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Computation of new diagonal graph Ramsey numbers

Richard M. Low[a], Ardak Kapbasov[a], Arman Kapbasov[b], Sergey Bereg[c]

[a]*Department of Mathematics and Statistics, San Jose State University, San Jose, CA 95192, USA*
[b]*Facebook, 1 Hacker Way, Menlo Park, CA 94025, USA*
[c]*Department of Computer Science, University of Texas at Dallas, Richardson, TX 75083, USA*

richard.low@sjsu.edu, ardak.kapbasov@yahoo.com, akapbasov@fb.com, sbereg@gmail.com

Abstract

For various connected simple graphs G, we extend the table of diagonal graph Ramsey numbers $R(G, G)$ in '*An Atlas of Graphs*.' This is accomplished by first converting the calculation of $R(G, G)$ into a satisfiability problem in propositional logic. Mathematical arguments and scientific computing are then used to calculate $R(G, G)$.

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1. Introduction

In 1929, Frank Ramsey [20] established an innocuous-looking theorem in his groundbreaking paper on formal logic. Although it was not apparent at the time, Ramsey's theorem would eventually form the cornerstone of Ramsey theory, a vibrant and rich area of extremal combinatorics.

The following general question [12] is investigated in Ramsey theory.

- If a particular mathematical structure (e.g., algebraic, combinatorial, or geometric) is arbitrarily partitioned into finite many classes, what kinds of substructures must always remain intact in at least one of the classes?

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Over many decades, Ramsey-type questions in mathematical structures such as the integers [16], graphs, and Euclidean space have been investigated. As of this writing, a keyword search for "Ramsey" yields 2926 entries in the MathSciNet database. The interested reader is directed to [12, 13] for a comprehensive overview of Ramsey theory. For a gentle introduction to Ramsey theory, [22] is recommended.

Applications of Ramsey theory can be found in number theory, algebra, geometry, topology, set theory, logic, ergodic theory, information theory and computer science. The reader is directed to Rosta's [23] survey for a detailed exposition of some of these applications.

The reader should note that the seeds of Ramsey theory were planted even before Ramsey introduced his theorem. Soifer's [25] beautifully written book is filled with deep mathematics and also provides a rich historical context of Ramsey theory.

2. Preliminaries

The focus of this paper is on calculating new diagonal Ramsey numbers in graph Ramsey theory.

First, we recall some standard definitions and notation from graph theory. All graphs are finite, simple and connected. Any notation and terminology which are not explicitly defined in this paper can be found in [12, 27]. For a graph G with vertex set $V(G)$ and edge set $E(G)$, the *order* and *size* of G are defined to be $|V(G)|$ and $|E(G)|$, respectively. The *complete graph* K_n is the simple graph on n vertices, where every pair of vertices are adjacent.

In graph Ramsey theory, the following definitions and notation are used.

Definition 1. *Let $k \geq 2$. A r-**coloring** of G is a coloring of $E(G)$, using a maximum of r colors.*

Notation 1. *Let G and H be simple connected graphs. If every 2-coloring of K_n yields a monochromatic subgraph G or a monochromatic subgraph H in K_n, then this is denoted by $K_n \rightarrow (G, H)$. If that is not the case, then the notation $K_n \nrightarrow (G, H)$ is used.*

Definition 2. *The Ramsey number $R(G, H)$ is defined to be the minimum n, where $K_n \rightarrow (G, H)$.*

Using graph-theoretic language, the finite version of Ramsey's theorem can be stated in the following way.

Theorem A. (*Ramsey* [*20*]). *Let $s, t \geq 2$. Then, there exists a minimal positive integer n such that every edge coloring of K_n (using two colors) contains a monochromatic K_s or a monochromatic K_t.*

Considerable work has been done in graph Ramsey theory. In addition to the calculation of Ramsey numbers in the classical theory, many different concepts have been introduced over time. They include Ramsey functions on graphs, many kinds of mixed Ramsey numbers, size Ramsey numbers, connected Ramsey numbers, anti-Ramsey numbers and Gallai-Ramsey numbers. These topics (and many others) can be found within the extensive mathematical literature. For an overview of classical graph Ramsey theory, the general surveys of Burr [1, 2], Radziszowski [19], Read and Wilson [21], and Sudakov [26] are invaluable. New directions and additional open questions in graph Ramsey theory are addressed in [4, 29, 30].

