Electronic Journal of Graph Theory and Applications **10** (2) (2022), 541–552

Electronic Journal of
Graph Theory and Applications

Perfect matching transitivity of circulant graphs

Isaac A. Reiter, Ju Zhou

Kutztown University of Pennsylvania, Kutztown, PA

ireit426@live.kutztown.edu, zhou@kutztown.edu

Abstract

A graph G is perfect matching transitive, shortly PM-transitive, if for any two perfect matchings M_1 and M_2 of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f_e(M_1) = M_2$, where $f_e(uv) = f(u)f(v)$. In this paper, the authors completely characterize the perfect matching transitivity of circulant graphs of order less than or equal to 10.

Keywords: automorphism, vertex-transitive, edge-transitive, perfect matching, perfect matching transitivity, PM-transitive, circulant graph.
Mathematics Subject Classification: 05C60
DOI: 10.5614/ejgta.2022.10.2.14

1. Introduction

An automorphism of a graph is a form of symmetry in which the graph is mapped onto itself while preserving the edge-vertex connectivity. Formally, an automorphism of a graph $G = (V(G), E(G))$ is a permutation f of the vertex set $V(G)$ such that the pair of vertices uv is an edge of G if and only if $f(u)f(v)$ is also an edge of G. In other words, it is a graph isomorphism from G to itself. Every graph automorphism f induces a mapping $f_e : E(G) \mapsto E(G)$ such that $f_e(uv) = f(u)f(v)$. For any vertex set $X \subseteq V(G)$ and edge set $M \subseteq E(G)$, denote $f(X) = \{f(v) : v \in X\}$ and $f_e(M) = \{f_e(uv) : uv \in M\}$.

A graph G is *vertex-transitive* [11] if for any two given vertices v_1 and v_2 of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f(v_1) = v_2$. In other words, a graph is vertex-transitive if its automorphism group acts transitively upon its vertices. A graph is vertex-transitive if and only if its complement graph is vertex-transitive (since the group actions are identical). For example, the finite Cayley graphs, the Petersen graph, and $C_n \times K_2$ with $n \geq 3$ are vertex-transitive.

Received: 1 April 2022, Revised: 17 June 2022, Accepted: 26 September 2022.

A graph G is *edge-transitive* if for any two given edges e_1 and e_2 of G, there is an automorphism of G that maps e_1 to e_2. In other words, a graph is edge-transitive if its automorphism group acts transitively upon its edges. The complete bipartite graph $K_{m,n}$, the Petersen graph, and the cubical graph $C_n \times K_2$ with $n = 4$ are edge-transitive.

A graph G is *symmetric* or *arc-transitive* if for any two pairs of adjacent vertices u_1v_1 and u_2v_2 of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f(u_1) = u_2$ and $f(v_1) = v_2$. In other words, a graph is symmetric if its automorphism group acts transitively upon ordered pairs of adjacent vertices, that is, upon edges considered as having a direction. The cubical graph $C_n \times K_2$ with $n = 4$ and Petersen graph are symmetric graphs.

Every connected symmetric graph must be both vertex-transitive and edge-transitive, and the converse is true for graphs of odd degree [2]. However, for graphs of even degree, there exist connected graphs which are vertex-transitive and edge-transitive, but not symmetric [3]. Every symmetric graph without isolated vertices is vertex-transitive, and every vertex-transitive graph is regular. However, not all vertex-transitive graphs are symmetric (for example, the edges of the truncated tetrahedron), and not all regular graphs are vertex-transitive (for example, the Frucht graph and Tietze's graph).

A lot of work has been done about the relationship between vertex-transitive graphs and edge-transitive graphs. Some of the related results can be found in [3]-[17]. In general, edge-transitive graphs need not be vertex-transitive. The Gray graph is an example of a graph which is edge-transitive but not vertex-transitive. Conversely, vertex-transitive graphs need not be edge-transitive. The graph $C_n \times K_2$, where $n \geq 5$ is vertex-transitive but not edge-transitive.

A graph G is *perfect matching transitive*, shortly *PM-transitive*, if for any two perfect matchings M_1 and M_2 of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f_e(M_1) = M_2$, where f_e is the mapping induced by f.

In [18], the author (Zhou) verified that some well known symmetric graphs such as C_{2n}, K_{2n}, $K_{n,n}$, and the Petersen graph are PM-transitive, constructed several families of PM-transitive graphs which are neither vertex-transitive nor edge-transitive, discussed some methods to generate new PM-transitive graphs, and proved that all the generated Petersen graphs except the Petersen graph are non-perfect matching transitive.

A circulant graph is a graph of n vertices $v_1, v_2, \ldots, v_n$ in which the ith vertex is adjacent to the $(i+j)$th and $(i-j)$th vertices for each j in a list l, where the addition and subtraction are taken by modulo n. In Section 2, the authors prove a collection of general results about the PM-transitivity of connected circulant graphs of even order $n \geq 4$. In Section 3, the authors characterize the PM-transitivity of connected circulant graphs of order 6. In Section 4, the authors characterize the PM-transitivity of connected circulant graphs of order 8. In Section 5, the authors characterize the PM-transitivity of connected circulant graphs of order 10.

2. PM-transitivity of Connected Circulant Graphs of Order 2n

For any integer $n \geq 2$, the circulant graph $Ci_{2n}(1, 2, \ldots, n)$ gives the complete graph K_{2n}, the circulant graph $Ci_{2n}(1)$ gives the cyclic graph C_{2n}, and the circulant graph $Ci_{2n}(1, 3, 5, \ldots, m)$, where m represents the largest odd integer less than or equal to n, gives the complete bipartite graph $K_{n,n}$. The following Theorem 2.1 is proven in [18].

