The Manhattan Product of Digraphs

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Abstract

We study the main properties of a new product of bipartite digraphs which we call Manhattan product. This product allows us to understand the subjacent product in the Manhattan street networks and can be used to built other networks with similar good properties. It is shown that if all the factors of such a product are (directed) cycles, then the digraph obtained is a Manhattan street network, a widely studied topology for modeling some interconnection networks. To this respect, it is proved that many properties of these networks, such as high symmetries, reduced diameter and the presence of Hamiltonian cycles, are shared by the Manhattan product of some digraphs. Moreover, we show that the Manhattan product of two Manhattan streets networks is also a Manhattan street network. Finally, some sufficient conditions for the Manhattan product of two Cayley digraphs to be also a Cayley digraph are given. Throughout our study we use some interesting recent concepts, such as the unilateral distance and related graph invariants.

Keywords: self-converse digraph, Manhattan street network, unilateral diameter, Cayley digraph, Hamiltonian cycle
Mathematics Subject Classification: 05C20, 05C25, 05C12, 05C45, 90B10

1. Introduction

The 2-dimensional Manhattan street network M_2 was introduced simultaneously, in different contexts, by Morillo, Fiol and Fàbrega [13] and Maxemchuk [12] as an unidirectional regular mesh structure resembling locally the topology of the avenues and streets of Manhattan (or *l'Eixample*, in downtown Barcelona), see Fig. 3. In fact, M_2 has a natural embedding in the torus and it has been

extensively studied in the literature as a model of interconnection network. For instance, its average distance was computed by Khasnabish [11], and Chung and Agrawal [3], and the generation of routing schemes was studied by Maxemchuk [12]. Moreover, Chung and Agrawal [3] derived its diameter. Varvarigos [15] evaluated again the mean internodal distance, provided a shortest path routing algorithm and proved some Hamiltonian properties. In [4, 7], the authors formally defined the n-dimensional Manhattan street network M_n and studied some of its structural properties. In particular, it was shown that M_n is a Cayley digraph, which can be seen as a subgroup of the n-dimensional version of the wallpaper group pgg. The spectra of Manhattan street networks was also studied by the authors and Mitjana in [6].

Before outlining the contents of the paper, recall that a digraph $G = (V, A)$ consists of a set of *vertices* $V = V(G)$, together with a set of *arcs* $A = A(G)$, which are ordered pairs of vertices, $A \subset V \times V = \{(u, v) : u, v \in V\}$. An arc (u, v) is usually depicted as an arrow with *tail* u (initial vertex) and *head* v (end vertex), that is, $u \to v$. The *in-neighborhood* $\Gamma_G^-(u)$ (respectively, *out-neighborhood* $\Gamma_G^+(u)$) of a vertex u is the set of vertices adjacent to (respectively, from) u. The *indegree* of vertex u is $\delta^-(u) = |\Gamma_G^-(u)|$, whereas its *outdegree* is $\delta^+(u) = |\Gamma_G^+(u)|$. Then, G is δ-*regular* when $\delta^-(u) = \delta^+(u) = \delta$ for every vertex $u \in V$. Given a digraph $G = (V, A)$, its *converse* digraph $\overline{G} = (V, \overline{A})$ is obtained from G by reversing all the orientations of the arcs in A, that is, $(u, v) \in \overline{A}$ if and only if $(v, u) \in A$. A digraph G is said to be *self-converse* when $\overline{G} \cong G$. In our study, the *unilateral distance* between two vertices $u, v \in V$ is defined to be

$$\mathrm{dist}_G^*(u, v) = \min\{\mathrm{dist}_G(u, v), \mathrm{dist}_G(v, u)\} = \min\{\mathrm{dist}_G(u, v), \mathrm{dist}_{\overline{G}}(u, v)\},$$

where dist_G is the standard (directional) distance for digraphs. From this concept, we also define the *unilateral eccentricity* ε^* and the *unilateral radius* r^* of a vertex u, and the *unilateral diameter* D^* of G, as expected:

$$\varepsilon^*(u) = \max_{v \in V}\{\mathrm{dist}_G^*(u, v)\}, \ r^*(G) = \min_{u \in V}\{\varepsilon^*(u)\}, \ \text{and} \ D^*(G) = \max_{u \in V}\{\varepsilon^*(u)\}.$$

For instance, if $G = C_N$, the directed cycle on N vertices, we have $r^*(C_N) = D^*(C_N) = \lfloor N/2 \rfloor$. Notice that all these parameters are obviously upper bounded by the corresponding parameters of the underlying graph (obtained from G by changing arcs for edges). Some constructions of digraphs with large order for a given maximum degree and unilateral diameter were given by Gómez, Canale and Muñoz [8, 9].

Other standard definitions and basic results about graphs and digraphs not defined here can be found in Bang-Jensen and Gutin [1], Chartrand and Lesniak [2] and White [16]. Imrich and Klavzar gave a good description of the most common graph products in [10].

In this paper, we first recall the definition and some of the properties of the Manhattan street network (where the Manhattan product takes its name from). The motivation for this product is that it allows us to understand the subjacent product in the well-known Manhattan street networks and it can be used to build new networks with similar good properties. Afterwards, we define the Manhattan product of (self-converse bipartite) digraphs, which was introduced by the authors in [5]. It is shown that when all the factors are (directed) cycles, then the obtained digraph is just the Manhattan street network. Moreover, we prove that the Manhattan product of two Manhattan streets networks is also a Manhattan street network. It is also proved that many properties of these

networks, such as high symmetries, reduced diameter and the presence of Hamiltonian cycles, are shared by the Manhattan product of some digraphs. Finally, we investigate when the Manhattan product of two Cayley digraphs is also a Cayley digraph and characterize the corresponding group.

2. Manhattan street networks

We recall the definition and some basic properties of a class of toroidal directed networks, commonly known as Manhattan street networks. For more information see [4, 7].

Given n *even* positive integers $N_1, N_2, \ldots, N_n$, the n-*dimensional Manhattan street network* $M_n = M(N_1, N_2, \ldots, N_n)$ is a digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$. Thus, each of its vertices is represented by an n-vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, with $0 \le u_i \le N_i - 1$, $i = 1, 2, \ldots, n$. The arc set $A(M_n)$ is defined by the following adjacencies (here called i-*arcs*):

$$(u_1, \ldots, u_i, \ldots, u_n) \ \to \ \big(u_1, \ldots, u_i + (-1)^{\sum_{j \ne i} u_j}, \ldots, u_n\big), \qquad i = 1, 2, \ldots, n.$$

Therefore, M_n is an n-regular digraph on $N = \prod_{i=1}^n N_i$ vertices.

