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On b-edge consecutive edge magic total labeling on trees

Eunike Setiawan[a], Kiki Ariyanti Sugeng[a,b], Denny Riama Silaban[a,b]

[a]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Indonesia, Indonesia*
[b]*Center for Research Collaboration on Graph Theory and Combinatorics, Indonesia*

{ike, kiki, denny}@sci.ui.ac.id

Abstract

Let $G = (V, E)$ be a simple, connected, and undirected graph, where V and E are the set of vertices and the set of edges of G. An edge magic total labeling on G is a bijection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$, provided that for every $uv \in E, w(uv) = f(u) + f(v) + f(uv) = K$ for a constant number K. Such a labeling is said to be a super edge magic total labeling if $f(V) = \{1, 2, \ldots, |V|\}$ and a b-edge consecutive edge magic total labeling if $f(E) = \{b+1, b+2, \ldots, b+|E|\}$ with $b \geq 1$. In this research, we give sufficient conditions for a graph G having a super edge magic total labeling to have a b-edge consecutive edge magic total labeling. We also give several classes of connected graphs which have both labelings.

Keywords: super edge magic total labeling, b-edge consecutive edge magic total labeling, tree graph
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1. Introduction

All graphs which are considered in this paper are simple, undirected, and connected. Let $G = (V, E)$ be a graph. An edge magic total labeling (EMTL), introduced in 1970 by Kotzig and Rosa [6], is a bijective mapping $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that the edge weight $w(uv) = f(u) + f(v) + f(uv)$ is constant for all $uv \in E$. This is a variation of magic labeling that was first introduced by Sedláček in 1963 [9].

In 1998, Enomoto et al. [1] introduced the super edge magic total labeling (SEMTL), which is an EMTL with the additional property that $f(V) = \{1, 2, \ldots, |V|\}$. Graphs that meet this labeling

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are called super edge magic total (SEMT) graphs. They also conjectured that every tree is SEMT. In 2008, Sugeng and Miller [11] observed that the consecutive edge labels do not have to start from 1 but can be shifted (such that they start) from a number $b+1$ with $b \geq 1$, thus introducing another variation of EMTL called the b-edge consecutive edge magic total labeling (b-ECEMTL). Graphs that meet this labeling are called b-edge consecutive edge magic total (b-ECEMT) graphs. In the b-ECEMTL, the vertex set of the graph will be partitioned into two subsets, namely the $\{1, 2, \ldots, b\}$ labeled subset and $\{b+|E|+1, b+|E|+2, \ldots, |V|+|E|\}$ labeled subset. This fact suggests that the b-ECEMT graph is a bipartite graph. However, Sugeng and Miller [11] already proved that for connected graphs, a b-ECEMT graph must be a tree. Sugeng and Silaban provide b-edge consecutive edge magic total labeling for some classes of regular trees [12] and disconnected graphs [10]. For graphs that don't admit SEMTL, Ngurah and Simanjuntak [7, 8] add isolated vertices such that the graph admits SEMTL. The number of isolated vertices added is called the super edge magic deficiency. Further results in b-ECEMTL and other variations of edge magic total labeling can be seen at [2].

The SEMTL can be viewed as the 0-ECEMTL, so it is intuitive that there is a relationship between the SEMT graphs and the b-ECEMT graphs. In this study, we give the sufficient condition for a SEMT graph to be a b-ECEMT graph for $b \geq 1$. The b-ECEMTL of families of graphs that meet this condition is given as a result of these sufficient conditions. In addition, we also show the b-ECEMTL of other families of trees that have not been proven to be SEMT graphs. In the Main Results, we will start the discussion by showing the SEMTL of those graphs.

2. Preliminary

The following results will be used in this paper.

Theorem 2.1. *[6] A caterpillar $G \cong S_{n_1,n_2,\ldots,n_k}$ admits a SEMTL.*

Theorem 2.2. *[3] A banana tree $G \cong BT(n_1, n_2, \ldots, n_k)$ with $n_1 = n_2 = \cdots = n_k = n$, $\lceil \frac{k}{2} \rceil \leq n \leq k-1$ admits a SEMTL.*

Theorem 2.3. *[3] A banana tree $G \cong BT(n_1, n_2, \ldots, n_k)$ with $n_i > n_{i+1}$ for each $1 \leq i \leq k$ admits a SEMTL.*

Theorem 2.4. *[4] A generalized comb $G \cong Cb_k(n_1, n_2, \ldots, n_k)$ with $n_1 = n_2 = \cdots = n_k = n$, $n \geq 2$, and $k \geq 2$ admits a SEMTL.*

Theorem 2.5. *[4] A generalized comb $G \cong Cb_k(2, 3, \ldots, k+1)$ admits a SEMTL.*

Theorem 2.6. *[5] For a connected bipartite graph G with partite sets X and Y, exactly one of the following is true:*

1. *G does not have a b-ECEMTL for any b;*
2. *G has only 0-ECEMTL and SEMTL;*
3. *G is a tree having a b-ECEMTL for each $b = 0$, $|X|$, $|Y|$, $|X+Y|$.*

3. Main Results

3.1. New SEMT Graphs

We will give the new results regarding SEMTL for banana tree and firecracker. Hussain, Baskoro, and Slamin (2009) proved that a banana tree graph $BT(n_1, n_2, \ldots, n_k)$ with some restrictions has SEMTL. See Theorem 2.3 for the result. In Theorem 3.1, we generalize their result for banana tree graph $BT(n_1, n_2, \ldots, n_k)$ which is formed by joining the vertex v_0 with exactly one leaf of a sequence of stars with order greater than or equal to the sequence index.