3. Calculating $R(G, H)$ using propositional logic

For a mathematical introduction to logic, the reader is directed to [11]. We begin by recalling two concepts from propositional logic. A *conjunctive normal form* (CNF) is a Boolean expression consisting of a conjunction of disjunctions of propositional statements or their negations. An example of a CNF would be

$$(A \vee \neg B \vee \neg C) \wedge (\neg A \vee \neg B) \wedge (\neg A \vee \neg B \vee C),$$

where A, B and C represent propositional statements and $\vee, \wedge$ and $\neg$ represent "OR," "AND" and "NOT," respectively. A Boolean expression is *satisfiable* if there is an assignment of "true" and "false" values to its propositional statements which makes the expression "true" (when evaluated using the standard truth table rules).

Cowen [9] used Mathematica's Boolean computational abilities to investigate some questions from graph Ramsey theory. In particular, he converted the problem of calculating $R(K_s, K_t)$ into a satisfiabilty problem. Here is an overview of his approach.

Cowen's approach: Let s, t, and n be integers with $2 < s, t < n$, and K_n be the complete graph with vertices numbered $1, 2, \ldots, n$. For each pair of integers (i, j), where $i, j \in V(K_n)$ with $1 \leq i < j \leq n$, construct a CNF f (called "RamseyTest"), using variables $r_{i,j}$ and $b_{i,j}$ which denote the edge $e_{i,j} \in E(K_n)$ being colored red or blue, respectively. The CNF f consists of two sets of clauses:

- (Coloring clauses). For each pair of integers (i, j) with $1 \leq i < j \leq n$, the clauses $r_{i,j} \vee b_{i,j}$ and $\neg r_{i,j} \vee \neg b_{i,j}$ provide a valid 2-coloring of K_n.
- (Non-monochromatic subgraph clauses). These clauses make sure that a 2-coloring of K_n does not contain a monochromatic K_s or K_t.

If f is not satisfied, then $K_n \rightarrow (K_s, K_t)$. In this case, $R(K_s, K_t) \leq n$. If f is satisfied, then $K_n \nrightarrow (K_s, K_t)$. In this case, $n + 1 \leq R(K_s, K_t)$.

Example 1. Suppose we wish to determine if $R(K_3, K_3) \leq 4$. Using the Wolfram Language in Mathematica [28], the (combined) coloring and non-monochromatic subgraph clauses are, respectively:

Here, | | denotes "OR," && denotes "AND," and ! denotes "NOT." The two sets of clauses are connected by && in f. $\Diamond$