The properties of M_n are listed in the following result of the authors:

Theorem 2.1. *[4] Let $M_n = M(N_1, N_2, \ldots, M_n)$ be an n-dimensional Manhattan street network. Then, the following statements hold:*

(a) *There exists an homomorphism from M_n to the symmetric digraph of the hypercube Q_n^*, so that M_n is a 2^n-partite and bipartite digraph.*

(b) *The n-dimensional Manhattan street network M_n is a vertex-symmetric digraph.*

(c) *For any N_1, N_2, the 2-dimensional Manhattan street network $M_2(N_1, N_2)$ is a line digraph.*

(d) *For $N_i > 4$, $i = 1, 2, \ldots, n$, the diameter of an n-dimensional Manhattan street network $M_n = M(N_1, N_2, \ldots, N_n)$ is*

$$D = \begin{cases} \frac{N_1}{2} + \frac{N_2}{2} + 1, & \text{if } N_1 \equiv N_2 \equiv 0 \bmod 4; \\ \frac{N_1}{2} + \frac{N_2}{2}, & \text{otherwise.} \end{cases}$$

(e) *The n-dimensional Manhattan street network M_n is Hamiltonian.*

3. The Manhattan product and its basic properties

In this section, we present an operation on (bipartite) digraphs which, as a particular case, gives rise to the Manhattan street networks. We begin with the more intuitive case of two factors. Let $G_i = (V_i, A_i)$ be two self-converse bipartite digraphs with independent vertex sets V_{i0} and V_{i1}, $i = 1, 2$. Then, the definition of the *Manhattan product $H = G_1 \# G_2$* is as follows [5]: As usual, the vertex set of H is the Cartesian (or direct) product $V_1 \times V_2$. For the set of arcs, one starts with the direct product of G_1 and G_2, usually denoted by $G_1 \square G_2$, and where there is an arc from (u_1, u_2) to (v_1, v_2) if either $u_1 = v_1$ and $u_2 \to v_2$ in G_2, or $u_2 = v_2$ and $u_1 \to v_1$ in G_1
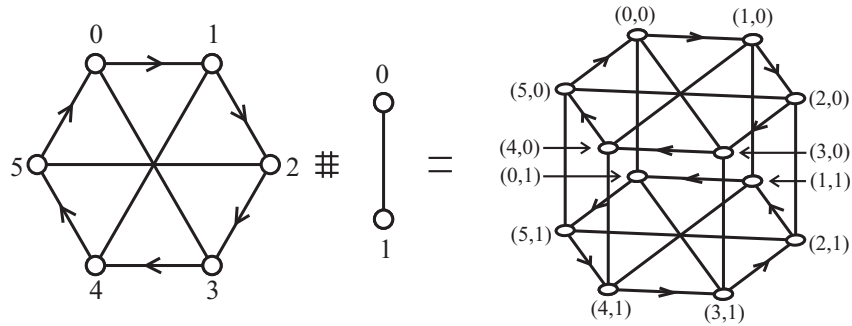
Figure 1. The Manhattan product $\mathrm{Cay}(\mathbb{Z}_6, \{1,3\}) \,\#\, K_2^*$ (undirected lines stand for pairs of arcs with opposite directions).

(see Fig.4 for an example of a direct product). Then, one modifies this by changing some of the arrow directions according to the following rule: Because G_1 and G_2 are bipartite, each vertex of each graph comes with a prescribed type which is either 0 or 1 depending on which independent set, V_{i0} or V_{i1}, it belongs to (since each G_i is self-converse, it does not matter which partite set is V_{i0} or V_{i1}). Then, an edge from (w, u_2) to (w, v_2) preserves its direction if w is of type 0, that is $w \in V_{10}$, and reverses its direction if it is of type 1, $w \in V_{11}$. Similarly, an edge from (u_1, w) to (u_2, w) preserves its direction if $w \in V_{20}$ and reverses it if $w \in V_{21}$. Fig. 1 shows an example of the Manhattan product of the circulant digraph on six vertices and steps 1 and 3 (in other words, the Cayley digraph on $\mathbb{Z}_6$ with generating set $\{1,3\}$) by the symmetric complete digraph on two vertices, $K_2^* = \mathrm{Cay}(\mathbb{Z}_2, \{1\})$. One can, then, define the product of more bipartite digraphs by iterating and checking that the product is both associative and commutative. This lead us to the following more formal definition.

Definition 3.1. *Let $G_i = (V_i, A_i)$ be n bipartite self-converse digraphs with independent sets $V_i = V_{i0} \cup V_{i1}$, $N_i = |V_i|$, $i = 1, 2, \ldots, n$. Let π be the characteristic function of $V_{i1} \subset V_i$ for any i, that is, for $i = 1, 2, \ldots, n$,*

$$\pi(u) := \begin{cases} 0 & \text{if } u \in V_{i0}, \\ 1 & \text{if } u \in V_{i1}. \end{cases}$$

Then, the Manhattan product *$H_n = G_1 \,\#\, G_2 \,\#\, \cdots \,\#\, G_n$ is the digraph with vertex set $V(H_n) = V_1 \times V_2 \times \cdots \times V_n$, and adjacencies*

$$(u_1, \ldots, u_i, \ldots, u_n) \quad \rightarrow \quad (u_1, \ldots, v_i, \ldots, u_n), \qquad i = 1, 2, \ldots, n,$$

where

$$v_i \in \Gamma^{\sigma}(u_i) := \begin{cases} \Gamma_{G_i}^{+}(u_i) & \text{if } \sigma = \sum_{j \neq i} \pi(u_j) \text{ is even}, \\ \Gamma_{G_i}^{-}(u_i) & \text{if } \sigma = \sum_{j \neq i} \pi(u_j) \text{ is odd}. \end{cases}$$

Thus, if every G_i is δ_i-regular for $i = 1, 2, \ldots, n$, then H_n is a δ-regular digraph with degree $\delta = \sum_{i=1}^{n} \delta_i$ and $N = \prod_{i=1}^{n} N_i$ vertices.

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Notice that the same definition applies for non-necessarily self-converse digraphs, but then the product obtained depends on the characteristic function π for each factor graph which, in turn, depends on the choice of 'colors' 0 and 1 for the independent sets of the corresponding factor graph. Then, in this case, one can obtain up to 2^n non-isomorphic products. By the way of example, take a digraph G having three vertices u, v, w and two arcs $u \to v$, $w \to v$, as shown in Fig. 2. When taking the product $G \# G$, the four arcs incident with vertex (v, v) may all be outgoing, $(G \# G)_1$, all incoming, $(G \# G)_2$, or two may be outgoing and two incoming, $(G \# G)_3$, depending on the 0-1 coloring of each of the two factors. In this particular case, the two factors are isomorphic, $G_1 = G_2 = G$, and we only get three non-isomorphic product digraphs and not four. Perhaps it would be interesting if one could find examples of factors that are not all self-converse but their Manhattan product is unique.
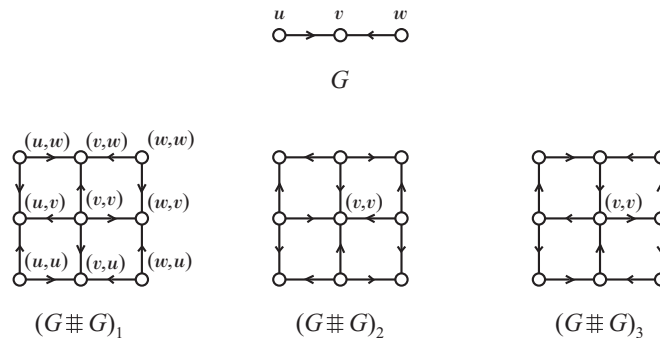

Figure 2. Three non-isomorphic Manhattan products.