Theorem 2.1. *For any integer $n \geq 2$, the circulant graphs $Ci_{2n}(1,2,...,n) \cong K_{2n}$, $Ci_{2n}(1) \cong C_{2n}$, and $Ci_{2n}(1,3,5,\ldots,m) \cong K_{n,n}$, where m represents the largest odd integer that is less than or equal to n, are PM-transitive.*

Theorem 2.2. *For any integer $n \geq 4$, the circulant graph $Ci_{2n}(1,2)$ is not PM-transitive.*

Proof. Let $M_1 = \{v_1v_3, v_2v_{2n}, v_4v_5, v_6v_7, v_8v_9, \ldots, v_{2n-2}v_{2n-1}\}$ and $M_2 = \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2n-1}v_{2n}\}$. Then M_1 and M_2 are two perfect matchings of G such that $G - M_1$ has 3-cycles ($v_2v_3v_4v_2$ being one such 3-cycle) while $G - M_2 \cong C_n \times K_2$ doesn't have 3-cycles. Therefore, G is not PM-transitive. □

Theorem 2.3. *For any integer $n \geq 4$, the circulant graph $Ci_{2n}(1,n)$ is not PM-transitive.*

Proof. Let $M_1 = \{v_1v_{n+1}, v_2v_{n+2}, \ldots, v_nv_{2n}\}$. If $n = 4$, then let $M_2 = \{v_8v_1, v_2v_3, v_4v_5, v_6v_7\}$. Otherwise, let $M_2 = \{v_{2n}v_1, v_2v_3, v_nv_{n+1}, v_{n+2}v_{n+3}, v_4v_{n+4}, v_5v_{n+5}, \ldots, v_{n-1}v_{2n-1}\}$. Then M_1 and M_2 are two perfect matchings of G such that $G - M_1$ is a $2n$-cycle and $G - M_2$ is a union of a 4-cycle $v_1v_2v_{n+2}v_{n+1}v_1$ and a $(2n-4)$-cycle $v_3v_4v_5\cdots v_nv_{2n}v_{2n-1}v_{2n-2}\cdots v_{n+3}v_3$. Therefore, $Ci_{2n}(1,n)$ is not PM-transitive. □

In this paper's proofs, the authors shall frequently use the phrase "without loss of generality, let $f(v_1) = v_1$." The following lemma justifies why we can make this assumption. We will define a perfect matching M of a circulant graph G to be *vertex-perfect-matching transitive* if for any two given vertices v_i and v_j of G, there is an automorphism $f : V(G) \mapsto V(G)$ such that $f(v_i) = v_j$ and $f_e(M) = M$.

Lemma 2.1. *Let G be a circulant graph of order $2n$, $n \geq 2$. Let M_1 and M_2 be two perfect matchings of G such that either M_1 or M_2 is vertex-perfect-matching transitive. If f is an automorphism $f : V(G) \mapsto V(G)$ such that $f_e(M_1) = M_2$, then we may assume without loss of generality that $f(v_1) = v_1$.*

Proof. Let $f : V(G) \mapsto V(G)$ be an automorphism of G such that $f_e(M_1) = M_2$.

If M_1 is vertex-perfect-matching transitive, let v_i be the vertex such that $f(v_i) = v_1$. Since M_1 is vertex-perfect-matching transitive, there exists an automorphism $g : V(G) \mapsto V(G)$ such that $g(v_1) = v_i$ and $g_e(M_1) = M_1$. Now, we define $h = f \circ g$, implying that h is an automorphism such that $h(v_1) = v_1$ and $h_e(M_1) = M_2$.

If M_2 is vertex-perfect-matching transitive, let v_i be the vertex such that $f(v_1) = v_i$. Since M_2 is vertex-perfect-matching transitive, there exists an automorphism $g : V(G) \mapsto V(G)$ such that $g(v_i) = v_1$ and $g_e(M_2) = M_2$. Now, we define $h = g \circ f$, implying that h is an automorphism such that $h(v_1) = v_1$ and $h_e(M_1) = M_2$.

In other words, any automorphism $f : V(G) \mapsto V(G)$ such that $f_e(M_1) = M_2$ induces an automorphism $h : V(G) \mapsto V(G)$ such that $h(v_1) = v_1$ and $h_e(M_1) = M_2$. Thus, we may assume without loss of generality that $f(v_1) = v_1$. □

Lemma 2.2. *Let G be a circulant graph of order $2n$, $n \geq 2$.*

(1) *If G contains the perfect matching $M_1 = \{v_1v_{n+1}, v_2v_{n+2}, v_3v_{n+3}, \ldots, v_nv_{2n}\}$, then M_1 is vertex-perfect-matching transitive.*

(2) *If G contains the perfect matching $M_2 = \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2n-1}v_{2n}\}$, then M_2 is vertex-perfect-matching transitive.*

Proof. First, consider G with M_1. Let $v_i, v_j \in V(G)$, and define $\delta_1 = j-i$. Let $f : V(G) \mapsto V(G)$ such that $f(v_m) = v_{m+\delta_1}$ for all $1 \leq m \leq 2n$. Then f is a rotation of G, and f is an automorphism such that $f(v_i) = v_j$ and $f_e(M) = M$. Thus, M_1 is vertex-perfect-matching transitive, and **(1)** is proven.

Second, consider G with M_2. Let $v_i, v_j \in V(G)$, and define $\delta_2 = j - i$. If δ_2 is even, then let $g : V(G) \mapsto V(G)$ such that $g(v_m) = v_{m+\delta_2}$ for all $1 \leq m \leq 2n$. Then g is a rotation of G. It is the case that g is an automorphism such that $g(v_i) = v_j$ and $g_e(M) = M$.

If δ_2 is odd, then let $h_1 : V(G) \mapsto V(G)$ such that $h_1(v_p) = v_{2n+1-p}$ for all $1 \leq p \leq 2n$. Then h_1 is a reflection of G, and h_1 is an automorphism. Also, h_1 maps odd-indexed vertices to even-indexed vertices and even-indexed vertices to odd-indexed vertices. Let $v_k = h_1(v_i)$ and define $\delta_3 = j - k$. Since h_1 switches the parity of all vertices, δ_3 is even. Let $h_2 : V(G) \mapsto V(G)$ such that $h_2(v_p) = v_{p+\delta_3}$ for all $1 \leq p \leq 2n$. Then h_2 is a rotation of G, and h_2 is an automorphism. Let $h : V(G) \mapsto V(G)$ such that $h = h_2 \circ h_1$. Then h is an automorphism such that $h(v_i) = v_j$. Furthermore, let $v_x v_{x+1} \in M$. Then h maps this edge to $v_{2n+1-x+\delta_3} v_{2n+1-x-1+\delta_3}$. Since δ_3 is even and x is odd, $2n+1-x-1+\delta_3$ is odd. Thus, $v_{2n+1-x+\delta_3} v_{2n+1-x-1+\delta_3} \in M$. Since h is bijective, it maps each edge in M to a unique edge in M. This implies that $h_e(M) = M$. Thus, M_2 is vertex-perfect-matching transitive, and **(2)** is proven. □

The condition of Lemma 2.1 is that one of the perfect matchings is vertex-perfect-matching transitive. Thus, whenever Lemma 2.1 is invoked, one of the perfect matchings in Lemma 2.2 will be present.