Theorem 3.1. *A banana tree $G \cong BT(n_1, n_2, \ldots, n_k)$ with $n_i \geq i$ for each $i \in \{1, 2, \ldots, k\}$ admits a SEMTL.*

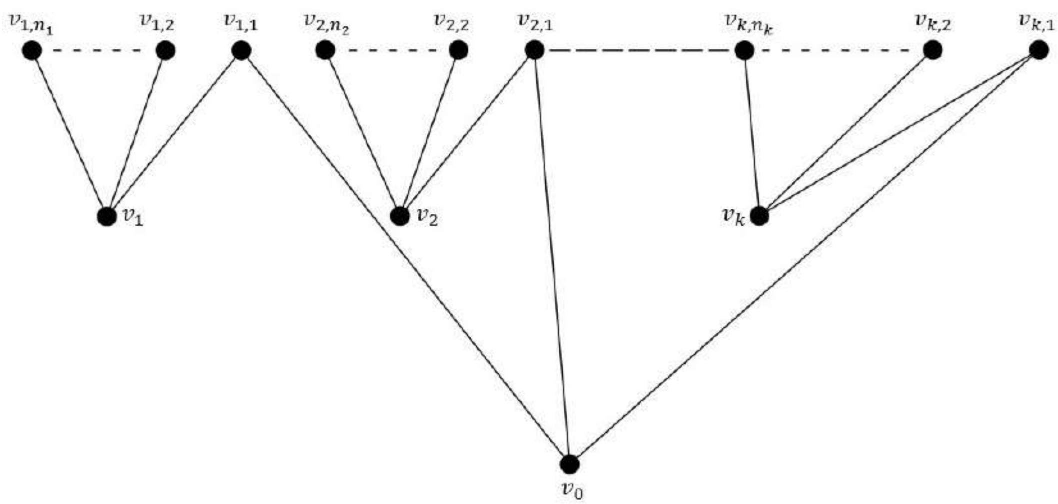

Figure 1. Vertices naming on $BT(n_1, n_2, \ldots, n_k)$.

Proof. Banana tree $BT(n_1, n_2, \ldots, n_k)$ has vertex set $V = \{v_i | 0 \leq i \leq k\} \cup (\bigcup_{i=1}^{k} \{v_{i,m} | 1 \leq m \leq n_i\})$ and edge set $E = \{v_i v_{i,m} | 1 \leq i \leq k, 1 \leq m \leq n_i\} \cup \{v_0 v_{i,1} | 1 \leq i < k\}$. An illustration of a banana tree is shown in Figure 1.

Suppose $\alpha_i = \sum_{j=1}^{i-1} n_j$ and $\sum_{i=1}^{j} n_i = 0$ if $j < 1$. Define $f : V \cup E \to \{1, \ldots, |V| + |E|\}$ as follows

$$
\begin{aligned}
f(v_i) &= \alpha_{k+1} + i + 1, \text{ for } 0 \leq i \leq k, \\
f(v_{i,1}) &= \alpha_i + i, \text{ for } 1 \leq i \leq k, \\
f(\{v_{i,m} | 2 \leq m \leq n_i\}) &= \{\alpha_i + 1, \alpha_i + 2, \ldots, \alpha_i + n_i\} \text{ except for } \{\alpha_i + i\}, \text{ for } 1 \leq i \leq k, \\
f(v_0 v_{i,m}) &= |V| + |E| - (\alpha_i + (i - 1)), \text{ for } 1 \leq i \leq k, \\
f(v_i v_{i,m}) &= |V| + |E| - (f(v_{i,m}) + (i - 1)), \text{ for } 1 \leq i \leq k \text{ and } 1 \leq m \leq n_i.
\end{aligned}
$$

From the definition of f given, it is clear that $f : V \cup E \to \{1, \ldots, |V| + |E|\}$ is a bijection with $f(V) = \{1, 2, \ldots, |V|\}$. Next, we prove that each edge in G has the same weight. Take an arbitrary $xy \in E$ and we show that $w(xy) = f(x) + f(y) + f(xy) = K$ for $K \in \mathbb{N}$. Suppose

$xy = v_i v_{j,m} \in E$, then from the definition of f, the weight of xy can be calculated by considering two cases as follows.