Cowen then improved the efficiency of his "RamseyTest" CNF by noting the following fact: In any 2-coloring of K_n, at least half of the edges from a particular vertex are of the same color (say, red). Thus, he created a new CNF f_0 called "QuickerRamseyTest" which utilized this observation. This modest optimization decreased the runtime of "RamseyTest" significantly. For example, f_0 determined that $R(K_4, K_4) = 18$ in approximately 4.5 seconds. In stark contrast to this, "RamseyTest" could not calculate $R(K_4, K_4)$ in a reasonable amount of time and had to be manually

```
Out[5]= (red[{1, 2}] || blue[{1, 2}]) &&
         (! red[{1, 2}] || ! blue[{1, 2}]) &&
         (red[{1, 3}] || blue[{1, 3}]) &&
         (! red[{1, 3}] || ! blue[{1, 3}]) &&
         (red[{1, 4}] || blue[{1, 4}]) &&
         (! red[{1, 4}] || ! blue[{1, 4}]) &&
         (red[{2, 3}] || blue[{2, 3}]) &&
         (! red[{2, 3}] || ! blue[{2, 3}]) &&
         (red[{2, 4}] || blue[{2, 4}]) &&
         (! red[{2, 4}] || ! blue[{2, 4}]) &&
         (red[{3, 4}] || blue[{3, 4}]) && (! red[{3, 4}] || ! blue[{3, 4}])

Out[6]= (! red[{1, 2}] || ! red[{1, 3}] || ! red[{2, 3}]) &&
         (! red[{1, 2}] || ! red[{1, 4}] || ! red[{2, 4}]) &&
         (! red[{1, 3}] || ! red[{1, 4}] || ! red[{3, 4}]) &&
         (! red[{2, 3}] || ! red[{2, 4}] || ! red[{3, 4}]) &&
         (! blue[{1, 2}] || ! blue[{1, 3}] || ! blue[{2, 3}]) &&
         (! blue[{1, 2}] || ! blue[{1, 4}] || ! blue[{2, 4}]) &&
         (! blue[{1, 3}] || ! blue[{1, 4}] || ! blue[{3, 4}]) &&
         (! blue[{2, 3}] || ! blue[{2, 4}] || ! blue[{3, 4}])
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terminated.

In [9], Cowen explored $R(G_1, G_2, G_3)$, where G_i were complete graphs. We ask the natural question "Is it feasible to adapt Cowen's approach to compute new diagonal graph Ramsey numbers $R(G, G)$, where G is any simple connected graph?"

Our hybrid approach: To further improve the efficiency of Cowen's "QuickerRamseyTest," we make additional use of K_n's symmetry when creating the 2-colorings of it. Let G be a simple connected graph. We want to decide if $K_n \to (G, G)$ or not. Let $k \leq n$. Now, fix k and consider $\mathcal{G}(k)$, the set of nonisomorphic simple graphs of order k. For a graph $H \in \mathcal{G}(k)$, color the edges of H in red and the edges of $\overline{H}$ (the complement of H) in blue. Then, embed H and $\overline{H}$ (with vertices $v_1, v_2, \ldots, v_k$) in K_n (with vertices $v_1, v_2, \ldots, v_n$).

As in Cowen's approach, create a CNF $f(H)$ using coloring clauses and non-monochromatic subgraph clauses. If $f(H)$ is not satisfied for all $H \in \mathcal{G}(k)$, then $K_n \to (G, G)$. In this case, $R(G, G) \leq n$. If $f(H)$ is satisfied for some $H \in \mathcal{G}(k)$, then $K_n \nrightarrow (G, G)$. In this case, $n + 1 \leq R(G, G)$.

Example 2. Let G be a simple connected graph where $|V(G)| = 7$, say. Suppose that we want to determine if $R(G, G) \leq 9$. There is a one-to-one correspondence between the set $\mathcal{G}(8)$ of nonisomorphic simple graphs of order eight and the set $\mathcal{C}(K_8)$ of nonequivalent 2-colorings of K_8.