In our case of self-converse factors, some of the basic properties of their Manhattan product, which are a generalization of the properties of the Manhattan street networks given in [4, 7], are presented in the next proposition:

Proposition 3.1. *The Manhattan product* $H_n = G_1 \# G_2 \# \cdots \# G_n$ *satisfies the following properties:*

(a) H_n *is a bipartite self-converse digraph.*

(b) *There exists an homomorphism from* H_n *to the symmetric digraph of the hypercube* Q_n^*. *Therefore,* H_n *is a bipartite and* 2^n-*partite digraph.*

(c) *Given* $k = 2, 3 \ldots, n - 1$ *and any* $n - k$ *fixed vertices* $x_i \in V_i$, $i = k + 1, k + 2, \ldots, n$, *the subdigraph of* H *induced by the vertex set*

$$\{(u_1, \ldots, u_k, x_{k+1}, \ldots, x_n) : u_1 \in V_1, \ldots, u_k \in V_k\}$$

is isomorphic to the Manhattan product $H_k = G_1 \# G_2 \# \cdots \# G_k$.

Proof. We only prove the properties (b) and (a) because (c) can be proved similarly.

(b) The homomorphism from H to Q_n^* is

$$(u_1, u_2, \ldots, u_n) \quad \mapsto \quad (\pi(u_1), \pi(u_2), \ldots, \pi(u_n)),$$

which transforms each vertex of H in a binary n-string as its image vertex in Q_n^*.

(a) As the Manhattan product is associative, we only need to deal with the case $H = G_1 \# G_2$. For a given mapping ϕ and a vertex set U, let $\phi(U)$ denote the set $\{\phi(u) : u \in U\}$. Since $G_i \cong \overline{G}_i$ for $i = 1, 2$, there exist isomorphisms ψ_i, such that $\psi_i(\Gamma_{G_i}^{\mp}(u_i)) = \Gamma_{G_i}^{\pm}(\psi_i(u_i))$ for all $u_i \in V_i$. As ψ_i is a mapping between stable sets, the parity π in $\overline{G}_i$ can be defined in such a way that $\pi(u_i)$ is even if and only if $\pi(\psi_i(u_i))$ is also even. Then, the mapping Ψ defined in H as $\Psi(u_1, u_2) := (\psi_1(u_1), \psi_2(u_2))$ is the automorphism from H to its converse $\overline{H}$. Indeed,

$$\begin{aligned}
\Psi\big(\Gamma_H^+(u_1, u_2)\big) &= \Psi\big((\Gamma_{G_1}^{\pi(u_2)}(u_1), u_2)\big) \cup \Psi\big((u_1, \Gamma_{G_2}^{\pi(u_1)}(u_2))\big) \\
&= \big(\psi_1(\Gamma_{G_1}^{\pi(u_2)}(u_1)), \psi_2(u_2)\big) \cup \big(\psi_1(u_1), \psi_2(\Gamma_{G_2}^{\pi(u_1)}(u_2))\big) \\
&= \big(\Gamma_{G_1}^{\pi(u_2)+1}(\psi_1(u_1)), \psi_2(u_2)\big) \cup \big(\psi_1(u_1), \Gamma_{G_2}^{\pi(u_1)+1}(\psi_2(u_2))\big) \\
&= \big(\Gamma_{G_1}^{\pi(\psi_2(u_2))+1}(\psi_1(u_1)), \psi_2(u_2)\big) \cup \big(\psi_1(u_1), \Gamma_{G_2}^{\pi(\psi_1(u_1))+1}(\psi_2(u_2))\big) \\
&= \big(\Gamma_{G_1}^-(\psi_1(u_1)), \psi_2(u_2)\big) \cup \big(\psi_1(u_1), \Gamma_{G_2}^-(\psi_2(u_2))\big) \\
&= \Gamma_H^-\big(\psi_1(u_1), \psi_2(u_2)\big) \\
&= \Gamma_H^-\big(\Psi(u_1, u_2)\big).
\end{aligned}$$

This completes the proof. $\qquad\square$

4. The Manhattan product and the Manhattan street networks

Here we show the relation between the digraphs obtained by the Manhattan product and the Manhattan street networks.

Proposition 4.1. *The Manhattan product of directed cycles with an even order N_i, $i = 1, 2, \ldots, n$, is a Manhattan street network. More precisely, if $G_i = C_{N_i}$, then*

$$C_{N_1} \# C_{N_2} \# \cdots \# C_{N_n} = M(N_1, N_2, \ldots, N_n).$$

Proof. For $i = 1, 2, \ldots, n$, each cycle C_{N_i} has set of vertices $V_i = \mathbb{Z}_{N_i}$, and adjacencies $\Gamma^+(u_i) = \{u_i + 1 \bmod N_i\}$ and $\Gamma^-(u_i) = \{u_i - 1 \bmod N_i\}$, such that V_{i0} and V_{i1} can be the sets of even and odd vertices, respectively. Thus, the set of vertices in the Manhattan product of directed cycles is $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$ and

$$(u_1, \ldots, u_i, \ldots, u_n) \quad \to \quad (u_1, \ldots, v_i, \ldots, u_n), \qquad i = 1, 2, \ldots, n,$$

where
- $v_i = u_i + 1$ if and only if $\sum_{j \neq i} \pi(u_j)$ is even and, hence, $\sum_{j \neq i} u_j$ is also even,
- $v_i = u_i - 1$ if and only if $\sum_{j \neq i} \pi(u_j)$ is odd and, hence, $\sum_{j \neq i} u_j$ is also odd,

which corresponds to the definition of the Manhattan street network. $\qquad\square$

As a consequence of the above proposition, we have the following result.

Corollary 4.1. *The Manhattan product of two Manhattan street networks is a Manhattan street network. More precisely, if $M^1 = M(N_1^1, N_2^1, \ldots, N_{n_1}^1)$ and $M^2 = M(N_1^2, N_2^2, \ldots, N_{n_2}^2)$, then*

$$M^1 \# M^2 = M,$$

where $M = M(N_1^1, \ldots, N_{n_1}^1, N_1^2, \ldots, N_{n_2}^2)$.