Theorem 2.4. *For any odd integer $n \geq 5$, the circulant graph $Ci_{2n}(1, 2, 3, \ldots, n-1)$ is not PM-transitive.*

Proof. If n is odd, let $M_1 = \{v_1v_2, v_3v_4, v_5v_6, \ldots, v_{2n-1}v_{2n}\}$ and $M_2 = \{v_{n+1}v_{n+2}, v_nv_{n+3}\} \cup (M_1 \backslash \{v_nv_{n+1}, v_{n+2}v_{n+3}\})$. M_1 and M_2 are two perfect matchings of G. Suppose f is an automorphism $f : V(G) \mapsto V(G)$ such that $f_e(M_1) = M_2$. By Lemma 2.1, we can assume, without loss of generality, that $f(v_1) = v_1$. Since $v_1v_2 \in M_2$, this implies that $f(v_2) = v_2$. Since $v_1v_{n+1} \notin E(G)$, $f(v_1)f(v_{n+1}) = v_1f(v_{n+1}) \notin E(G)$. Since v_{n+1} is the only vertex not adjacent to v_1, $f(v_{n+1}) = v_{n+1}$. Similarly, $v_2v_{n+2} \notin E(G)$ implies that $f(v_2)f(v_{n+2}) = v_2f(v_{n+2}) \notin E(G)$. Since v_{n+2} is the only vertex not adjacent to v_2, $f(v_{n+2}) = v_{n+2}$. Notice that $v_{n+1}v_{n+2} \notin M_1$ but $f(v_{n+1})f(v_{n+2}) = v_{n+1}v_{n+2} \in M_2$, contradicting $f_e(M_1) = M_2$. Thus, G is not PM-transitive. □

3. PM-transitivity of Connected Circulant Graphs of Order 6

In this section, we characterize the PM-transitivity of connected circulant graphs of order 6. The circulant graphs of order 6 include $Ci_6(1)$, $Ci_6(2)$, $Ci_6(3)$, $Ci_6(1, 2)$, $Ci_6(1, 3)$, $Ci_6(2, 3)$, and $Ci_6(1, 2, 3)$, where $Ci_6(2)$ and $Ci_6(3)$ are disconnected.

Theorem 3.1. *If G is a connected PM-transitive circulant graph of order 6, then G is congruent to $Ci_6(1)$, $Ci_6(1, 2)$, $Ci_6(1, 3)$, or $Ci_6(1, 2, 3)$.*

Proof. If $G \cong Ci_6(1) = C_6$, $G \cong Ci_6(1,3) = K_{3,3}$, or $G \cong Ci_6(1,2,3) = K_6$, then G is PM-transitive by Theorem 2.1. We just need to consider the following two cases.

Case 1. $G \cong Ci_6(1,2)$ is PM-transitive.

Define $\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$ to be the *outer edges* and all other edges to be the *inner edges*. Let M be a perfect matching of G. Notice that the six inner edges form two C_3 subgraphs. If M contains three inner edges, then one of these subgraphs will have two edges in M. This results in one vertex being covered twice, a contradiction. Thus, M must contain at least one outer edge.

Without loss of generality, let $v_5v_6 \in M$. To cover the remaining vertices, either $v_1v_2, v_3v_4 \in M$ or $v_1v_3, v_2v_4 \in M$. Let $M_1 = \{v_1v_2, v_3v_4, v_5v_6\}$ and $M_2 = \{v_1v_3, v_2v_4, v_5v_6\}$.

Now, it simply remains to find an automorphism that maps M_1 to M_2. Let $f : V(G) \to V(G)$ such that $f(v_1) = v_1$, $f(v_2) = v_3$, $f(v_3) = v_2$, $f(v_4) = v_4$, $f(v_5) = v_6$, and $f(v_6) = v_5$. Then f is an automorphism such that $f_e(M_1) = M_2$. Thus, G is PM-transitive.

Case 2. $G \cong Ci_6(2,3)$ is not PM-transitive.

Let $M_1 = \{v_1v_4, v_2v_5, v_3v_6\}$ and $M_2 = \{v_1v_5, v_2v_4, v_3v_6\}$.Then M_1 and M_2 are two perfect matchings of G such that $G - M_1$ is a union of two 3-cycles while $G - M_2$ is a 6-cycle. Therefore, G is not PM-transitive.

In summary, the connected PM-transitive circulant graphs of order 6 are congruent to $Ci_6(1)$, $Ci_6(1,2)$, $Ci_6(1,3)$, or $Ci_6(1,2,3)$. In other words, they are congruent to C_6, $K_{2,2,2}$, $K_{3,3}$, or K_6. □

4. PM-transitivity of Connected Circulant Graphs of Order 8

In this section, we characterize the PM-transitivity of connected circulant graphs of order 8. The circulant graphs of order 8 include $Ci_8(1)$, $Ci_8(2)$, $Ci_8(3)$, $Ci_8(4)$, $Ci_8(1,2)$, $Ci_8(1,3)$, $Ci_8(1,4)$, $Ci_8(2,3)$, $Ci_8(2,4)$, $Ci_8(3,4)$, $Ci_8(1,2,3)$, $Ci_8(1,2,4)$, $Ci_8(1,3,4)$, $Ci_8(2,3,4)$, and $Ci_8(1,2,3,4)$, where $Ci_8(2)$, $Ci_8(4)$ and $Ci_8(2,4)$ are disconnected. Furthermore, Theorem 4.1 contains 4 statements of congruence that reduce the number of cases needed to prove Theorem 4.2.

Theorem 4.1. *For the connected circulant graph or order 8, the following congruence statements hold.*

(1) $Ci_8(1) \cong Ci_8(3)$
(2) $Ci_8(1,2) \cong Ci_8(2,3)$
(3) $Ci_8(1,4) \cong Ci_8(3,4)$
(4) $Ci_8(1,2,4) \cong Ci_8(2,3,4)$

Proof. To prove each congruence statement, it is sufficient to define an automorphism f from the vertices of the first graph to the vertices of the second graph.

