(w_1) > 0$. Hence, there is a valid movement $\sigma_2 \in \Gamma[f_t \circ \sigma_1]$ such that $\sigma_2(v_1) = u_1$ and $\sigma_2(v_1)$ and the result follows. See Figure 15 (right). \square

Theorem 5.5. Let $P = (v_1, v_2, \dots, v_{r+1})$ be a path of G such that each edge is a bridge and $v_1, v_{r+1} \in V_t$. If $[v_1]$ is a no trivial equivalence class of G_B and $r \leq b_{v_2}(v_1)$, then v_1 is an exchange-vertex of each edge of P .

Proof. To begin with, observe that each vertex of P has weight 1. For each $i \in [r]$ we have $b_{v_{i+1}}(v_i) \geq r$, then there exists a valid movement σ_i such that $\sigma_i(v_1) = v_i$ and $\sigma_i(v_2) = v_{i+1}$. By Lemma 5, v_1 is an exchange-vertex of $\sigma_i(v_1)\sigma_i(v_2) = v_i v_{i+1}$ because $b_{v_2}[f_t \circ \sigma_i] \geq r + 1 - i$, see Figure 16.

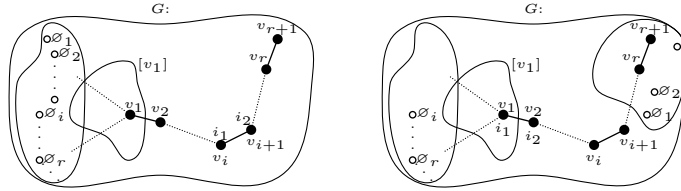


Figure 16. If $[v]$ is not a trivial equivalence class, then v_1 is an exchange-vertex of $v_i v_{i+1}$.

□

Corollary 5.3. Let $P = (v_1, v_2, \dots, v_{r+1})$ be a path of G such that each edge is a bridge and $v_1, v_{r+1} \in V_t$. If $[v_1]$ is a no trivial equivalence class of G_B and $r \leq b_{v_2}(v_1)$, then $v_1 v_{r+1} \in \Phi[f_t]$.

Proof. By Theorem 5.5, v_1 is an exchange-vertex of each edge of P . By Theorem 5.2, the result follows. □

Next, we analyze the case where $[v]$ is a trivial equivalence class, but degree at least 3.

Theorem 5.6. Let $P = (v_1, v_2, \dots, v_r)$ be a path of G such that each edge is a bridge and $v_1, v_r \in V_t$. If $[v_1]$ is a trivial equivalence class of G_B , degree at least 3 and $r \leq b_{v_2}(v_1)$, then v_1 is an exchange-vertex of each edge of $\{v_2, v_3, \dots, v_r\}$.

Proof. Observe that the condition $r \leq b_{v_2}(v_1)$ implies at least two empty vertices in the direction of v_1 with respect to v_2 . Via a valid movement, we can obtain at least two empty vertices in at least one branches at v_1 . Now, we use exactly the same argument as in Theorem 5.5 and the result follows, see Figure 17.

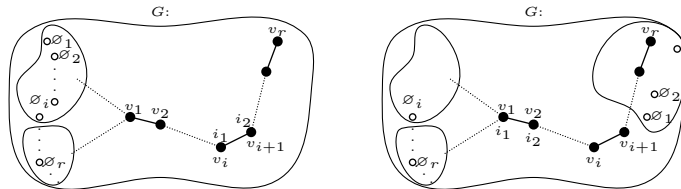


Figure 17. If $[v]$ is a trivial equivalence class, then v_1 is an exchange-vertex of $v_i v_{i+1}$.

□

In the previous proof we remark if f_t has at least two branches at v_1 with at least an empty vertex each, then v_1 is an exchange vertex of (v_1v_2) . See the vertices $[v_{17}]$ and $[v_7]$ of the example of Figure 18.