Proof. Both M^1 and M^2 are bipartite digraphs with vertex sets $V^\alpha = \mathbb{Z}_{N_1^\alpha} \times \mathbb{Z}_{N_2^\alpha} \times \cdots \times \mathbb{Z}_{N_{n_\alpha}^\alpha}$, $\alpha = 1, 2$; whereas $M^1 \# M^2$ has vertex set $V = V^1 \times V^2$. Let $V(M)$ be the vertex set of M. Then, we claim that the natural mapping $\Psi : V \to V(M)$, defined by $\Psi(\boldsymbol{u}^1, \boldsymbol{u}^2) = (u_1^1, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2)$ is an isomorphism between the corresponding digraphs. In proving this, let V_0^α and V_1^α be the stable sets of M^α constituted, respectively, by the vertices $\boldsymbol{u}^\alpha = (u_1^\alpha, \ldots, u_{n_\alpha}^\alpha)$, whose sum of components $\sum_{k=1}^{n_\alpha} u_k^\alpha$ is even or odd. With this convention, each vertex $(\boldsymbol{u}^1, \boldsymbol{u}^2)$ of the Manhattan product $M^1 \# M^2$ is adjacent to the vertices $(\boldsymbol{v}^1, \boldsymbol{u}^2)$ and $(\boldsymbol{u}^1, \boldsymbol{v}^2)$ where, for the first ones,

- $\boldsymbol{v}^1 \in \Gamma_{M^1}^+(\boldsymbol{u}^1)$ if $\pi(\boldsymbol{u}^2)$, and hence $\sum_{k=1}^{n_2} u_k^2$, is even;
- $\boldsymbol{v}^1 \in \Gamma_{M^1}^-(\boldsymbol{u}^1)$ if $\pi(\boldsymbol{u}^2)$, and hence $\sum_{k=1}^{n_2} u_k^2$, is odd.

In the first case, for $i = 1, 2, \ldots, n_1$,

$$
\begin{aligned}
(\boldsymbol{v}^1, \boldsymbol{u}^2) \quad &\overset{\Psi}{\mapsto} \quad \left(u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j \neq i} u_j^1}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2 \right) \\
&= \quad \left(u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j \neq i} u_j^1 + \sum_{k=1}^{n_2} u_k^2}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2 \right).
\end{aligned}
$$

Analogously, in the second case, for $i = 1, 2, \ldots, n_2$,

$$
\begin{aligned}
(\boldsymbol{v}^1, \boldsymbol{u}^2) \quad &\overset{\Psi}{\mapsto} \quad \left(u_1^1, \ldots, u_i^1 - (-1)^{\sum_{j \neq i} u_j^1}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2 \right) \\
&= \quad \left(u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j \neq i} u_j^1 + \sum_{k=1}^{n_2} u_k^2}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2 \right).
\end{aligned}
$$

Considering altogether both cases, we obtain (through all the i-arcs, $1 \leq i \leq n_1$) the vertices adjacent to $\Psi(\boldsymbol{u}^1, \boldsymbol{u}^2) = (u_1^1, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2)$ in M. The adjacencies through the other i-arcs, $n_1 + 1 \leq i \leq n_1 + n_2$ come from the vertices $(\boldsymbol{u}^1, \boldsymbol{v}^2)$. $\qquad \square$

The result of the previous proposition can also be seen as a corollary of Proposition 4.1 and the associative property. Indeed, we have

$$
\begin{aligned}
M^1 \# M^2 &= M(N_1^1, N_2^1, \ldots, N_{n_1}^1) \# M(N_1^2, N_2^2, \ldots, N_{n_2}^2) \\
&= (C_{N_1}^1 \# C_{N_2}^1 \# \cdots \# C_{N_{n_1}}^1) \# (C_{N_1}^2 \# C_{N_2}^2 \# \cdots \# C_{N_{n_2}}^2) \\
&= C_{N_1}^1 \# C_{N_2}^1 \# \cdots \# C_{N_{n_1}}^1 \# C_{N_1}^2 \# C_{N_2}^2 \# \cdots \# C_{N_{n_2}}^2 \\
&= M(N_1^1, N_2^1, \ldots, N_{n_1}^1, N_1^2, N_2^2, \ldots, N_{n_2}^2) = M.
\end{aligned}
$$

5. The diameter

In this section, we derive a precise bound of the Manhattan product of digraphs in terms of the unilateral diameters of its factors. In this context and by the way of comparison, notice that, according to the result in Theorem 2.1(d), the diameter D of the Manhattan street network $M_2 = M(N_1, N_2)$ always satisfies the upper bound

$$D \leq D_1^* + D_2^* + 1,$$

where D_1^* and D_2^* are, respectively, the unilateral diameters of cycles C_{N_1} and C_{N_2}. Now, we show that a very similar bound applies for the general case.

Theorem 5.1. *Let G_1 and G_2 be digraphs with neither sources nor sinks (vertices with indegree or outdegree zero) and (finite) unilateral diameters D_1^* and D_2^*, respectively. Then, the Manhattan product $H = G_1 \# G_2$ has diameter D satisfying*

$$D \leq D_1^* + D_2^* + 2.$$

Proof. Let $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ be two generic vertices of H. We want to find a path from $\boldsymbol{u}$ to $\boldsymbol{v}$ with length not exceeding the claimed bound. With this aim, each vertex is denoted by $(\partial_1^{\pm}, \partial_2^{\pm})$, where $\partial_i = \text{dist}_{G_i}^*(u_i, v_i)$, $i = 1, 2$, are referred to as their *distance-components* (with respect to $\boldsymbol{v}$ and, hence, such a destiny vertex is just $(0,0)$). The superscript $+$ or $-$ of ∂_1 indicates whether its parity is either the right one or not to follow the shortest path from u_2 to v_2 in G_2, and similarly for the superscript of ∂_2. Since the distances ∂_1 and ∂_2 can be either even or odd, and their parity can either agree $(+)$ or not $(-)$ with the shortest path of their mates, there are sixteen possible cases to be investigated which, by symmetry, are reduced to ten as follows:

(a1) $(2x^+, 2y^+)$;

(a2) $(2x^+, 2y^-) \equiv (2x^-, 2y^+)$;

(a3) $(2x^-, 2y^-)$.

(b1) $(2x^+, 2y+1^+) \equiv (2x+1^+, 2y^+)$;

(b2) $(2x^+, 2y+1^-) \equiv (2x+1^-, 2y^+)$;

(b3) $(2x^-, 2y+1^+) \equiv (2x+1^+, 2y^-)$;

(b4) $(2x^-, 2y+1^-) \equiv (2x+1^-, 2y^-)$.

(c1) $(2x+1^+, 2y+1^+)$;

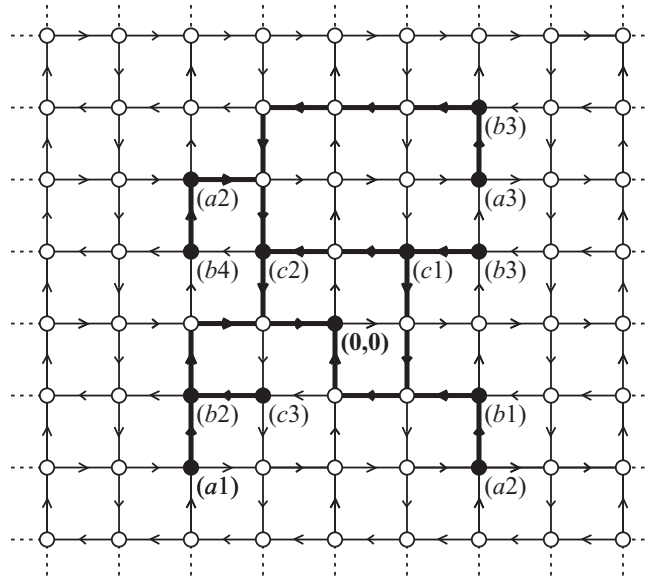(c2) $(2x+1^+, 2y+1^-) \equiv (2x+1^-, 2y+1^+)$;

Figure 3. The different paths in the proof of Theorem 5.1

(c3) $(2x + 1^-, 2y + 1^-)$.