Consider a simple graph $H \in \mathcal{G}(8)$ with $k = 8$ vertices (see Figure 1), with $\overline{H}$ being the complement of H. Associated with H is a partial 2-coloring $\mathcal{C}_H$ of K_9, where the edges of H and $\overline{H}$ are red and blue, respectively. A CNF $f(H)$ is created using clauses for the 2-colorings of K_9 containing $\mathcal{C}_H$, along with clauses for the non-monochromatic subgraphs ($\cong G$) of K_9. Then, $f(H)$ is tested for satisfiability. Since there are 12346 nonisomorphic simple graphs with eight vertices, 12346 CNFs $f(H_i)$ in total need to be constructed and tested for satisfiability. If all of the $f(H_i)$ are unsatisfied, then $R(G, G) \leq 9$. If at least one of the $f(H_i)$ is satisfied, then $R(G, G) > 9$. $\diamondsuit$

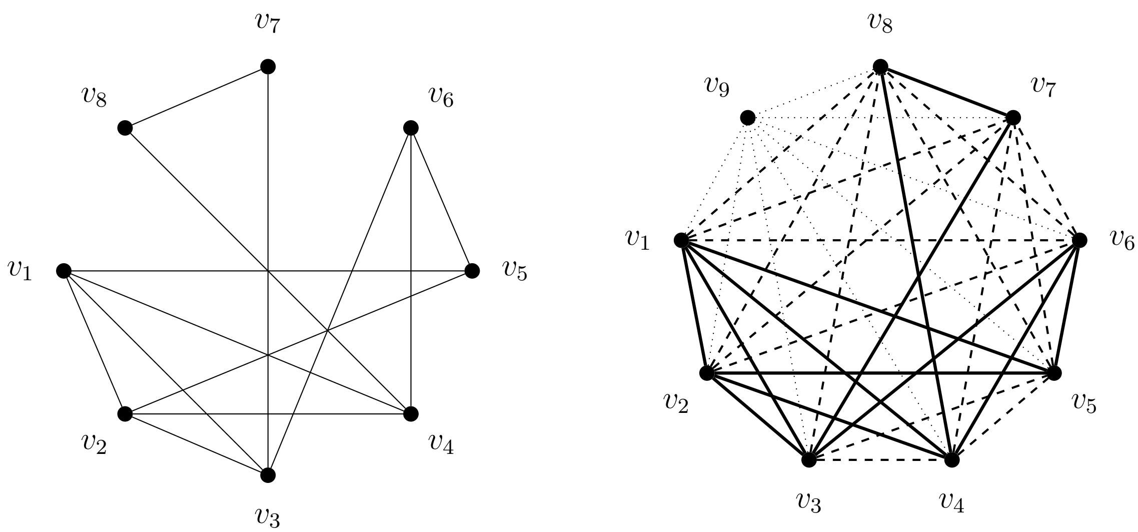


Figure 1: The graph on the left is an $H \in \mathcal{G}(8)$. The graph on the right is a partial 2-coloring of K_9. There, the solid edges are red, the dashed edges are blue and the dotted edges are not predetermined.

From a computational point of view, one wishes to make the most of symmetry when creating the 2-colorings of K_n. Fix $k \leq n$. Recall that $\mathcal{G}(n)$ and $\mathcal{C}(K_n)$ denote the set of nonisomorphic simple graphs of order n and the set of nonequivalent 2-colorings of K_n, respectively. As alluded to in Example 2, there is a bijection between $\mathcal{G}(n)$ and $\mathcal{C}(K_n)$. To construct only the nonequivalent 2-colorings of K_n, we would have to encode the set $\mathcal{G}(n)$ within our computational programs. Unfortunately, even for somewhat small values of n (≥ 9), this cannot be accomplished in a practical way (see Table 1), since $|\mathcal{G}(n)|$ is very large.

In our hybrid approach, we do not create all of the 2-colorings of K_n nor just the nonequivalent ones. Instead, we take the middle ground and construct the set of 2-colorings of K_n which contain c, for each $c \in \mathcal{C}(K_k)$. This improves upon Cowen's approach in different ways. First, the number of 2-colorings of K_n which need to be checked (when calculating $R(G, G)$) is greatly reduced. Second, the number of propositional variables and clauses in our K_n 2-coloring CNF is decreased.

After collecting computational runtime data, we decided to create two sets of CNFs, namely F_7 and F_8. In particular, for $k = 7$ and 8, each CNF in F_k is made up of a set of clauses for the subgraphs ($\cong G$) of K_n, along with a set of clauses for the 2-colorings of K_n containing a distinct $c \in \mathcal{C}(K_k)$. As such, $|F_7| = 1044$ and $|F_8| = 12346$ (see Table 1).

$\mathcal{G}(7)$ and $\mathcal{G}(8)$ were chosen for two reasons:

- $|\mathcal{G}(9)| = 274668$. Thus, 274668 CNFs would need to be tested for satisfiability. This would drastically increase the computational runtime.