Case 1. $i = 0, m = 1, j \in \{1, 2, \ldots, k\}$

$$\begin{aligned}
w(v_0v_{j,1}) &= f(v_0) + f(v_0v_{j,1}) + f(v_{j,1}) \\
&= \left(\sum_{i=1}^{k} n_i + 0 + 1\right) + \left(|V| + |E| - \left(\sum_{i=1}^{j-1} n_i + (j-1)\right)\right) + \left(\sum_{i=1}^{j-1} n_i + j\right) \\
&= \sum_{i=1}^{k} n_i + 2 + |V| + |E| \\
&= (|V| - k - 1 + 2) + (2|V| - 1) \\
&= 3|V| - k.
\end{aligned}$$

Case 2. $j \in \{1, 2, \ldots, k\}, m \in \{1, 2, \ldots, n_j\}$

$$\begin{aligned}
w(v_jv_{j,m}) &= f(v_j) + f(v_jv_{j,m}) + f(v_{j,m}) \\
&= \left(\sum_{i=1}^{k} n_i + j + 1\right) + (|V| + |E| - (f(v_{j,m}) + (j-1))) + f(v_{j,m}) \\
&= \sum_{i=1}^{k} n_i + 2 + |V| + |E| \\
&= (|V| - k - 1 + 2) + (2|V| - 1) \\
&= 3|V| - k.
\end{aligned}$$

From the results of both cases above, it has been shown that G admits SEMT labeling with magic constant $3|V| - k$. □

In Figure 2 we give an example of SEMTL on $BT(2, 4, 3, 4)$ with $f(V) = \{1, 2, \ldots, 18\}$, $f(E) = \{19, 20, \ldots, 35\}$, and magic constant $K = 50$.

In Theorem 3.2, we show that a firecracker $G \cong F(n_1, n_2, \ldots, n_k)$, formed by joining one leaf from a sequence of stars with increasing order, has SEMTL.

Theorem 3.2. *A firecracker $G \cong F(n_1, n_2, \ldots, n_k)$ with $n_i \geq n_{i+1}$ for each $i \in \{1, 2, \ldots, k-1\}$ admits a SEMTL.*

Proof. From the definition of firecracker, it is obtained that this graph has the vertex set $V = \{v_i | 1 \leq i \leq k\} \cup \{v_{i,j} | 1 \leq i \leq k, 1 \leq j \leq n_i\}$ and the edge set $E = \{v_iv_{i,j} | 1 \leq i \leq k, 1 \leq j \leq n_i\} \cup \{v_{i,1}v_{i+1,1} | 1 \leq i \leq k-1\}$. The naming for the vertices and edges on the firecracker can be seen in Figure 3.

Suppose A_i is the summation of all n_j where j is even and $j \leq i$, B_i is the summation of all n_j where j is odd and $j \leq i$, and C_i is the summation of all n_j where $j \leq i$. Then, $A_i = n_2 + n_4 + \cdots + n_t$ for even $t \leq i$, $B_i = n_1 + n_3 + \cdots + n_s$ for odd $s \leq i$, and $C_i = n_1 + n_2 + n_3 + \cdots + n_i$. Suppose that $K = |V| + |E| + A_k + \lceil k/2 \rceil + 2$. Define $f : V \cup E \to \{1, ..., |V| + |E|\}$ as follows

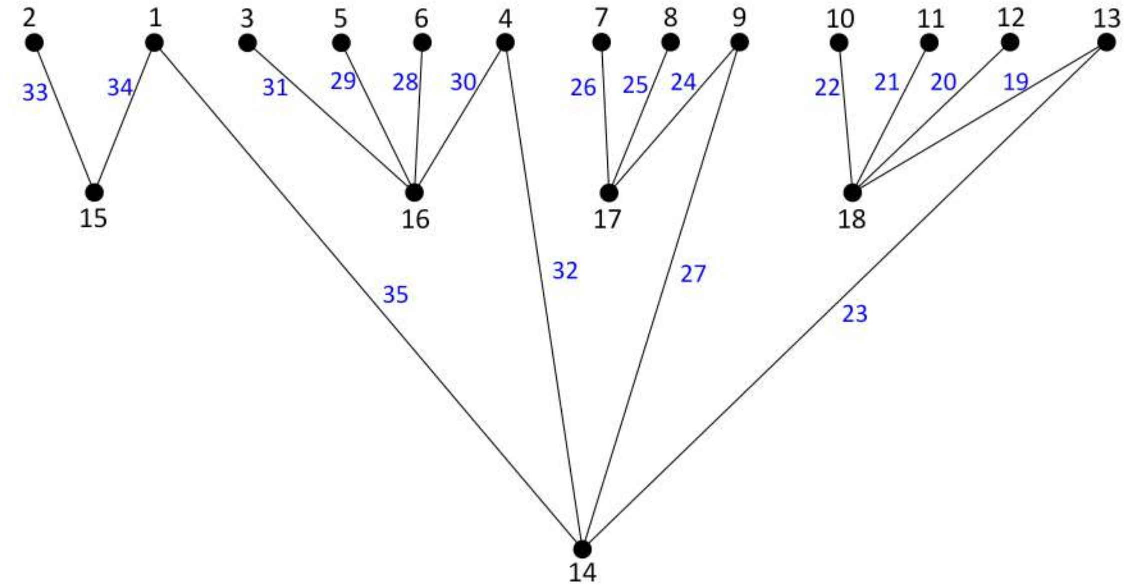

Figure 2. SEMTL on $BT(2,4,3,4)$.