Then, a path from (u_1, u_2) to (v_1, v_2) corresponds to a path from $(\partial_1^\pm, \partial_2^\pm)$ to $(0, 0)$ as follows (see Fig. 3 when considering the Manhattan product of two cycles, that is, a Manhattan street network M_2):

(a1) $(2x^+, 2y^+) \xrightarrow{2y} (2x, 0^+) \xrightarrow{2x} (0, 0)$;

(a2) $(2x^+, 2y^-) \xrightarrow{2y-1} (2x^+, 1^+) \xrightarrow{2x} (0^+, 1) \xrightarrow{1} (0, 0)$;

(a3) $(2x^-, 2y^-) \xrightarrow{1} (b1) (2x^+, 2z + 1^+) \xrightarrow{2x} \cdots$;
$ \xrightarrow{1} (b3) (2x^-, 2z + 1^+) \xrightarrow{2x+1} \cdots$;

(b1) $(2x^+, 2y + 1^+) \xrightarrow{2x} (0^+, 2y + 1) \xrightarrow{2y+1} (0, 0)$;

(b2) $(2x + 1^-, 2y^+) \xrightarrow{2x+1} (0^+, 2y) \xrightarrow{2y} (0, 0)$;

(b3) $(2x^-, 2y + 1^+) \xrightarrow{2x+1} (1^+, 2y + 1^-) \xrightarrow{2y+1} (1, 0^+) \xrightarrow{1} (0, 0)$;

(b4) $(2x^-, 2y + 1^-) \xrightarrow{1} (a1) (2x^+, 2z^+) \xrightarrow{2z} \cdots$;
$ \xrightarrow{1} (a2) (2x^-, 2z^+) \xrightarrow{2x-1} \cdots$;

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$(c1)$ $(2x + 1^+, 2y + 1^+) \xrightarrow{2y+2} (2x + 1^-, 1^+) \xrightarrow{2x+1} (0^+, 1) \xrightarrow{1} (0, 0);$

$(c2)$ $(2x + 1^+, 2y + 1^-) \xrightarrow{2y+1} (2x + 1, 0^+) \xrightarrow{2x+1} (0, 0);$

$(c3)$ $(2x + 1^-, 2y + 1^-) \xrightarrow{1} (b1) (2x + 1^+, 2z^+) \xrightarrow{2z} \cdots;$
$ \xrightarrow{1} (b2) (2x + 1^-, 2z^+) \xrightarrow{2x+1} \cdots.$

Note that, sometimes, we need to do some steps following a given (forced) orientation, so that the condition about having no vertices with null outdegree or indegree is necessary (and, clearly, also sufficient) for reaching the vertex desired. Then, the result follows by considering the maximum length of each of these paths taking into account that, if the diameter D_i^* is even (respectively, odd) the unilateral distance between two vertices u_i and v_i of G_i, $i = 1, 2$, belonging to distinct (respectively, equal) independent sets is at most $D_i^* - 1$. For instance, let us check in detail one particular case of $(a3)$ where the upper bound can be reached, the other cases being analyzed similarly. With this aim, let us assume that D_1^* and D_2^* are even and consider the path

$$(a3) \ (2x^-, 2y^-) \xrightarrow{1} (b3) \ (2x^-, 2z + 1^+) \xrightarrow{2x+1} (1^+, 2z + 1^-) \xrightarrow{2z+1} (1, 0^+) \xrightarrow{1} (0, 0).$$

First, the origin vertex $\boldsymbol{u} = (u_1, u_2)$ has distance-components $2x^- \leq D_1^*$ and $2y^- \leq D_2^*$. After 1 step, the vertex reached has second distance-component $2z + 1 \leq D_2^*$ (because of the parity of D_2^*). Then, we take $2x + 1 \leq D_1^* + 1$ steps to reach a vertex with first component $v_1' \in \Gamma_{G_1}^+(v_1)$ (and, hence, distance-component 1). Afterwards, $2z + 1 \leq D_2^*$ steps are done to reach the vertex with second components v_2, as required. Finally we take 1 extra step to reach the destiny vertex $\boldsymbol{v} = (v_1, v_2)$. Consequently, a total of $2x + 2z + 2 \leq D_1^* + D_2^* + 2$ steps are required. $\qquad \square$

Notice that the above result implicitly gives a necessary and sufficient condition for the Manhattan product to be strongly connected (that is, with finite diameter). Moreover, as its proof is constructive, it provides a routing algorithm in H from some given routing algorithms of its factors.

6. Symmetries

Now we study the symmetries of the digraphs obtained by the Manhattan product.

Proposition 6.1. *Let G_i be vertex-symmetric self-converse bipartite digraphs, $i = 1, 2, \ldots, n$. Then, the Manhattan product $H = G_1 \# G_2 \# \cdots \# G_n$ is a vertex-symmetric self-converse digraph.*

Proof. As before, we only need to study the case $H = G_1 \# G_2$, where $G_i = (V_i, A_i)$, $V_i = V_{i0} \cup V_{i1}$, $i = 1, 2$, are digraphs satisfying the hypotheses. Also, by Proposition 3.1(a), we already know that H is isomorphic to its converse. Now, let us prove that there exists an automorphism Φ in H, which transforms any vertex (u_1, u_2) into any vertex (v_1, v_2). Depending on whether u_i and v_i, $i = 1, 2$, are in the same or distinct partite sets $V_{i\alpha}$, $\alpha = 0, 1$, there are different cases to be considered:

(i) If both pairs of components are in the same stable sets, $u_1, v_1 \in V_{i\alpha}$ and $u_2, v_2 \in V_{i\beta}$, we know that there exist automorphisms ϕ_i in G_i, $\phi_i(\Gamma_{G_i}^{\pm}(w_i)) = \Gamma_{G_i}^{\pm}(\phi_i(w_i))$ for every $w_i \in V_i$, such that $\phi_i(u_i) = v_i$, $i = 1, 2$. Then, we define

$$\Phi(w_1, w_2) := (\phi_1(w_1), \phi_2(w_2)) \quad \text{for every } (w_1, w_2) \in V(H),$$

satisfying $\Phi(u_1, u_2) = (v_1, v_2)$ and, since $\pi(w_i) = \pi(\phi_i(w_i))$ (ϕ_i leaves invariant each partite set), we have:

$$
\begin{aligned}
\Phi\big(\Gamma_H^+(w_1, w_2)\big) &= \Phi\big((\Gamma_{G_1}^{\pi(w_2)}(w_1), w_2)\big) \cup \Phi\big((w_1, \Gamma_{G_2}^{\pi(w_1)}(w_2))\big) \\
&= \big(\phi_1(\Gamma_{G_1}^{\pi(w_2)}(w_1)), \phi_2(w_2)\big) \cup \big(\phi_1(w_1), \phi_2(\Gamma_{G_2}^{\pi(w_1)}(w_2))\big) \\
&= \big(\Gamma_{G_1}^{\pi(w_2)}(\phi_1(w_1)), \phi_2(w_2)\big) \cup \big(\phi_1(w_1), \Gamma_{G_2}^{\pi(w_1)}(\phi_2(w_2))\big) \\
&= \big(\Gamma_{G_1}^{\pi(\phi_2(w_2))}(\phi_1(w_1)), \phi_2(w_2)\big) \cup \big(\phi_1(w_1), \Gamma_{G_2}^{\pi(\phi_1(w_1))}(\phi_2(w_2))\big) \\
&= \Gamma_H^+\big(\phi_1(w_1), \phi_2(w_2)\big) \\
&= \Gamma_H^+\big(\Phi(w_1, w_2)\big),
\end{aligned}
$$

so that Φ is an automorphism of H.

(ii) If the pairs of components are in different stable sets, $u_1 \in V_{i\alpha}$, $v_1 \in V_{i\overline{\alpha}}$ and $u_2 \in V_{i\beta}$, $v_2 \in V_{i\overline{\beta}}$, and since G_i, $i = 1, 2$, are self-converse, there exist automorphisms ψ_i from G_i to $\overline{G_i}$, $\psi_i(\Gamma_{G_i}^{\pm}(w_i)) = \Gamma_{G_i}^{\mp}(\psi_i(w_i))$ for every $w_i \in V_i$, such that $\psi_i(u_u) = v_i$. We, then, define Φ as

$$\Psi(w_1, w_2) := (\psi_1(w_1), \psi_n(w_2)) \quad \text{for every } (w_1, w_2) \in V(H),$$

satisfying $\Phi(u_1, u_2) = (v_1, v_2)$ and, since $\pi(w_i) = 0 \iff \pi(\psi_i(w_i)) = 1$ (ψ_i interchanges the partite sets), we get

$$
\begin{aligned}
\Phi\big(\Gamma_H^+(w_1, w_2)\big) &= \Phi\big((\Gamma_{G_1}^{\pi(w_2)}(w_1), w_2)\big) \cup \Phi\big((w_1, \Gamma_{G_2}^{\pi(w_1)}(w_2))\big) \\
&= \big(\psi_1(\Gamma_{G_1}^{\pi(w_2)}(w_1)), \psi_2(w_2)\big) \cup \big(\psi_1(w_1), \psi_2(\Gamma_{G_2}^{\pi(w_1)}(w_2))\big) \\
&= \big(\Gamma_{G_1}^{\pi(w_2)+1}(\psi_1(w_1)), \psi_2(w_2)\big) \cup \big(\psi_1(w_1), \Gamma_{G_2}^{\pi(w_1)+1}(\psi_2(w_2))\big) \\
&= \big(\Gamma_{G_1}^{\pi(\psi_2(w_2))}(\psi_1(w_1)), \psi_2(w_2)\big) \cup \big(\psi_1(w_1), \Gamma_{G_2}^{\pi(\phi_1(w_1))}(\psi_2(w_2))\big) \\
&= \Gamma_H^+\big(\psi_1(w_1), \psi_2(w_2)\big) \\
&= \Gamma_H^+\big(\Phi(w_1, w_2)\big),
\end{aligned}
$$

which proves again that Φ is an automorphism of H.

(iii) Finally, in the case when one pair of components are in the same stable set and the other components are in distinct stable sets, say, $u_1, v_1 \in V_{i\alpha}$ and $u_2 \in V_{i\beta}$, $v_2 \in V_{i\overline{\beta}}$, we can apply the same ideas of (i) and (ii). Namely, with the same notation as before, we prove that the mapping Φ, defined as

$$\Psi(w_1, w_2) := (\phi_1(w_1), \psi_n(w_2)) \quad \text{for every } (w_1, w_2) \in V(H),$$

is the required automorphism of H. This completes the proof. $\qquad\square$

7. Cayley digraphs and the Manhattan product

In this section, we investigate when the Manhattan product of Cayley digraphs is also a Cayley digraph. This generalizes the case studied in [4, 7] of Manhattan street networks, where the factors of the product are directed cycles (see Proposition 4.1), that is, Cayley digraphs of the cyclic groups. Because of the associative property of such a product, we only need to study the case of two factors.

Theorem 7.1. *Let $G_1 = \mathrm{Cay}(\Gamma_1, \Delta_1)$ be a bipartite Cayley digraph of the group Γ_1 with generating set $\Delta_1 = \{a_1, \ldots, a_p\}$ and set of generating relations R_1, such that there exists a group automorphism ψ_1 satisfying $\psi_1(a_i) = a_i^{-1}$, for $i = 1, \ldots, p$. Let $G_2 = \mathrm{Cay}(\Gamma_2, \Delta_2)$ be a bipartite Cayley digraph of the group Γ_2 with generating set $\Delta_2 = \{b_1, \ldots, b_q\}$ and set of generating relations R_2, such that there exists a group automorphism ψ_2 satisfying $\psi_2(b_j) = b_j^{-1}$, for $j = 1, \ldots, q$. Then, the Manhattan product $H = G_1 \,\#\, G_2$ is the Cayley digraph of the group*

$$\Gamma = \langle \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \,|\, R'_1, R'_2, (\alpha_i \beta_j)^2 = (\alpha_i \beta_j^{-1})^2 = 1, i \neq j \rangle, \tag{1}$$

where R'_1 is the same set of generating relations as R_1 changing a_i by α_i (and similarly for R'_2).

Proof. Since, for every $u_1 \in \Gamma_1$ and $i = 1, \ldots, p$,

$$\psi_1(u_1 a_i) = \psi(u_1)\psi(a_i) = \psi(u_1)a_i^{-1},$$

then ψ_1 is an (involutive) isomorphism for G_1 to $\overline{G}_1$ preserving colors. The same statement holds for ψ_2 and G_2. Moreover, since G_1, G_2 are vertex-symmetric, Proposition 6.1 applies and H is also vertex-symmetric. In fact, we will see that its automorphism group contains a regular subgroup. With this aim, note first that, by using the previous automorphisms, we have the following natural way of defining the adjacencies of H (with 'colors' denoted by α_i, $1 \leq i \leq p$, and β_j, $1 \leq j \leq q$):

$$(u_1, u_2) \xrightarrow{\alpha_i\text{-arc}} (u_1, u_2) * \alpha_i = \left(u_1 \psi_1^{\pi(u_2)}(a_i), u_2\right),$$

$$(u_1, u_2) \xrightarrow{\beta_j\text{-arc}} (u_1, u_2) * \beta_j = \left(u_1, u_2 \psi_2^{\pi(u_1)}(b_j)\right).$$