- Suppose that $\mathcal{G}(4), \mathcal{G}(5)$ or $\mathcal{G}(6)$ is used. Then, Cowen's f_0 actually fixes more monochromatic edges in a 2-coloring of K_n (for $15 \leq n \leq 30$, approximately), compared to the CNFs in our hybrid approach. In this case, it is better to use Cowen's f_0 to calculate $R(G, G)$.

Testing all of the CNFs in F_7 (or F_8) for satisfiability results in a longer runtime than f_0, on smaller K_n. However, it can be sped up by running the individual $f(H_i)$ in parallel. Since parallelization typically requires an abundance of RAM, the full potential of our hybrid approach (using F_7 or F_8) can only be realized on a high performance computing platform (HPC). That said, even without parallelization, the use of F_7 (or F_8) improves on Cowen's f_0.

The calculations in this paper were obtained using several independent computing platforms:

- 3.68 GHz AMD Ryzen-9 3950X, 128GB RAM
- 3.3 GHz Intel i7-5820K, 32GB RAM
- 1.60 GHz Quad-Core Intel (R), 8GB RAM

Example 3. We calculated $R(G603, G603)$ (see Figure 2) using F_8, as well as f_0. Our hybrid approach determined that $R(G603, G603) = 15$ in approximately 32000 minutes, while f_0 was manually terminated after 33800 minutes. Here, the functions in F_8 were sequentially tested (not in parallel) for satisfiability. The reader should note that $R(G603, G603)$ was previously unknown. $\diamondsuit$

From experimentation, f_0 appears to have a maximum subgraph clause size of roughly 3 million. Under that threshold, f_0 runs much faster than testing all the functions in F_7 (or F_8), even when parallelized. In calculating $R(G, G)$, either f_0, F_7 or F_8 is used, depending on the subgraph clause size for G.

Our constrained coloring approach: Suppose we want to decide if $R(G, G) \leq n$ or not. Let H be a graph with known $R(H, H) = k$, where $k \leq n$. Then, any 2-coloring of K_n must contain a monochromatic (say, red) H. This reduces the number of 2-colorings of K_n which need to be examined, when calculating $R(G, G)$. Additional constraints can be imposed by embedding other monochromatic graphs within any 2-coloring of K_n and/or by mathematical arguments. We create a new CNF which uses a partial 2-coloring of K_n (inputted by the user).

The constrained coloring approach is a natural one and can provide a surprising decrease in computational runtimes. In Example 3, we mentioned that $R(G603, G603) = 15$ was determined in 30808 minutes, using our hybrid approach. However, using our constrained coloring approach, we are able to calculate $R(G603), G603)$ in approximately 5.4 minutes. The proofs of both Theorems 1 and 2 use the constrained coloring approach.

With the hybrid and constrained coloring approaches, we obtain new diagonal graph Ramsey numbers. These results are found in Section 4.

Table 1: The number of nonisomorphic simple graphs of order n.

n	$\lvert\mathcal{G}(n)\rvert$
1	1
2	2
3	4
4	11
5	34
6	156
7	1044
8	12346
9	274668
10	12005168
11	1018997864
12	165091172592
13	50502031367952
14	29054155657235488
15	31426485969804308768

4. New diagonal graph Ramsey numbers

The following known results give lower bounds for $R(G, G)$. They are used in our calculations.