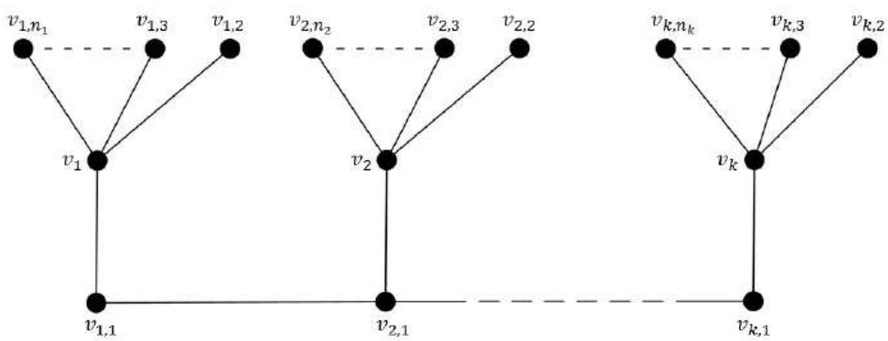

Figure 3. Vertices naming on $F(n_1, n_2, \ldots, n_k)$.

$$f(v_i) = \begin{cases} A_i + \frac{i+1}{2}, & \text{for odd } i, \\ B_i + \frac{i}{2} + \left(A_k + \left\lceil \frac{k}{2} \right\rceil\right), & \text{for even } i, \end{cases}$$

$$f(v_{i,1}) = \begin{cases} f(v_2) - 1, & \text{for } i = 1, \\ f(v_{i-1}) + \left(f(v_i) - f(v_{i-1,1})\right), & \text{for } i \neq 1. \end{cases}$$

$$f\left(\{v_{1,j} | 2 \leq j \leq n_1\}\right) = \{f(v_2) - 2, f(v_2) - 3, \ldots, f(v_2) - n_1\},$$

$$f\left(\{v_{i,j} | 2 \leq j \leq n_1\}\right) = \{f(v_{i-1}) + 1, f(v_{i-1}) + 2, \ldots, f(v_{i-1}) + n_i\} \text{except for} \{f(v_{i,1})\},$$
$$\text{for } 2 \leq i \leq k,$$

$$f(v_{i,1}v_{i+1,1}) = |V| + |E| - (C_i + i - 1), \text{ for } 1 \leq i \leq k-1,$$

$$f(v_i v_{i,j}) = K - (f(v_i) + f(v_{i,j})), \text{ for } 1 \leq i \leq k \text{ and } 1 \leq j \leq n_i.$$

From the definition of f, it is clear that $f : V \cup E \to \{1, \dots, |V| + |E|\}$ is a bijection with $f(V) = \{1, 2, \dots, |V|\}$. Now, we prove that every edge in G has the same weight. Let $xy \in E$ and we will show that $w(xy) = f(x) + f(y) + f(xy)$ equals to a constant. This proof will also be divided into two cases.

Case 1. $xy = v_i v_{i,j}$
From the definition of f, we have that $w(v_i v_{i,j}) = f(v_i) + f(v_i v_{i,j}) + f(v_{i,j}) = f(v_i) + (K - (f(v_i) + f(v_{i,j}))) + f(v_{i,j}) = K$.

Case 2. $xy = v_{i,1} v_{i+1,1}$
From the definition of f, we have that $w(v_{i,1} v_{i+1,1}) = f(v_{i,1}) + f(v_{i,1} v_{i+1,1}) + f(v_{i+1,1}) = f(v_{i,1}) + f(v_{i,1} v_{i+1,1}) + (f(v_i) + (f(v_{i+1}) - f(v_{i,1}))) = f(v_{i,1} v_{i+1,1}) + f(v_i) + f(v_{i+1})$. Notice that $|E| = |V| - 1$ and $|V| = C_k + k$. If i is odd, then

$$\begin{aligned}
w(v_{i,1} v_{i+1,1}) &= f(v_{i,1} v_{i+1,1}) + f(v_i) + f(v_{i+1}) \\
&= (|V| + |E| - (C_i + i - 1)) + \left(A_i + \frac{i+1}{2}\right) + \left(B_{i+1} + \frac{i+1}{2} + \left(A_k + \left\lceil \frac{k}{2} \right\rceil\right)\right) \\
&= |V| + |E| + ((1 - i) + (i + 1)) + (A_i + B_{i+1} - C_i) + (A_k + \lceil k/2 \rceil) \\
&= |V| + |E| + 2 + (A_k + \lceil k/2 \rceil) + (A_i + B_{i+1} - C_i) \\
&= K + ((n_2 + n_4 + \cdots + n_{i-1}) + (n_1 + n_3 + \cdots + n_i) - C_i) \\
&= K + (C_i - C_i) \\
&= K.
\end{aligned}$$