Let us now prove that the mappings ϕ_{1i}, ϕ_{2j}, for $1 \leq i \leq p$ and $1 \leq j \leq q$, defined by $\phi_{1i}(u_1, u_2) = (a_i u_1, \psi_2(u_2))$ and $\phi_{2j}(u_1, u_2) = (\psi_1(u_1), b_j u_2)$ are all color-preserving isomorphisms of H. Indeed, for all $1 \leq i, j \leq p$, we have

$$\phi_{1i}\left((u_1, u_2) * \alpha_j\right) = \phi_{1i}\left(u_1 \psi_1^{\pi(u_2)}(a_j), u_2\right) = \left(a_i u_1 \psi_1^{\pi(u_2)}(a_j), \psi_2(u_2)\right)$$
$$= \left(a_i u_1 \psi_1^{\pi(\psi_2(u_2))}(a_j), \psi_2(u_2)\right) = \left(a_i u_1, \psi_2(u_2)\right) * \alpha_j = \phi_{1i}(u_1, u_2) * \alpha_j,$$

where we used that $\pi(u_2) = \pi(\psi_2(u_2))$, because u_2 can be expressed as the product of the generators b_j and $\pi(b_j) = \pi(\psi_2(b_j)) = \pi(b_j^{-1})$, for all $1 \leq j \leq q$. Moreover, for all $1 \leq i \leq p$ and $1 \leq j \leq q$, we also have

$$\phi_{1i}\left((u_1, u_2) * \beta_j\right) = \phi_{1i}\left(u_1, u_2 \psi_2^{\pi(u_1)}(b_j)\right) = \left(a_i u_1, \psi_2(u_2)\psi_2^{\pi(u_1)+1}(b_j)\right)$$
$$= \left(a_i u_1, \psi_2(u_2)\psi_2^{\pi(a_i u_1)}(b_j)\right) = \left(a_i u_1, \psi_2(u_2)\right) * \beta_j = \phi_{1i}(u_1, u_2) * \beta_j,$$

where we used that u_1 and $a_i u_1$ belong to different stable sets of G_1. Similarly, we obtain

$$\phi_{2i}\big((u_1, u_2) * \alpha_j\big) = \phi_{2i}(u_1, u_2) * \alpha_j, \quad 1 \le i \le q,\ 1 \le j \le p,$$
$$\phi_{2i}\big((u_1, u_2) * \beta_j\big) = \phi_{2i}(u_1, u_2) * \beta_j, \quad 1 \le i, j \le q.$$

Now to see that the permutation group $\Gamma = \langle \phi_{1i}, \phi_{2j} \,|\, 1 \le i \le p, 1 \le j \le q \rangle$ acts transitively on $\Gamma_1 \times \Gamma_2$, that is, on the vertex set of H, it is enough to show that any vertex (u_1, u_2) can be mapped into vertex (e_1, e_2) (where e_1 and e_2 stand for the identity elements of Γ_1 and Γ_2, respectively) since, as stated previously, H is vertex-symmetric. To this end, as Δ_1 is a generating set, u_1^{-1} can be expressed in the form, say, $u_1^{-1} = a_{i_1} a_{i_2} \cdots a_{i_r}$. Then,

$$\phi_{1i_1} \phi_{1i_2} \cdots \phi_{1i_r}(u_1, u_2) = \big(a_{i_1} a_{i_2} \cdots a_{i_r} u_1, \psi_2^r(u_2)\big) = \big(e_1, \psi_2^r(u_2)\big) = (e_1, v_2),$$

where $v_2 = u_2^{(-1)^r}$ is either u_2 or u_2^{-1} according to the parity of r. In any case, as Δ_2 is also a generating set, the inverse of this element can be written as, say, $v_2^{-1} = b_{j_1} b_{j_2} \cdots b_{j_s}$. Then,

$$\phi_{2j_1} \phi_{2j_2} \cdots \phi_{2j_s}(e_1, v_2) = \big(\psi_1^s(e_1), e_2\big) = (e_1, e_2).$$

Thus, as claimed, the group Γ is a regular subgroup of the automorphism group of H. According to Sabidussi's characterization [14], which states that a graph is a Cayley graph of a certain group if and only if its group of automorphisms admits a regular subgroup, the Manhattan product is the Cayley digraph of Γ with generators $\alpha_i \equiv \phi_{1i}$ and $\beta_j \equiv \phi_{2j}$, $1 \le i \le p$, $1 \le j \le q$. Regarding the structure of Γ, let us check only one of the defining relations in (1), as the others can be proved similarly.

$$(\phi_{1i}\phi_{2j})^2(u_1, u_2) = \phi_{1i}\phi_{2j}\phi_{i1}\phi_{2j}(u_1, u_2) = \phi_{1i}\phi_{2j}\phi_{1i}\big(\psi_1(u_1), b_j u_2\big)$$
$$= \phi_{1i}\phi_{2j}\big(a_i\psi_1(u_1), b_j^{-1}\psi_2(u_2)\big) = \phi_{1i}\big(a_i^{-1}\psi_1^2(u_1), \psi_2(u_2)\big)$$
$$= \big(\psi_1^2(u_1), \psi_2^2(u_2)\big) = (u_1, u_2).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

This result can be compared with the well-known following one (see White [16]): If G_1 and G_2 are, respectively, Cayley digraphs of groups $\Gamma_1 = \langle a_1, \dots, a_p \,|\, R_1 \rangle$ and $\Gamma_2 = \langle b_1, \dots, b_q \,|\, R_2 \rangle$, then its direct product $G_1 \square G_2$ is the Cayley digraph of group

$$\Gamma = \Gamma_1 \times \Gamma_2$$
$$= \langle \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \,|\, R_1', R_2', \alpha_i \beta_j = \beta_j \alpha_i,\ 1 \le i \le p,\ 1 \le j \le q \rangle,$$

with the same notation as before. As an example of direct product of Cayley digraphs, see Fig. 4, to be compared with the Manhattan product of the same digraphs shown in Fig. 1.

8. An alternative definition

The results of the preceding section, specifically the structure of the color-preserving automorphisms, suggest to study some alternative definitions of the Manhattan product of digraphs when they satisfy some conditions. More precisely, if each of the factors G_i, $i = 1, 2, \dots, n$, of the Manhattan product has an involutive automorphism from G_i to $\overline{G}_i$, we have the following result.
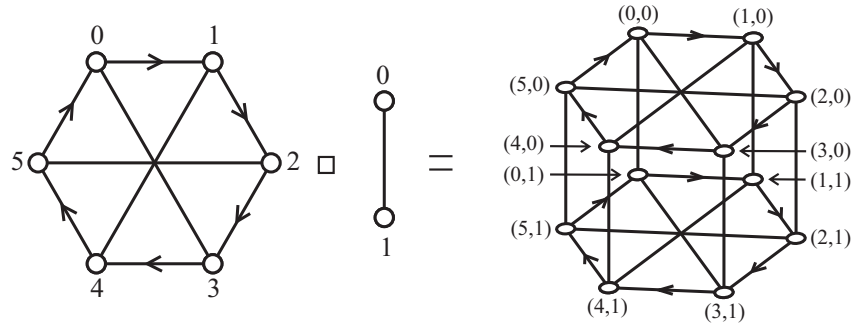
Figure 4. The direct product $\mathrm{Cay}\,(\mathbb{Z}_6, \{1,3\})\,\Box\, K_2^*$ (undirected lines stand for pairs of arcs with opposite directions).