Theorem B. *(Chvátal-Harary [5]). Let G and H be two graphs with no isolated vertices. Then, $R(G, H) \geq (\chi(G) - 1)(n(H) - 1) + 1$, where $\chi(G)$ is the chromatic number of G and $n(H)$ is the order of the largest connected component of H.*

Theorem C. *(Chvátal-Harary [6]). $R(G, G) > (s \cdot 2^{|E(G)|-1})^{1/|V(G)|}$, where s is the number of automorphisms of G.*

Theorem D. *(Burr-Erdős [3]). Let $|V(G)| \geq 4$. Then, $R(G, G) \geq \lfloor (4 \cdot |V(G)| - 1)/3 \rfloor$ for any connected G, and $R(G, G) \geq 2 \cdot |V(G)| - 1$ for any connected non-bipartite G.*

In addition to [21], the following summary of known diagonal graph Ramsey numbers is given on page 62 of [19]:

- $R(G, G)$, for all G without isolates on at most 4 vertices.
- $R(G, G)$, for all G without isolates and with at most 7 edges.
- $R(G, G)$, for all G on 5 vertices and with 7 or 8 edges.

Since we wish to calculate new diagonal graph Ramsey numbers, our attention is focused on connected simple graphs G, where $|V(G)| \geq 6$ and $|E(G)| \geq 8$.

In [21], it was conjectured that $R(G603, G603) = 15 = R(G606, G606)$. See Figure 2. We prove this in Theorems 1 and 2.

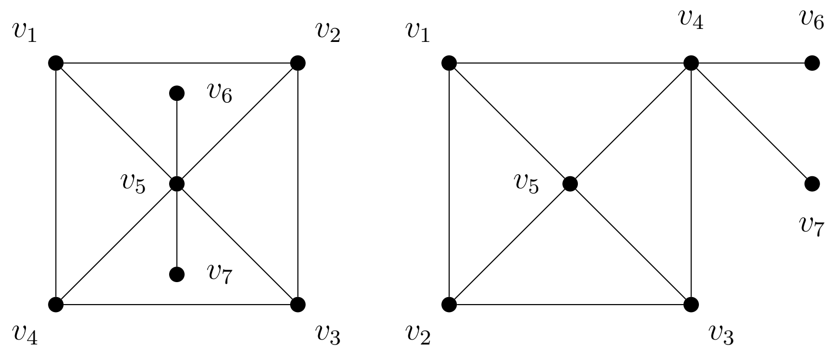


Figure 2: Graphs $G603$ and $G606$, respectively.

Theorem 1. $R(G603, G603) = 15$.

Proof. In [21], it is stated that the diagonal Ramsey number of graph $G603 - \{v_7\}$ is 15. So, $15 \leq R(G603, G603)$.

We wish to show that $R(G603, G603) \leq 15$. Let $v_1, v_2, \ldots, v_{15}$ be an ordering of the vertices of K_{15}. Assume that $K_{15} \nrightarrow (G603, G603)$. Then, there exists a 2-coloring $\mathcal{C}$ of K_{15} where every $G603$ subgraph is not monochromatic. In particular, there exists a $G603$ subgraph H which has a monochromatic $G603 - \{v_7\}$ and edge v_5v_7 of opposite color under $\mathcal{C}$. Otherwise, we reach a desired contradiction. Without loss of generality, H has vertices and edges as described in Figure 2, the $G603 - \{v_7\}$ in H is red and edge v_5v_7 is blue. This implies that edges v_5v_k, for $8 \leq k \leq 15$, are blue. Otherwise, there would be a red $G603$ subgraph under $\mathcal{C}$; thus giving a desired contradiction.

Let K be the graph $G128$, as found in [21]. There, it is stated that $R(K, K) = 8$. Now, consider the vertices $v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}$ and v_{14}. Thus, there is a red subgraph K, say $\{v_7v_8, v_8v_9, v_9v_{10}, v_{10}v_7, v_{10}v_{11}, v_{11}v_{12}, v_{12}v_7\}$, under $\mathcal{C}$. Otherwise, there would be a blue $G603$ subgraph under $\mathcal{C}$; thus giving a desired contradiction.

Next, consider the edges $v_{12}v_1$, $v_{12}v_2$, $v_{12}v_3$ and $v_{12}v_4$. If all four of these edges are red, then there would be a red $G603$ subgraph under $\mathcal{C}$; thus giving a desired contradiction. Without loss of generality, the edge $v_{12}v_1$ is blue. Figure 3 illustrates the partial 2-coloring $\mathcal{C}$ of K_{15}.