If i is even, then

$$\begin{aligned}
w(v_{i,1} v_{i+1,1}) &= f(v_{i,1} v_{i+1,1}) + f(v_i) + f(v_{i+1}) \\
&= (|V| + |E| - (C_i + i - 1)) + \left(B_i + \frac{i}{2} + \left(A_k + \left\lceil \frac{k}{2} \right\rceil\right)\right) + \left(A_{i+1} + \frac{i+2}{2}\right) \\
&= |V| + |E| + ((1 - i) + (i + 1)) + (A_{i+1} + B_i - C_i) + (A_k + \lceil k/2 \rceil) \\
&= |V| + |E| + 2 + (A_k + \lceil k/2 \rceil) + (A_{i+1} + B_i - C_i) \\
&= K + ((n_2 + n_4 + \cdots + n_i) + (n_1 + n_3 + \cdots + n_{i-1}) - C_i) \\
&= K + (C_i - C_i) \\
&= K.
\end{aligned}$$

From both cases above, it is proven that G has SEMTL with magic constant $K = |V| + |E| + A_k + \left\lceil \frac{k}{2} \right\rceil + 2$. □

We give the example of SEMTL for $F(2, 3, 3, 4, 4)$ with $f(V) = \{1, 2, \dots, 21\}$, $f(E) = \{22, 23, \dots, 41\}$, and the magic constant $K = 53$ in Figure 4 as well as SEMTL for $F(3, 4, 5, 5)$ with $f(V) = \{1, 2, \dots, 21\}$, $f(E) = \{22, 23, \dots, 41\}$, and the magic constant $K = 54$ in Figure 5.

3.2. Sufficient Conditions for SEMT Graphs to be b-ECEMT

Kang, Kim, and Park [5] have provided the sufficient condition for a graph that have b-ECEMTL to have SEMTL. In Theorem 3.3, we will give the opposite, which is the sufficient condition for a

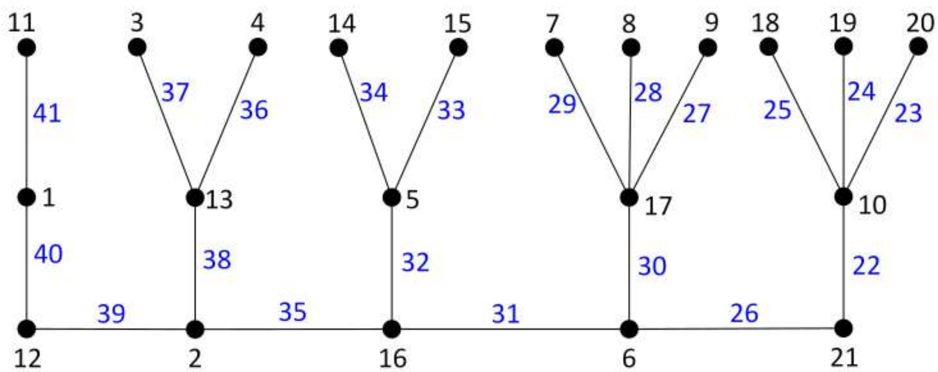

Figure 4. SEMTL on $F(2, 3, 3, 4, 4)$.

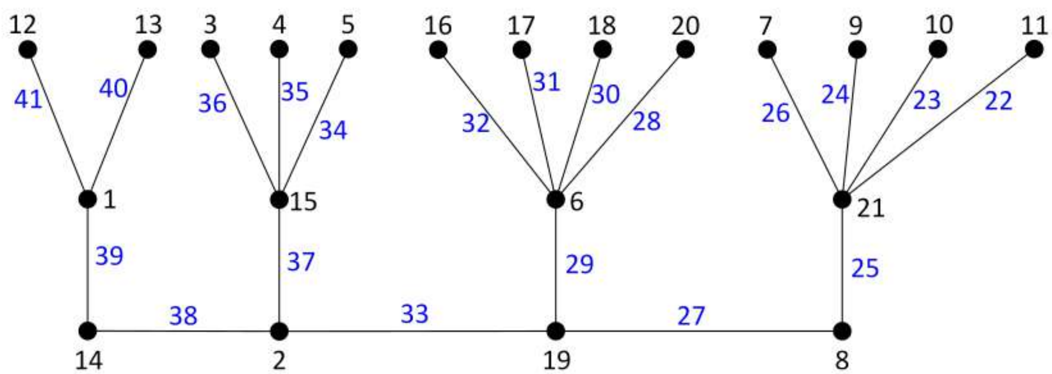

Figure 5. SEMTL on $F(3, 4, 5, 5)$.

graph that has SEMTL to have b-ECEMTL. Note that we only consider for the case $b \geq 1$, since if $b = 0$ then 0-ECEMTL is the same with SEMTL.