Let f be defined such that $f(v_1) = v_1$, $f(v_2) = v_4$, $f(v_3) = v_7$, $f(v_4) = v_2$, $f(v_5) = v_5$, $f(v_6) = v_8$, $f(v_7) = v_3$, and $f(v_8) = v_6$. For each of the 4 congruence statements, f is an automorphism that maps the vertices of the graph on the left side to the vertices of the graph on the right side. □

Theorem 4.2. *If G is a connected PM-transitive circulant graph of order 8, then G is congruent to $Ci_8(1)$, $Ci_8(1,3)$, or $Ci_8(1,2,3,4)$.*

Proof. If $G \cong Ci_8(1) \cong C_8$, $G \cong Ci_8(1,3) \cong K_{4,4}$ or $G \cong Ci_8(1,2,3,4) \cong K_8$, then G is PM-transitive by Theorem 2.1. If $G \cong Ci_8(1,2)$, then G is not PM-transitive by Theorem 2.2. If $G \cong Ci_8(1,4) \cong Ci_8(3,4)$, then G is not PM-transitive by Theorem 2.3. We just need to consider the following three cases.

Case 1. $C \cong Ci_8(1,2,3)$ is not PM-transitive.

Let $M_1 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8\}$ and $M_2 = \{v_1v_2, v_3v_4, v_5v_8, v_6v_7\}$. Then M_1 and M_2 are two perfect matchings of G. Suppose that f is an automorphism $f : V(G) \to V(G)$ such that $f_e(M_1) = M_2$. Without loss of generality, let $f(v_1) = v_1$. This forces $f(v_2) = v_2$.

Consider v_8. Since v_8 is adjacent to v_1 and v_2, $f(v_8)$ cannot be v_5 or v_6. If $f(v_8) = v_3$, then this forces $f(v_5) = v_4$. Now $v_1v_5 \notin E(G)$ and $v_1v_4 \in E(G)$, contradicting the supposition that f is an automorphism. If $f(v_8) = v_4$, then this forces $f(v_5) = v_3$. Now $v_1v_5 \notin E(G)$ and $v_1v_3 \in E(G)$, a contradiction. If $f(v_8) = v_7$, then this forces $f(v_5) = v_8$. Now $v_1v_5 \notin E(G)$ and $v_1v_8 \in E(G)$, a contradiction. If $f(v_8) = v_8$, then this forces $f(v_5) = v_7$. Now $v_1v_5 \notin E(G)$ and $v_1v_7 \in E(G)$, a contradiction. Therefore, G is not PM-transitive.

Case 2. $G \cong Ci_8(1,2,4)$ is not PM-transitive.

Let $M_1 = \{v_1v_3, v_2v_8, v_4v_6, v_5v_7\}$ and $M_2 = \{v_1v_5, v_2v_6, v_3v_7, v_4v_8\}$. Then M_1 and M_2 are two perfect matchings of G such that $G - M_1$ has four 3-cycles while $G - M_2 \cong Ci_8(1,2)$ has eight 3-cycles. Therefore, G is not PM-transitive.

Case 3. $G \cong Ci_8(1,3,4)$ is not PM-transitive.

Let $M_1 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8\}$ and $M_2 = \{v_1v_5, v_2v_6, v_3v_7, v_4v_8\}$. Then M_1 and M_2 are two perfect matchings of G such that $G - M_1$ has 3-cycles while $G - M_2 \cong Ci_8(1,3) \cong K_{4,4}$ doesn't have 3-cycles. Therefore, G is not PM-transitive.

In summary, the connected PM-transitive circulant graphs of order 8 are congruent to $Ci_8(1)$, $Ci_8(1,3)$, or $Ci_8(1,2,3,4)$. In other words, the connected PM-transitive circulant graphs of order 8 are congruent to C_8, $K_{4,4}$, or K_8. □

5. PM-transitivity of Connected Circulant Graphs of Order 10

In this section, we characterize the PM-transitivity of connected circulant graphs of order 10. The circulant graphs of order 10 include $Ci_{10}(1)$, $Ci_{10}(2)$, $Ci_{10}(3)$, $Ci_{10}(4)$, $Ci_{10}(5)$, $Ci_{10}(1,2)$, $Ci_{10}(1,3)$, $Ci_{10}(1,4)$, $Ci_{10}(1,5)$, $Ci_{10}(2,3)$, $Ci_{10}(2,4)$, $Ci_{10}(2,5)$, $Ci_{10}(3,4)$, $Ci_{10}(3,5)$, $Ci_{10}(4,5)$, $Ci_{10}(1,2,3)$, $Ci_{10}(1,2,4)$, $Ci_{10}(1,2,5)$, $Ci_{10}(1,3,4)$, $Ci_{10}(1,3,5)$, $Ci_{10}(1,4,5)$, $Ci_{10}(2,3,4)$, $Ci_{10}(2,3,5)$, $Ci_{10}(2,4,5)$, $Ci_{10}(3,4,5)$, $Ci_{10}(1,2,3,4)$, $Ci_{10}(1,2,3,5)$, $Ci_{10}(1,2,4,5)$, $Ci_{10}(1,3,4,5)$, $Ci_{10}(2,3,4,5)$, and $Ci_{10}(1,2,3,4,5)$, where $Ci_{10}(2)$, $Ci_{10}(4)$, $Ci_{10}(2,4)$, $Ci_{10}(5)$ are disconnected. Furthermore, Theorem 5.1 contains 10 statements of congruence that reduce the number of cases needed to prove Theorem 5.2.

Theorem 5.1. *For the connected circulant graph of order 10, the following congruence statements hold.*

(1) $Ci_{10}(1,2) \cong Ci_{10}(3,4)$
(2) $Ci_{10}(1,4) \cong Ci_{10}(2,3)$
(3) $Ci_{10}(2,5) \cong Ci_{10}(4,5)$
(4) $Ci_{10}(1,5) \cong Ci_{10}(3,5)$

(5) $Ci_{10}(1,2,3) \cong Ci_{10}(1,3,4)$
(6) $Ci_{10}(1,2,4) \cong Ci_{10}(2,3,4)$
(7) $Ci_{10}(1,2,5) \cong Ci_{10}(3,4,5)$
(8) $Ci_{10}(1,4,5) \cong Ci_{10}(2,3,5)$
(9) $Ci_{10}(1,2,4,5) \cong Ci_{10}(2,3,4,5)$
(10) $Ci_{10}(1,2,3,5) \cong Ci_{10}(1,3,4,5)$.

Proof. To prove each congruence statement, it is sufficient to define an automorphism f from the vertices of the first graph to the vertices of the second graph.