Finally, several independent computer programs (using the constrained coloring approach described in Section 3) were written to check for a monochromatic subgraph $G603$ in all possible 2-colorings of K_{15}, under the edge coloring constraints of $\mathcal{C}$. In all instances, we obtain a monochromatic subgraph $G603$; thus giving a desired contradiction.

Hence, our assumption that $K_{15} \nrightarrow (G603, G603)$ was wrong. This, along with the fact that $15 \leq R(G603, G603)$, implies that $R(G603, G603) = 15$. $\square$

Theorem 2. $R(G606, G606) = 15$.

Proof. In [21], it is stated that the diagonal Ramsey number of graph $G606 - \{v_7\}$ is 15. So, $15 \leq R(G606, G606)$.

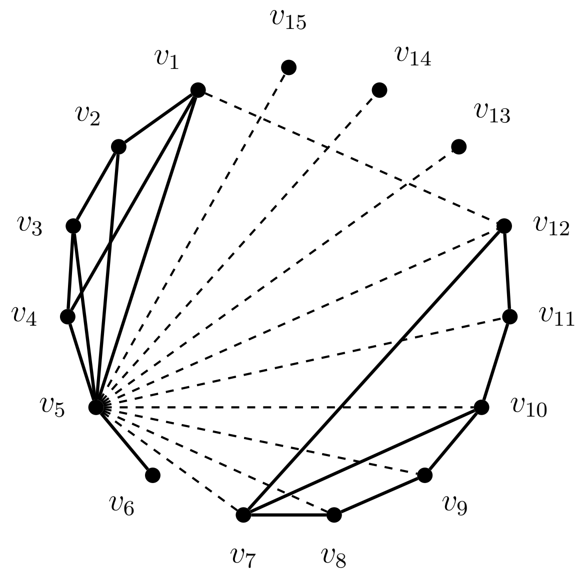


Figure 3: This is the partial 2-coloring $\mathcal{C}$ of K_{15} described in the proof of Theorem 1. The solid edges are red and the dashed edges are blue.

We wish to show that $R(G606, G606) \leq 15$. Let $v_1, v_2, \ldots, v_{15}$ be an ordering of the vertices of K_{15}. Assume that $K_{15} \nrightarrow (G606, G606)$. Then, there exists a 2-coloring $\mathcal{C}$ of K_{15} where every $G606$ subgraph is not monochromatic. In particular, there exists a $G606$ subgraph H which has a monochromatic $G606 - \{v_7\}$ and edge v_4v_7 of opposite color under $\mathcal{C}$. Otherwise, we reach a desired contradiction. Without loss of generality, H has vertices and edges as described in Figure 2, the $G606 - \{v_7\}$ in H is red and edge v_4v_7 is blue. This implies that edges v_4v_k, for $8 \leq k \leq 15$, are blue. Otherwise, there would be a red $G606$ subgraph under $\mathcal{C}$; thus giving a desired contradiction.

Let K be the graph $G325$, as found in [21]. There, it is stated that $R(K, K) = 9$. Now, consider the vertices $v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}$ and v_{15}. Thus, there is a red subgraph K, say $\{v_8v_9, v_9v_{10}, v_{10}v_{11}, v_{11}v_8, v_9v_7, v_{11}v_{12}, v_{11}v_{13}\}$, under $\mathcal{C}$. Otherwise, there would be a blue $G606$ subgraph under $\mathcal{C}$; thus giving a desired contradiction.

Next, consider the edges $v_{15}v_3$ and $v_{14}v_3$. If both of these edges are red, then there would be a red $G606$ subgraph under $\mathcal{C}$; thus giving a desired contradiction. Without loss of generality, the edge $v_{15}v_3$ is blue. Now, consider the edges $v_{13}v_1$ and $v_{12}v_1$. If both of these edges are red, then there would be a red $G606$ subgraph under $\mathcal{C}$; thus giving a desired contradiction. Without loss of generality, the edge $v_{13}v_1$ is blue. Lastly, consider the edges $v_{10}v_2$ and v_8v_2. If both of these edges are red, then there would be a red $G606$ subgraph under $\mathcal{C}$; thus giving a desired contradiction. Without loss of generality, the edge $v_{10}v_2$ is blue. Figure 4 illustrates the partial 2-coloring $\mathcal{C}$ of K_{15}.

Finally, several independent computer programs (using the constrained coloring approach described in Section 3) were written to check for a monochromatic subgraph $G606$ in all possible 2-colorings of K_{15}, under the edge coloring constraints of $\mathcal{C}$. In all instances, we obtain a monochromatic subgraph $G606$; thus giving a desired contradiction.