Theorem 3.3. *Suppose that G is a bipartite graph with partite sets V_1 and V_2 and has SEMTL $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ with magic constant K. If $f(v) \leq |V_1|$ for every $v \in V_1$ or $f(v) > |V_1|$ for every $v \in V_2$, then G has a b-ECEMTL with $b = |V_1|$.*

Proof. We first prove that $f(u) \leq |V_1|$ for every $u \in V_1$ or $f(v) > |V_1|$ for every $v \in V_2$ causes $f(V_1) = \{1, 2, \ldots, |V_1|\}$ and $f(V_2) = \{|V_1| + 1, |V_1| + 2, \ldots, |V|\}$. Note that f is a SEMTL, so $f(V) = \{1, 2, \ldots, |V|\}$. If $f(u) \leq |V_1|$ for every $u \in V_1$, then $1 \leq f(u) \leq |V_1|$ for $u \in V_1$. Since f is bijective, it is clear that $f(V_1) = \{1, 2, \ldots, |V_1|\}$, so $f(V_2) = \{|V_1| + 1, |V_1| + 2, \ldots, |V|\}$. If $f(v) > |V_1|$ for every $v \in V_2$, then $|V_1| + 1 \leq f(v) \leq |V|$ for $v \in V_2$. Because V_1 and V_2 are partite sets of G, then $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, thus $|V| = |V_1| + |V_2|$. Then, we can conclude that $|V_1| + 1 \leq f(v) \leq |V_1| + |V_2|$ for $v \in V_2$. Since f is bijective, it is clear that $f(V_2) = \{|V_1| + 1, |V_1| + 2, \ldots, |V|\}$, so $f(V_1) = \{1, 2, \ldots, |V_1|\}$.

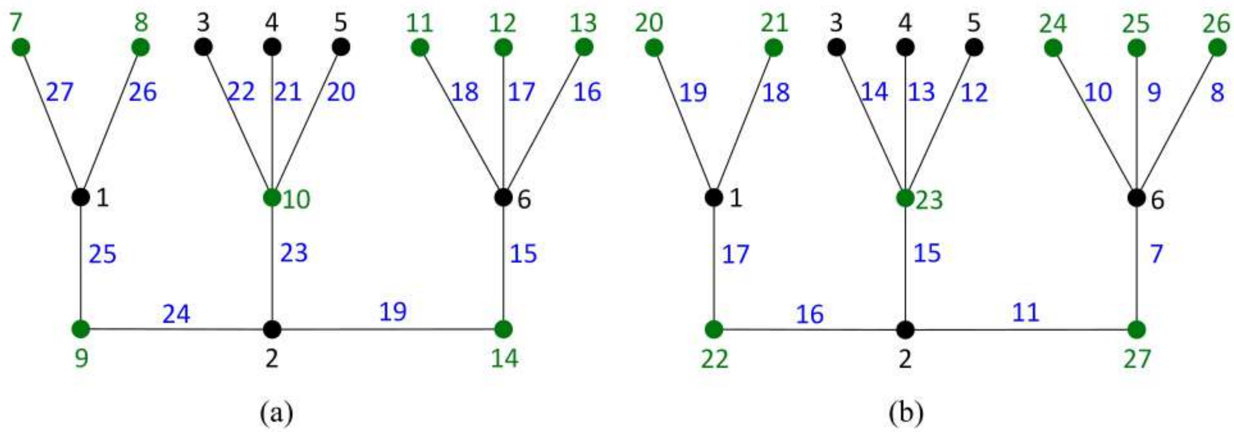

Figure 6. (a) SEMTL and (b) 6-ECEMTL on $F(2,3,3)$.

Now, we construct a new labeling $g : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ as follows

$$g(v) = \begin{cases} f(v), & \text{for } v \in V_1, \\ f(v) + |E|, & \text{for } v \in V_2, \end{cases}$$
$$g(uv) = f(uv) - |V_2|, \text{ for } uv \in E.$$

It will be proved that g is bijective with $g(E) = \{b+1, b+2, \ldots, b+|E|\}$ and $w(uv) = g(u) + g(v) + g(uv)$ is a constant natural number for every $uv \in E$.