Corollary 5.4. *Let $P = (v_1, v_2, \dots, v_r)$ be a path of G such that each edge is a bridge and $v_1, v_r \in V_t$. If $[v_1]$ is a trivial equivalence class of G_B , degree at least 3 and $r \leq b_{v_2}(v_1)$, then $v_2v_r \in \Phi[f_t]$.*

Theorem 5.7. *Let $[v]$ be a trivial equivalence class of G_B with degree at least 3. If $x \in V_t$ (and $y \in V_t$) such that it is in direction of v_1 with respect to v (and v_2 with respect to v , respectively) and at distance to v less than r . If $b_{v_1}(v) \geq r$ (and $b_{v_2}(v) \geq r$) then $(xy) \in \Phi[f_t]$.*

Proof. By Corollary 5.4, we have $(vv_1), (vv_2) \in \Phi[f_t]$. It remains to prove that $(v_1v_2) \in \Phi[f_t]$ owing to the fact that $(xy) = (v_1x) \circ (v_2y) \circ (v_1v_2) \circ (v_1x) \circ (v_2y)$.

If vv_1 and vv_2 satisfy the hypothesis of Theorem 5.1, the result follows. Without loss of generality, vv_2 doesn't have such hypothesis. Since $r \geq 2$, there exists a vertex v_3 adjacent to v for which $b_vv_3 \geq 2$. Hence, there is a valid movement $\sigma \Gamma[f_t]$ such that $\sigma(v_3) = v$ and $\sigma(v) = v_2$. Therefore, $b_v[f_t \circ \sigma](v_2) > 0$ and $b_v[f_t \circ \sigma](v_3) > 0$ and then v is an exchange-vertex of v_1 and v_2 and finally $(v_1v_2) \in \Phi[f_t]$. \square

Through Theorems 5.5 and 5.6 we can determine the vertices of C_v where $[v]$ has degree at least 3 in G_B . We need the following useful definition related to C_v .

Definition 5.5. *Let $[v]$ be a vertex of degree at least 3 of G_B . We denote by $C_{[v]}$ to the subset of vertices of G having the following property: for every pair of their vertices (x, y) , the transposition (xy) is in $\Phi[f_t]$ according to Corollaries 5.3 and 5.4 and Theorems 5.4 and 5.7.*

Figure 18 shows the four sets $C_{[v]}$ of a graph G . Note that $b_{[v_4]}([v_3]) = 1$, so $C_{[v_3]}$ contains v_2, v_3 and v_4 . Similarly $b_{[v_{10}]}([v_{11}]) = 2$, so $C_{[v_{11}]}$ contains v_9, v_{10}, v_{11} and v_{14} . The vertices $[v_7]$ and $[v_{17}]$ of G_B are trivial equivalence classes. For $C_{[v_7]}$ we have $b_{[v_6]}([v_7]) = 2$, $b_{[v_8]}([v_7]) = 1$ and $b_{[v_{15}]}([v_7]) = 3$ then $v_6 \in C_{[v_7]}$ and $v_{15}, v_{16} \in C_{[v_7]}$. Moreover, there are two branches at v_7 with empty vertices, then $v_7 \in C_{[v_7]}$. Finally, $C_{[v_{17}]}$ contains v_{18} and v_{19} because $b_{[v_{16}]}([v_{17}]) = 0$, $b_{[v_{19}]}([v_{17}]) = 3$ and $b_{[v_{19}]}([v_{17}]) = 3$.

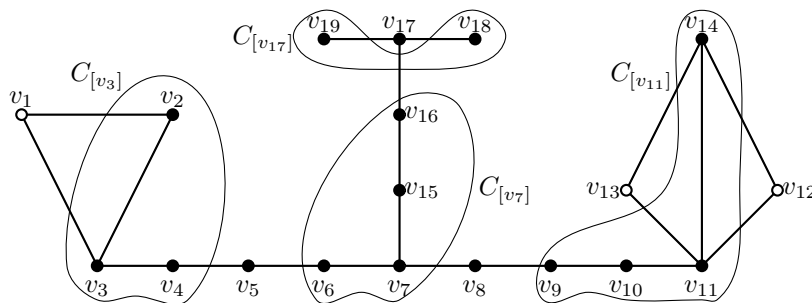


Figure 18. The graph G as an example with the set of vertices $C_{[v_3]}$, $C_{[v_7]}$, $C_{[v_{11}]}$ and $C_{[v_{17}]}$.

Note the following facts. $S_{C_{[v]}} \subseteq \Phi[f_t]$ and if v has degree at least 3, then $C_{[v]} \subseteq C_v$. Let $V_{\geq 3}$ be the set of vertices of degree at least 3.