Let f be defined such that $f(v_1) = v_1$, $f(v_2) = v_4$, $f(v_3) = v_7$, $f(v_4) = v_{10}$, $f(v_5) = v_3$, $f(v_6) = v_6$, $f(v_7) = v_9$, $f(v_8) = v_2$, $f(v_9) = v_5$, and $f(v_{10}) = v_8$. For each of the 10 congruence statements, f is an automorphism that maps the vertices of the graph on the left side to the vertices of the graph on the right side. □

Theorem 5.2. *If G is a connected PM-transitive circulant graph of order 10, then G is congruent to $Ci_{10}(1)$, $Ci_{10}(1,4)$, $Ci_{10}(1,3,5)$, or $Ci_{10}(1,2,3,4,5)$.*

Proof. If $G \cong Ci_{10}(1) \cong Ci_{10}(3) \cong C_{10}$, $G \cong Ci_{10}(1,3,5) \cong K_{5,5}$, or $G \cong Ci_{10}(1,2,3,4,5) \cong K_{10}$, then G is PM-transitive by Theorem 2.1. If $G \cong Ci_{10}(1,2)$, then G is not PM-transitive by Theorem 2.2. If $G \cong Ci_{10}(1,5) \cong Ci_{10}(3,5)$, then G is not PM-transitive by Theorem 2.3. If $G \cong Ci_{10}(1,2,3,4) \cong K_{2,2,2,2,2}$, then G is not PM-transitive by Theorem 2.4. We just need to distinguish the following ten cases.

Case 1. $G \cong Ci_{10}(1,3)$ is not PM-transitive.

Let $M_1 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$ and $M_2 = \{v_1v_2, v_3v_{10}, v_4v_7, v_5v_8, v_6v_9\}$. Then M_1 and M_2 are two perfect matchings of G.

Let G_1 be the graph formed by identifying the vertices in each edge of M_1. In other words, G_1 is the graph formed by identifying v_1 with v_2, v_3 with v_4, v_5 with v_6, v_7 with v_8, and v_9 with v_{10}. Similarly, let G_2 be the graph formed by identifying the vertices in each edge of M_2. Notice that G_1 is K_5 and that G_2 is K_5 minus an edge. Therefore, G is not PM-transitive.

Case 2. $G \cong Ci_{10}(1,4)$ is PM-transitive.

Consider the following four perfect matchings: $M_1 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$, $M_2 = \{v_1v_2, v_5v_6, v_8v_9, v_3v_7, v_4v_{10}\}$, $M_3 = \{v_1v_2, v_5v_6, v_7v_8, v_3v_9, v_4v_{10}\}$, and $M_4 = \{v_1v_2, v_3v_7, v_4v_8, v_5v_9, v_6v_{10}\}$. We shall show that every perfect matching M of G is automorphic to one of these perfect matchings. To show this, we shall define $\{v_{10}v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9, v_9v_{10}\}$ to be the *outer edges* and all other edges to be the *inner edges*.

If M contains no inner edges, then $M = M_1$ or $M = \{v_{10}v_1, v_2v_3, v_4v_5, v_6v_7, v_8v_9\}$. It is easy to see that M is automorphic to M_1.

In the following, we assume that M contains at least one inner edge. Without loss of generality, let $v_4v_{10} \in M$. Notice that v_1, v_2, and v_3 cannot be covered by only using outer edges in M. Thus, M has at least two inner edges.

If M has exactly two inner edges, then not all of v_1, v_2, v_3 can be covered by inner edges. Without loss of generality, let $v_1v_2 \in M$. To cover v_3, either $v_3v_7 \in M$ or $v_3v_9 \in M$. If the former is true, then $v_8v_9, v_5v_6 \in M$. Now, M is automorphic to M_2. If the latter is true, then $v_7v_8, v_5v_6 \in M$. Now, M is automorphic to M_3.

Suppose that M has exactly three inner edges. If neither v_1v_2 nor v_2v_3 are in M, then three more inner edges are needed to cover $\{v_1, v_2, v_3\}$, a contradiction. Thus, without loss of generality let $v_1v_2 \in M$. To cover v_5, either $v_5v_6 \in M$ or $v_5v_9 \in M$. If the former is true, then $\{v_3, v_7, v_8, v_9\}$ cannot be covered by two inner edges in M. If the latter is true, then $v_3v_7 \in M$ in order to cover v_3. Now, v_6 and v_8 cannot be covered by an outer edge in M, a contradiction. Thus, M cannot have three inner edges.

If M has exactly four inner edges, then consider the following. Suppose that the single outer edge in M is v_1v_2. Now, there is no inner edge that can cover v_8. Thus, by symmetry $v_1v_2 \notin M$ and $v_2v_3 \notin M$. If the single outer edge in M is v_8v_9, then $v_3v_7 \in M$ to cover v_3. To cover v_2, $v_2v_6 \in M$. To cover v_1, $v_1v_5 \in M$. Now, M is automorphic to M_4. By symmetry, if the single outer edge in M is v_5v_6 then M is automorphic to M_4. If the single outer edge is v_7v_8, then $v_3v_9 \in M$ to cover v_3. To cover v_2, $v_2v_6 \in M$. To cover v_1, $v_1v_5 \in M$. Now, M is automorphic to M_4. By symmetry, if the single outer edge in M is v_6v_7 then M is automorphic to M_4.

Suppose M has five inner edges. To cover v_3, either $v_3v_7 \in M$ or $v_3v_9 \in M$. If the former is true, then $v_5v_9 \in M$ to cover v_9. To cover v_8, $v_2v_8 \in M$. No edge covers both v_1 and v_6, a contradiction. If the latter is true, then $v_2v_8 \in M$ to cover v_8. To cover v_7, $v_1v_7 \in M$. Now, v_5 and v_6 cannot be covered by an inner edge, a contradiction. Thus, M cannot have five inner edges.