It is known that f is a SEMTL, so f is bijective with $f(V) = \{1, 2, \ldots, |V|\}$. Because for $u \in V_1$, then we have $g(u) = f(u)$, thus we have $g(V_1) = \{1, 2, \ldots, |V_1|\}$. From the conditions given to the theorem it is obtained that $f(V_2) = \{|V_1| + 1, |V_1| + 2, \ldots, |V_1| + |V_2|\}$, thus from the definition of g we obtain that $g(V_2) = \{|V_1| + 1 + |E|, |V_1| + 2 + |E|, \ldots, |V_1| + |V_2| + |E|\}$. Because $f(V) = \{1, 2, \ldots, |V|\}$ and $f(V \cup E) = \{1, 2, \ldots, |V|, |V| + 1, \ldots, |V| + |E|\}$ then it's clear that $f(E) = \{|V| + 1, |V| + 2, \ldots, |V| + |E|\}$. From the definition of g we also obtain that $g(E) = \{|V|+1-|V_2|, |V|+2-|V_2|, \ldots, |V|+|E|-|V_2|\}$. It can be seen that $b = |V_1| = |V|-|V_2|$ so $g(E) = \{b+1, b+2, \ldots, b+|E|\}$ and $g(V_2) = \{b+|E|+1, b+|E|+2, \ldots, b+|E|+|V_2|\}$. From the explanation above, we can conclude that g is bijective with $g(E) = \{b+1, b+2, \ldots, b+|E|\}$.

The conditions provided state that for every $uv \in E$, then $u \in V_1$ and $v \in V_2$ or vice versa. Because f is a SEMTL then for every $uv \in E$, $w(uv) = f(u) + f(v) + f(uv) = K$ a constant number. Suppose that $uv \in E$, with $u \in V_1$ and $v \in V_2$. From the definition of g we get that the edge weight for g i.e., $w^*(uv) = g(u) + g(v) + g(uv) = f(u) + f(v) + |E| + f(uv) - |V_2| = K + |E| - |V_2|$ is a constant number. Note that for every $x \in E$, x is incident with one of the members of V_2 thus $|E| \geq |V_2|$ and $K + |E| - |V_2| \geq K > 0$. Therefore, it is proved that g is a b-ECEMTL with $b = |V_1|$ and magic constant $K^* = K + |E| - |V_2|$. $\square$

In Figure 6 (a) we give an example of SEMTL on $F(2,3,3)$ with $f(v) \leq 6 = |V_1|$ for every $v \in V_1$. The vertices in V_1 are colored black while the vertices in V_2 are colored green. In Figure 6 (b) we give a 6-ECEMTL obtained from the construction given at the proof for Theorem 3.3.

Next, we will show the corollary obtained from Theorem 3.3. In Theorem 2.6, Kang, Kim, and Park [5] proved that if G is connected bipartite graph having a b-ECEMTL, then G is a tree. Therefore, in Corollary 3.1 we give some trees that meet Theorem 3.3.

Corollary 3.1. *The following graphs, namely*

1. *A caterpillar* $G \cong S_{n_1,n_2,\ldots,n_k}$ *with* $n_i \in \mathbb{N}$ *for every* $1 \leq i \leq k$
2. *A banana tree* $G \cong BT(n_1, n_2, \ldots, n_k)$ *with* $n_1 = n_2 = \cdots = n_k = n$, $\lceil \frac{k}{2} \rceil \leq n \leq k-1$
3. *A generalized comb* $G \cong Cb_k(n_1, n_2, \ldots, n_k)$ *with* $n_1 = n_2 = \cdots = n_k = n$, $n \geq 2$, *and* $k \geq 2$
4. *A generalized comb* $G \cong Cb_k(2, 3, \ldots, k+1)$
5. *A banana tree* $G \cong BT(n_1, n_2, \ldots, n_k)$ *with* $n_i \geq i$ *for every* $1 \leq i \leq k$
6. *A firecracker* $G \cong F(n_1, n_2, \ldots, n_k)$ *with* $n_i \geq n_{i+1}$ *for every* $1 \leq i < k$

have a b*-ECEMTL with* $b = |V_1|$.

Proof. 1.) From Theorem 2.1, we obtain that $S_{n_1,n_2,\ldots,n_k}$ has a SEMTL f. Kotzig and Rosa [6] provide definition for f as follows. Rearrange $S_{n_1,n_2,\ldots,n_k}$ in such a way that the vertices are arranged into two rows, with the edges joining vertices from different rows and no two edges intersecting each other. Suppose that $v_1, v_2, \ldots, v_p$ and $w_1, w_2, \ldots, w_q$, respectively, are the vertices in the first row and the second row sequentially from left to right. Define f as follows.

$$\begin{aligned} f(v_i) &= i, \text{ for } 1 \leq i \leq p, \\ f(w_j) &= p + j, \text{ for } 1 \leq j \leq q, \\ f(v_i w_j) &= 2p + 2q - i - j + 1. \end{aligned}$$

Let $V_1 = \{v_1, v_2, \ldots, v_p\}$ and $V_2 = \{w_1, w_2, \ldots, w_q\}$, then from the definition given, it is clear that these two sets are the partite sets of $S_{n_1,n_2,\ldots,n_k}$. We also obtain that $|V_1| = p$, so $f(V_1) = \{1, 2, \ldots, |V_1|\}$ and $f(V_2) = \{|V_1|+1, |V_1|+2, \ldots, |V|\}$. From the results above, it is proven that G meets the conditions of Theorem 3.3. Thus $S_{n_1,n_2,\ldots,n_k}$ has a b-ECEMTL with $b = |V_1|$.