We define the relation R over $V_{\geq 3}$ as follows:

vRv if and only if there exists a sequence $\{u = u_1, u_2, \dots, u_{r+1} = v\}$ of vertices of $V_{\geq 3}$ such that for each $i \in [r]$, $C_{[u_i]} \cap C_{[u_{i+1}]} \neq \emptyset$. Clearly, R is an equivalence relation. We denote by $R(v)$ the equivalence class of v .

Theorem 5.8. *If $v \in V_{\geq 3}$, then*

$$C_v = \bigcup_{u \in R(v)} C_{[u]}.$$

Proof. Let $u \in R(v)$ be and a sequence $\{u = u_1, u_2, \dots, u_{r+1} = v\}$ of vertices of $V_{\geq 3}$ such that for each $i \in [r]$, $C_{[u_i]} \cap C_{[u_{i+1}]} \neq \emptyset$. Let $v_i \in C_{[u_i]}$. Let $v_0 \in C_{[u]}$ then for each $j \in [r]$, we have $(v_{j-1}v_j) \in \Phi[f_t]$. Then, $(v_0v_r) \in \Phi[f_t]$. Since $v_r \in C_{[v]} \subseteq C_v$, then $v_0 \in C_v$ and $C_{[u]} \subseteq C_v$.

Now, suppose that $v_0 \in C_v$ and $v_0 \in C_{[u]}$ for all $u \in R(v)$. Let $u_1 \in R(u)$ such that $[u_1]$ is the closest vertex to $[v_0]$ in $G_{\mathbf{B}}$. Then $C_{[u_1]} \cap C_{[v_0]} = \emptyset$. Since $v_0 \in C_v$ there exist valid movements σ and σ_1 such that $\sigma(v) = v_0$ and $\sigma_1(u_1) = v_0$ but it is not possible according to Theorems 5.5 and 5.6 because v_0 would have to be a vertex of C_{u_1} . \square

Finally, note that if $x, y \in C_v$, then $(xy) \in \Phi[f_t]$ and then $S_{C_v} \subset \Phi[f_t]$. In consequence, we can prove the main theorem for no-saturated configurations.

Theorem 5.9. *Let $R(v_1), R(v_2), \dots, R(v_m)$ be the equivalence classes of R . Then*

$$\Phi[f_t] = \prod_{i=1}^m S_{C_{v_i}} \times S_{V_{\emptyset}}.$$

Proof. By construction, $S_{V_{\emptyset}}, S_{C_v} \leq \Phi[f_t]$, then $\prod_{i=1}^m S_{C_{v_i}} \times S_{V_{\emptyset}} \leq \Phi[f_t]$.

Let $\sigma \in \Phi[f_t]$ be such that $\sigma \notin \prod_{i=1}^m S_{C_{v_i}}$, i.e., $\sigma(x) = y$ for $x \in C_{v_i}$ and $y \in C_{v_j}$ with $i \neq j$. The vertices v_i and v_j are exchange-vertices for some edges xw and yz , respectively, then $(xw), (yz) \in \Phi[f_t]$.

On the other hand, if $\sigma(w) = z$ then $\sigma \in \Phi[f_t]$ and v_i is an exchange-vertex for xw in $f_t \circ \sigma$, then v_i is an exchange-vertex for yz in f_t . Hence, $y \in C_{v_i}$. A contradiction since $C_{v_i} \cap C_{v_j} = \emptyset$.

Now, suppose $\sigma(w) \neq z$, then $\sigma^{-1}(z) \neq w$. And we have $\sigma(x) = y$ then $\sigma^{-1}(z) \neq x$ and $\sigma(w) \neq y$. Therefore, $\sigma_1 := \sigma \circ (xw) \circ \sigma^{-1} \circ (yz) \circ \sigma \circ (xw) \in \Phi[f_t]$ and satisfies that $\sigma_1(x) = y$ and $\sigma_1(y) = z$. As before, this is a contradiction. Then $\sigma \in \prod_{i=1}^m S_{C_{v_i}}$ and $\Phi[f_t] \leq \prod_{i=1}^m S_{C_{v_i}} \times S_{V_{\emptyset}}$. \square

To end this section, Figure 18 shows a graph with a no-saturated configuration f_t for which its Wilson group is $S_{C_{v_3}} \times S_{C_{v_7}} \times S_{C_{v_{11}}} \times S_{C_{v_{17}}} \times S_{V_{\emptyset}}$.

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