Proposition 8.1. *Let ψ_i be an involutive automorphism from G_i to $\overline{G}_i$, for $i = 1, 2, \ldots, n$. Then, the Manhattan product $H = G_1 \,\#\, G_2 \,\#\, \cdots \,\#\, G_n$ is the digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$ and the following adjacencies:*

$$(u_1, u_2, \ldots, u_i, \ldots, u_n) \rightsquigarrow (\psi_1(u_1), \psi_2(u_2), \ldots, v_i, \ldots, \psi_n(u_n)), \ i = 1, 2, \ldots, n,$$

where $v_i \in \Gamma^+(u_i)$.

Proof. We write, respectively, the adjacencies of the first definition and the alternative one as

$$(u_1, \ldots, u_i, \ldots, u_n) \rightarrow \left(u_1, \ldots, \Gamma^{\sum_{j \neq i} \pi(u_j)}(u_i), \ldots, u_n\right), \ i = 1, 2, \ldots, n, \tag{2}$$

$$(u_1, \ldots, u_i, \ldots, u_n) \rightsquigarrow \left(\psi_1(u_1), \ldots, \Gamma^+(u_i), \ldots, \psi_n(u_n)\right), \ i = 1, 2, \ldots, n. \tag{3}$$

The isomorphism from the digraph obtained by the first definition to the digraph obtained by the alternative one is

$$\Phi(u_1, \ldots, u_i, \ldots, u_n) = \\ \left(\psi_1^{\sum_{j \neq 1} \pi(u_j)}(u_1), \ldots, \psi_i^{\sum_{j \neq i} \pi(u_j)}(u_i), \ldots, \psi_n^{\sum_{j \neq n} \pi(u_j)}(u_n)\right).$$

Indeed, let us see that this mapping preserves the adjacencies. First, by (2), we have

$$\Phi\left(\Gamma^+(u_1, \ldots, u_i, \ldots, u_n)\right) = \tag{4} \\ \left(\psi_1^{\sum_{j \neq 1} \pi(u_j)+1}(u_1), \ldots, \psi_i^{\sum_{j \neq i} \pi(u_j)}\left(\Gamma^{\sum_{j \neq i} \pi(u_j)}(u_i)\right), \ldots, \psi_n^{\sum_{j \neq n} \pi(u_j)+1}(u_n)\right).$$

Whereas, by (3), we have

$$\Gamma^+\left(\Phi(u_1, \ldots, u_i, \ldots, u_n)\right) = \tag{5} \\ \left(\psi_1^{\sum_{j \neq 1} \pi(u_j)+1}(u_1), \ldots, \Gamma^+\left(\psi_i^{\sum_{j \neq i} \pi(u_j)}(u_i)\right), \ldots, \psi_n^{\sum_{j \neq n} \pi(u_j)+1}(u_n)\right).$$

To check that, for every $i = 1, 2, \ldots, n$, the i-th entry in (4) and (5) represents the same set, we distinguish two cases:

- If $\sum_{j \neq i} \pi(u_j) = \sigma$ is an even number, then $\psi_i^\sigma = Id$ (as ψ_i is involutive) and $Id\big(\Gamma^+(u_i)\big) = \Gamma^+\big(Id(u_i)\big)$.

- If $\sum_{j \neq i} \pi(u_j) = \sigma$ is an odd number, then $\psi_i^\sigma = \psi_i$ and $\psi_i\big(\Gamma^-(u_i)\big) = \Gamma^+\big(\psi_i(u_i)\big)$ (as ψ_i is an automorphism from G_i to $\overline{G}_i$). $\hspace{2cm}\square$

In the case of the Manhattan street network M_n, $G_i = C_i$ (Proposition 4.1), $i = 1, 2, \ldots, n$. Then, a simple way of choosing the involutive automorphisms is $\psi_i(u_i) = -u_i \bmod N_i$ (in fact, it is readily checked that any isomorphism from C_i to $\overline{C}_i$ is involutive). This gives the following definition of M_n [4, 7]: The Manhattan street network $M_n = M_n(M_1, \ldots, M_n)$ is the digraph with vertex set $\mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_n}$ and adjacencies

$$(u_1, \ldots, u_i, \ldots, u_n) \quad \rightsquigarrow \quad (-u_1, \ldots, u_i + 1, \ldots, -u_n), \qquad i = 1, 2, \ldots, n.$$

9. Hamiltonian Cycles

Next, we give a result on the Hamiltonicity of the Manhattan product of two digraphs with Hamiltonian paths, as a generalization of a theorem in [4, 7] about the Hamiltonicity of the Manhattan street network.

Theorem 9.1. *If G_1 and G_2 have both a Hamiltonian path, then their Manhattan product $H = G_1 \# G_2$ is Hamiltonian.*

Proof. We construct a Hamiltonian cycle in H, from the Hamiltonian paths in G_1 and G_2, say $1 \to 2 \to \cdots \to N_1$ and $1' \to 2' \to \cdots \to N_2$, respectively. With this aim, we appropriately joint N_2 Hamiltonian paths (some of them without an arc) of N_2 subdigraphs isomorphic to G_1 or $\overline{G}_1$ (see Proposition 3.1(c)). Such paths are joined by using three copies of Hamiltonian paths (two of them with alternative arcs removed) of subdigraphs isomorphic to G_2 or $\overline{G}_2$. See the self-explanatory Fig. 5. $\hspace{1cm}\square$
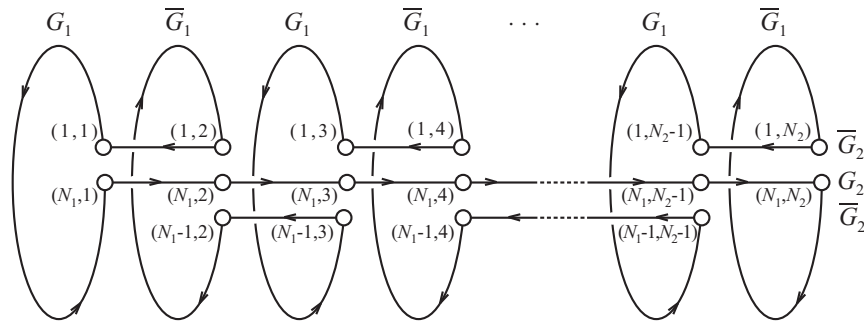
Figure 5. A Hamiltonian cycle in the Manhattan product $G_1 \# G_2$.

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