Hence, our assumption that $K_{15} \nrightarrow (G606, G606)$ was wrong. This, along with the fact that $15 \leq R(G606, G606)$, implies that $R(G606, G606) = 15$. $\square$

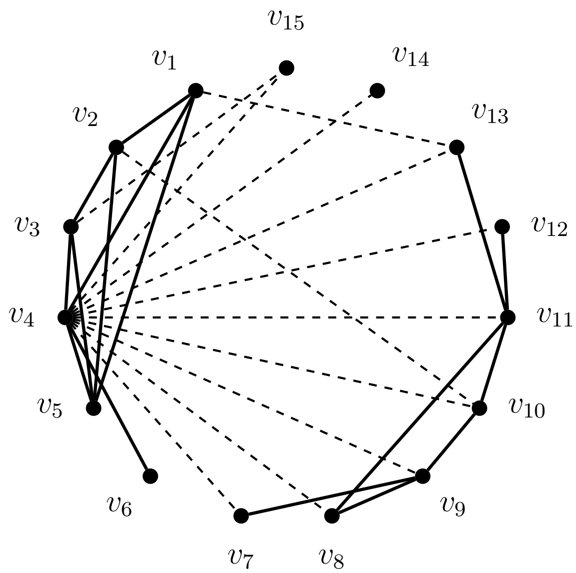
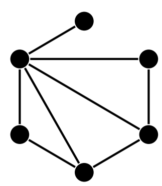
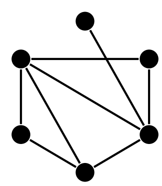
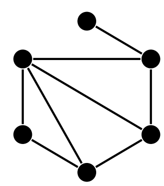
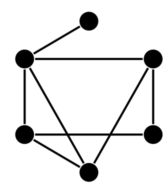
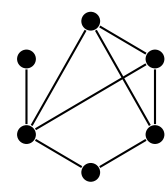
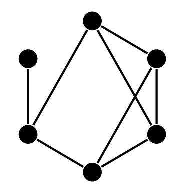
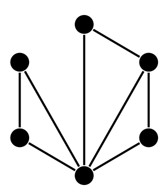
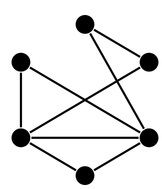
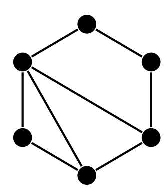
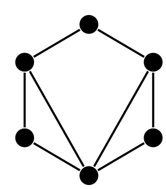
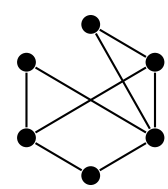
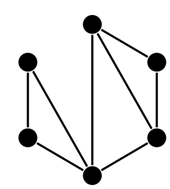
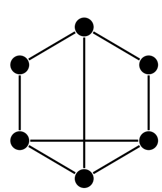
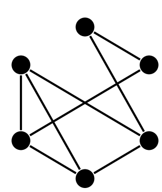
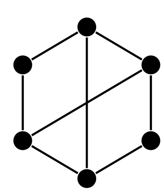
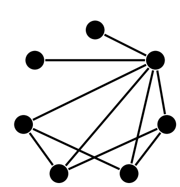
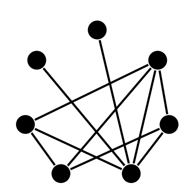
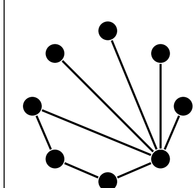


Figure 4: This is the partial 2-coloring $\mathcal{C}$ of K_{15} described in the proof of Theorem 2. The solid edges are red and the dashed edges are blue.

Table 2 gives additional diagonal graph Ramsey numbers, which were previously unknown when [21] was published, and which are not found in the current mathematical literature. These new results were obtained using the computational methods described in Section 3 of this paper.

Table 2: Some new diagonal graph Ramsey numbers. The "Gxxx" labels in this table correspond to classification numbers used in [21].

$G136$	$G137$	$G139$	$G140$	$G141$	$G143$
					
12	11	11	11	11	11
$G144$	$G145$	$G147$	$G148$	$G149$	$G150$
					
12	11	11	11	11	13
$G151$	$G153$	$G154$	$G603$	$G606$	$C_4^{(4)}$
					
11	12	11	15	15	11

5. Miscellany

While exploring various graph Ramsey theory problems with Mathematica, Cowen [8] proved the following theorem. This beautiful result is what originally motivated our research project.

Theorem E. (*Cowen* [8]). *Suppose $R(K_s, K_t) = n$. Then, $G = K_n - \{uv\}$ has a red/blue edge coloring where there is neither a red K_s nor a blue K_t.*

To conclude this paper, we extend Theorem E to r colors.

Theorem 3. *Suppose $R(K_{x_1}, K_{x_2}, \ldots, K_{x_k}) = n$. Then, $G = K_n - \{uv\}$ has an r-coloring of the edges where no monochromatic K_{x_i} exist in G.*

Proof. Let $G = K_n - \{uv\}$. Since $G - \{u\}$ is K_{n-1}, it has an r-coloring of the edges where no monochromatic K_{x_i} exist in G. To complete the coloring of G, we color an edge ux, $x \in V \setminus \{u, v\}$ using the color of edge vx. We now claim that no monochromatic K_{x_i} exist in G. Suppose (to the contrary) a monochromatic K_{x_i} exists in G. Then, K_{x_i} does not contain both u and v. Without loss of generality, $u \notin K_{x_i}$. Then, K_{x_i} is a monochromatic subgraph of $G - \{u\}$. This gives a desired contradiction and establishes the claim. $\square$

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