Table 1. f, g, and h are automorphisms

uv	$f(u)f(v)$	$g(u)g(v)$	$h(u)h(v)$
v_1v_2	v_1v_2	v_1v_2	v_1v_2
v_2v_3	v_2v_8	v_2v_3	v_2v_8
v_3v_4	v_8v_9	v_3v_9	v_8v_4
v_4v_5	v_9v_5	v_9v_5	v_4v_{10}
v_5v_6	v_5v_6	v_5v_6	$v_{10}v_6$
v_6v_7	v_6v_7	v_6v_7	v_6v_7
v_7v_8	v_7v_3	v_7v_8	v_7v_3
v_8v_9	v_3v_4	v_8v_4	v_3v_9
v_9v_{10}	v_4v_{10}	v_4v_{10}	v_9v_5
$v_{10}v_1$	$v_{10}v_1$	$v_{10}v_1$	v_5v_1
v_1v_5	v_1v_5	v_1v_5	v_1v_{10}
v_5v_9	v_5v_4	v_5v_4	$v_{10}v_9$
v_9v_3	v_4v_8	v_4v_3	v_9v_8
v_3v_7	v_8v_7	v_3v_7	v_8v_7
v_7v_1	v_7v_1	v_7v_1	v_7v_1
v_2v_6	v_2v_6	v_2v_6	v_2v_6
v_6v_{10}	v_6v_{10}	v_6v_{10}	v_6v_5
$v_{10}v_4$	$v_{10}v_9$	$v_{10}v_9$	v_5v_4
v_4v_8	v_9v_3	v_9v_8	v_4v_3
v_8v_2	v_3v_2	v_8v_2	v_3v_2

Now we prove that M_1 is automorphic to M_2, M_3, and M_4, respectively. For M_1 and M_2, we define $f : V(G) \to V(G)$ such that $f(v_i) = v_i$ if $i = 1, 2, 5, 6, 7, 10$, $f(v_3) = v_8$, $f(v_4) = v_9$,

$f(v_8) = v_3$, and $f(v_9) = v_4$. To prove that f is an automorphism, Table 1 shows that f preserves all 20 edges (the edges in the second column form $E(G)$). Also, f maps M_1 to M_2, that is, $f_e(M_1) = M_2$.

For M_1 and M_3, we define $g : V(G) \to V(G)$ such that $g(v_i) = v_i$ if $i = 1, 2, 3, 5, 6, 7, 8, 10$, $g(v_4) = v_9$, and $g(v_9) = v_4$. To prove that g is an automorphism, Table 1 shows that g preserves all 20 edges (the edges in the third column form $E(G)$). Also, g maps M_1 to M_3, that is, $g_e(M_1) = M_3$.

For M_1 and M_4, we define $h : V(G) \to V(G)$ such that $h(v_i) = v_i$ if $i = 1, 2, 4, 6, 7, 9$, $h(v_3) = v_8$, $h(v_5) = v_{10}$, $h(v_8) = v_3$, and $h(v_{10}) = v_5$. To prove that h is an automorphism, Table 1 shows that h preserves all 20 edges (the edges in the third column form $E(G)$). Also, h maps M_1 to M_4, that is, $h_e(M_1) = M_4$.

Therefore, G is PM-transitive.

Case 3. $G \cong Ci_{10}(2, 5)$ is not PM-transitive.

Let $M_1 = \{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\}$ and $M_2 = \{v_1v_3, v_5v_7, v_4v_9, v_2v_{10}, v_6v_8\}$. Then M_1 and M_2 are two perfect matchings of G such that $G - M_1$ is a union of two 5-cycles while $G - M_2$ is a union of a 4-cycle and a 6-cycle. Therefore, G is not PM-transitive.

Case 4. $G \cong Ci_{10}(1, 2, 3)$ is not PM-transitive.

Let $M_1 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$ and $M_2 = \{v_1v_4, v_7v_{10}, v_3v_6, v_9v_2, v_5v_8\}$. Then M_1 and M_2 are two perfect matchings of G. Suppose that f is an automorphism $f : V(G) \to V(G)$ such that $f_e(M_1) = M_2$. Without loss of generality, let $f(v_1) = v_1$. This forces $f(v_2) = v_4$.

Consider v_{10}. If $f(v_{10}) = v_2$, then this forces $f(v_9) = v_9$. Now $v_2v_9 \in E(G)$ and $v_4v_9 \notin E(G)$, contradicting the supposition that f is an automorphism. If $f(v_{10}) = v_3$, then this forces $f(v_9) = v_6$. Now $v_1v_9 \in E(G)$ and $v_1v_6 \notin E(G)$, a contradiction. If $f(v_{10}) = v_5$, then $v_1v_{10} \in E(G)$ and $v_1v_5 \notin E(G)$, a contradiction. If $f(v_{10}) = v_6$, then $v_1v_{10} \in E(G)$ and $v_1v_6 \notin E(G)$, a contradiction. If $f(v_{10}) = v_7$, then $v_1v_{10} \in E(G)$ and $v_1v_7 \notin E(G)$, a contradiction. If $f(v_{10}) = v_8$, then $v_2v_{10} \in E(G)$ and $v_4v_8 \notin E(G)$, a contradiction. If $f(v_{10}) = v_9$, then $v_2v_{10} \in E(G)$ and $v_4v_9 \notin E(G)$, a contradiction. If $f(v_{10}) = v_{10}$, then $v_2v_{10} \in E(G)$ and $v_4v_{10} \notin E(G)$, a contradiction. Therefore, G is not PM-transitive.

Case 5. $G \cong Ci_{10}(1, 2, 4)$ is not PM-transitive.

Let $M_1 = \{v_1v_5, v_2v_8, v_3v_9, v_4v_{10}, v_6v_7\}$ and $M_2 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$. Then M_1 and M_2 are two perfect matchings of G. Suppose that f is an automorphism $f : V(G) \to V(G)$ such that $f_e(M_1) = M_2$. Without loss of generality, let $f(v_1) = v_1$. This forces $f(v_5) = v_2$.

Consider v_7. Since $v_1v_7, v_5v_7 \in E(G)$, it is the case that $f(v_1)f(v_7) = v_1f(v_7) \in E(G)$ and $f(v_5)f(v_7) = v_2f(v_7) \in E(G)$. Since v_3 and v_{10} are the only vertices adjacent to both v_1 and v_2, the image of v_7 is either v_3 or v_{10}.

If $f(v_7) = v_3$, then this forces $f(v_6) = v_4$. Now, any proposed preimage of v_{10} will contradict the fact that f is an automorphism. Specifically, if $f(v_2) = v_{10}$, then $v_2v_5 \notin E(G)$ contradicts $v_{10}v_2 \in E(G)$. If $f(v_3) = v_{10}$, then $v_3v_6 \notin E(G)$ contradicts $v_{10}v_4 \in E(G)$. If $f(v_4) = v_{10}$, then $v_4v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_8) = v_{10}$, then $v_8v_5 \notin E(G)$ contradicts $v_{10}v_2 \in E(G)$. If $f(v_9) = v_{10}$, then $v_9v_6 \notin E(G)$ contradicts $v_{10}v_4 \in E(G)$. If $f(v_{10}) = v_{10}$, then $v_{10}v_5 \notin E(G)$ contradicts $v_{10}v_2 \in E(G)$.

If $f(v_7) = v_{10}$, then this forces $f(v_6) = v_9$. Now, any proposed preimage of v_3 will contradict the fact that f is an automorphism. Specifically, if $f(v_2) = v_3$, then $v_5v_2 \notin E(G)$ contradicts $v_2v_3 \in E(G)$. If $f(v_3) = v_3$, then $v_7v_3 \in E(G)$ contradicts $v_{10}v_3 \notin E(G)$. If $f(v_4) = v_3$, then

$v_1v_4 \notin E(G)$ contradicts $v_1v_3 \in E(G)$. If $f(v_8) = v_3$, then $v_1v_8 \notin E(G)$ contradicts $v_1v_3 \in E(G)$. If $f(v_9) = v_3$, then $v_7v_9 \in E(G)$ contradicts $v_{10}v_3 \notin E(G)$. If $f(v_{10}) = v_3$, then $v_5v_{10} \notin E(G)$ contradicts $v_2v_3 \in E(G)$. Therefore, G is not PM-transitive.

Case 6. $G \cong Ci_{10}(1, 2, 5)$ is not PM-transitive.

Let $M_1 = \{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\}$ and $M_2 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$. Then M_1 and M_2 are two perfect matchings of G. Suppose that f is an automorphism $f : V(G) \to V(G)$ such that $f_e(M_1) = M_2$. Without loss of generality, let $f(v_1) = v_1$. This forces $f(v_6) = v_2$.

Now, any proposed preimage of v_{10} will contradict the fact that f is an automorphism. Specifically, if $f(v_2) = v_{10}$, then $v_2v_6 \notin E(G)$ contradicts $v_{10}v_2 \in E(G)$. If $f(v_3) = v_{10}$, then $v_3v_6 \notin E(G)$ contradicts $v_{10}v_2 \in E(G)$. If $f(v_4) = v_{10}$, then $v_4v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_5) = v_{10}$, then $v_5v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_7) = v_{10}$, then $v_7v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_8) = v_{10}$, then $v_8v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_9) = v_{10}$, then $v_9v_6 \notin E(G)$ contradicts $v_{10}v_2 \in E(G)$. If $f(v_{10}) = v_{10}$, then $v_{10}v_6 \notin E(G)$ contradicts $v_{10}v_2 \in E(G)$. Therefore, G is not PM-transitive.

Case 7. $G \cong Ci_{10}(1, 4, 5)$ is not PM-transitive.

Let $M_1 = \{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\}$ and $M_2 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$. Then M_1 and M_2 are two perfect matchings of G. Suppose that f is an automorphism $f : V(G) \to V(G)$ such that $f_e(M_1) = M_2$. Without loss of generality, let $f(v_1) = v_1$. This forces $f(v_6) = v_2$.

Now, any proposed preimage of v_{10} will contradict the fact that f is an automorphism. Specifically, if $f(v_2) = v_{10}$, then $v_2v_6 \in E(G)$ contradicts $v_{10}v_2 \notin E(G)$. If $f(v_3) = v_{10}$, then $v_3v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_4) = v_{10}$, then $v_4v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_5) = v_{10}$, then $v_5v_6 \in E(G)$ contradicts $v_{10}v_2 \notin E(G)$. If $f(v_7) = v_{10}$, then $v_7v_6 \in E(G)$ contradicts $v_{10}v_2 \notin E(G)$. If $f(v_8) = v_{10}$, then $v_8v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_9) = v_{10}$, then $v_9v_1 \notin E(G)$ contradicts $v_{10}v_1 \in E(G)$. If $f(v_{10}) = v_{10}$, then $v_{10}v_6 \in E(G)$ contradicts $v_{10}v_2 \notin E(G)$. Therefore, G is not PM-transitive.

Case 8. $G \cong Ci_{10}(2, 4, 5)$ is not PM-transitive.

Let $M_1 = \{v_1v_3, v_2v_7, v_4v_8, v_5v_9, v_6v_{10}\}$ and $M_2 = \{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\}$. Then M_1 and M_2 are two perfect matchings of G. Suppose that f is an automorphism $f : V(G) \to V(G)$ such that $f_e(M_1) = M_2$. Without loss of generality, let $f(v_1) = v_1$. This forces $f(v_3) = v_6$.

Consider v_7. Since $v_1v_7, v_3v_7 \in E(G)$, it is the case that $f(v_1)f(v_7) = v_1f(v_7) \in E(G)$ and $f(v_3)f(v_7) = v_6f(v_7) \in E(G)$. Since there are no vertices adjacent to both v_1 and v_6, this is a contradiction. Therefore, G is not PM-transitive.

Case 9. $G \cong Ci_{10}(1, 2, 3, 5)$ is not PM-transitive.

Let $M_1 = \{v_1v_3, v_2v_4, v_5v_6, v_7v_8, v_9v_{10}\}$ and $M_2 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$. Then M_1 and M_2 are two perfect matchings of G. Suppose that f is an automorphism $f : V(G) \to V(G)$ such that $f_e(M_1) = M_2$. Without loss of generality, let $f(v_1) = v_1$. This forces $f(v_3) = v_2$.

Consider v_7. Since $v_1v_7, v_3v_7 \notin E(G)$, it is the case that $f(v_1)f(v_7) = v_1f(v_7) \notin E(G)$ and $f(v_3)f(v_7) = v_2f(v_7) \notin E(G)$. Since every vertex is adjacent to at least one of v_1 and v_2, this is a contradiction. Therefore, G is not PM-transitive.

Case 10. $G \cong Ci_{10}(1, 2, 4, 5)$ is not PM-transitive.

Let $M_1 = \{v_1v_5, v_2v_8, v_3v_9, v_4v_{10}, v_6v_7\}$ and $M_2 = \{v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}\}$. Then M_1 and M_2 are two perfect matchings of G. Suppose that f is an automorphism $f : V(G) \to V(G)$ such that $f_e(M_1) = M_2$. Without loss of generality, let $f(v_1) = v_1$. This forces $f(v_5) = v_2$.

Consider v_7. Since $v_1v_7, v_5v_7 \in E(G)$, it is the case that $f(v_1)f(v_7) = v_1f(v_7) \in E(G)$ and $f(v_5)f(v_7) = v_2f(v_7) \in E(G)$. Since v_3, v_6, v_7, and v_{10} are the only vertices adjacent to both v_1 and v_2, the image of v_7 is either v_3, v_6, v_7, or v_{10}. If $f(v_7) = v_3$, then this forces $f(v_6) = v_4$. Now, $v_1v_6 \in E(G)$ contradicts $v_1v_4 \notin E(G)$. If $f(v_7) = v_6$, then this forces $f(v_6) = v_5$. Now, $v_5v_6 \in E(G)$ contradicts $v_2v_5 \notin E(G)$. If $f(v_7) = v_7$, then this forces $f(v_6) = v_8$. Now, $v_1v_6 \in E(G)$ contradicts $v_1v_8 \notin E(G)$. If $f(v_7) = v_{10}$, then this forces $f(v_6) = v_9$. Now, $v_5v_6 \in E(G)$ contradicts $v_2v_9 \notin E(G)$. Therefore, G is not PM-transitive.

In summary, the connected PM-transitive circulant graphs of order 10 are congruent to $Ci_{10}(1)$, $Ci_{10}(1,4) \cong Ci_{10}(2,3)$, $Ci_{10}(1,3,5)$, or $Ci_{10}(1,2,3,4,5)$. That is to say, the connected PM-transitive circulant graphs of order 10 are congruent to C_{10}, $Ci_{10}(1,4) \cong Ci_{10}(2,3)$, $K_{5,5}$, or K_{10}. □

Acknowledgement

The authors are grateful for the support of the FPDC Grant in conducting this research.

References

[1] B.R. Alspach, The classification of Hamiltonian generalized Petersen graphs, *Journal of Combinatorial Theory, Series B* **34** (1983), 293-312. https://www.sciencedirect.com/science/article/pii/0095895683900424

[2] L. Babai, Automorphism groups, isomorphism, reconstruction, *Handbook of Combinatorics* (1996).

[3] Z. Bouwer, Vertex and edge-transitive, but not 1-transitive graphs, *Canadian Mathematical Bulletin* **13** (1970), 231-237. https://www.cambridge.org/core/journals/canadian-mathematical-bulletin/article/vertex-and-edge-transitive-but-not-1transitive-graphs/A8D20CEBB3C8D150712B72040A422757

[4] N. Biggs, Algebraic Graph Theory (2nd ed.), Cambridge: Cambridge University Press., (1993), 118. https://www.cambridge.org/core/books/algebraic-graph-theory/6C70471342F19680068C35EF174075DC

[5] M. Conder, Trivalent symmetric graphs on up to 768 vertices, *Journal of Combinatorial Mathematics and Combinatorial Computing* **20** (2000), 41-63. https://www.researchgate.net/publication/2617744_Trivalent_Symmetric_Graphs_On_Up_To_768_Vertices

[6] R. Diestel and I. Leader, A conjecture concerning a limit of non-Cayley graphs, *Journal of Algebraic Combinatorics* **14**(1) (2001), 17-25. https://link.springer.com/article/10.1023/A%3A1011257718029

[7] A. Eskin, D. Fisher, and K. Whyte, Quasi-isometries and rigidity of solvable groups, 2005. https://arxiv.org/abs/math/0511647

[8] R.M. Foster, Geometrical circuits of electrical networks, *Transactions of the American Institute of Electrical Engineers* **51** (1932), 309-317.
https://ieeexplore.ieee.org/document/5056068

[9] R.M. Foster, I.Z. Bouwer, W.W. Chernoff, B. Monson, and Z. Star, The Foster Census: R.M. Foster's Census of Connected Symmetric Trivalent Graphs, 1988.
https://search.library.wisc.edu/catalog/999595788402121

[10] R. Frucht, J.E. Graver, and M.E. Watkins, The groups of the generalized Petersen graphs, *Proc. Cambridge Philos. Soc.* **70** (1971), 211-218.
https://www.cambridge.org/core/journals/mathematical-proceedings-of-the-cambridge-philosophical-society/article/abs/groups-of-the-generalized-petersen-graphs/33EC7C273E8897088CEE71060A3659CD

[11] C. Godsil and G. Royle, Algebraic Graph Theory, *Graduate Texts in Mathematics,* New York: Springer-Verlag (2001), 207.
https://link.springer.com/book/10.1007/978-1-4613-0163-9

[12] J.L. Gross and J. Yellen, Handbook of Graph Theory, CRC Press. (2004), 491.
https://www.routledge.com/Handbook-of-Graph-Theory/Gross-Yellen-Zhang/p/book/9781439880180

[13] D.F. Holt, A graph which is edge-transitive but not arc-transitive, *Journal of Graph Theory* **5**(2) (1981), 201-204.
https://onlinelibrary.wiley.com/doi/10.1002/jgt.3190050210

[14] D.A. Holton and J. Sheehan, Generalized Petersen and permutation graphs, §9.13 in *The Petersen Graph*, Cambridge, England: Cambridge University Press (1993), 315-317.
https://www.abebooks.com/9780521435949/Petersen-Graph-Australian-Mathematical-Society-0521435943/plp

[15] J. Lauri and R. Scapellato, Topics in graph automorphisms and reconstruction, *London Mathematical Society Lecture Note Series*, Cambridge University Press (2003), 20-21.
https://www.cambridge.org/core/books/topics-in-graph-automorphisms-and-reconstruction/3699EDC473353370EFA6FD93FA26A160

[16] P. Potočnik, P. Spiga, and G. Verret, Cubic vertex-transitive graphs on up to 1280 vertices, *Journal of Symbolic Computation* **50** (2013), 465-477.
https://www.sciencedirect.com/science/article/pii/S0747717112001563

[17] Z. Ryjáček, On a closure concept in claw-free graphs, *Journal of Combinatorial Theory, Series B* **70**(2) (1997), 217-224.
https://www.sciencedirect.com/science/article/pii/S0095895696917323

[18] J. Zhou, Characterization of perfect matching transitive graphs, *Electronic Journal of Graph Theory and Applications* **6**(2) (2018), 362-369.
https://www.ejgta.org/index.php/ejgta/article/view/583