2.) From Theorem 2.2, we obtain that $BT(n_1, n_2, \ldots, n_k)$ with $n_1 = n_2 = \cdots = n_k = n$, $\lceil \frac{k}{2} \rceil \leq n \leq k-1$, has a SEMTL. Hussain, Baskoro, and Slamin [3] provide the definition for SEMTL f as follows

$$\begin{aligned} f(v_0) &= (n+1)k + 1 - \left\lfloor \frac{k}{2} \right\rfloor, \\ f(v_i) &= \begin{cases} nk + i, & \text{for } 1 \leq i \leq \lceil \frac{k}{2} \rceil, \\ nk + 1 + i, & \text{for } \lceil \frac{k}{2} \rceil < i \leq k, \end{cases} \\ f(v_{i,1}) &= \begin{cases} (n+1)i - \lceil \frac{k}{2} \rceil, & \text{for } 1 \leq i \leq \lceil \frac{k}{2} \rceil, \\ (n+1)i - n - \lceil \frac{k}{2} \rceil, & \text{for } \lceil \frac{k}{2} \rceil < i \leq k, \end{cases} \\ f(\{v_{i,j} | 2 \leq j \leq n\}) &= \{(i-1)n + 1, \ldots, (i-1)n + n\} \text{ except for } \{f(v_{i,1})\}, \text{ for } 1 \leq i \leq k. \end{aligned}$$

Let $V_1 = \{\bigcup_{i=1}^{k}\{v_{i,m}|1 \leq m \leq n_i\}\}$ and $V_2 = \{v_0, v_1, \ldots, v_k\}$. Note that $E = \{v_i v_{i,m}|1 \leq i \leq k, 1 \leq m \leq n_i\} \cup \{v_0 v_{i,1}|1 \leq i < k\}$, so it is clear that V_1 and V_2 are partite sets. Because $n_1 = n_2 = \cdots = n_k = n$, then $|V_1| = nk$. From the definition given for f it is clear that $f(v_i) > nk = |V_1|$ for every $i \in \{1, 2, \ldots, k\}$. Then it is proven that $BT(n_1, n_2, \ldots, n_k)$ with $n_1 = n_2 = \cdots = n_k = n$, $\lceil \frac{k}{2} \rceil \leq n \leq k-1$ meets the conditions of Theorem 3.3. Thus G has a b-ECEMTL with $b = |V_1|$.

3.) From Theorem 2.4, we obtain that $Cb_k(n_1, n_2, \ldots, n_k)$ with $n_1 = n_2 = \cdots = n_k = n$, $n \geq 2$, $k \geq 2$, has a SEMTL. In a similar way with 1) and 2), it can be proved that $Cb_k(n_1, n_2, \ldots, n_k)$ with $n_1 = n_2 = \cdots = n_k = n$, $n \geq 2$, $k \geq 2$ has a b-ECEMTL with $b = |V_1|$.

4.) From Theorem 2.5, we obtain that $Cb_k(2, 3, \ldots, k+1)$ has a SEMTL. It can be proved that $Cb_k(2, 3, \ldots, k+1)$ has a b-ECEMTL with $b = |V_1|$.

5.) From Theorem 3.1, we obtain that $BT(n_1, n_2, \ldots, n_k)$ with $n_i \geq i$ for each $i \in \{1, 2, \ldots, k\}$, has a SEMTL. It can be proved that $BT(n_1, n_2, \ldots, n_k)$ with $n_i \geq i$ for each $i \in \{1, 2, \ldots, k\}$ has a b-ECEMTL with $b = |V_1|$.

6.) From Theorem 3.2, we obtain that $F(n_1, n_2, \ldots, n_k)$ with $n_i \geq n_{i+1}$ for each $i \in \{1, 2, \ldots, k-1\}$, has a SEMTL. It can be proved that $F(n_1, n_2, \ldots, n_k)$ with $n_i \geq n_{i+1}$ for each $i \in \{1, 2, \ldots, k-1\}$ has a b-ECEMTL with $b = |V_1|$. □

4. Conclusions

In this study, we gave the SEMTL for two classes of trees, banana tree and firecracker. A banana tree $BT(n_1, n_2, \ldots, n_k)$ has a SEMTL when $n_i \geq i$ for every $i \in \{1, 2, \ldots, k\}$, while a firecracker $F(n_1, n_2, \ldots, n_k)$ has a SEMTL when $n_i \geq n_{i+1}$ for every $i \in \{1, 2, \ldots, k-1\}$.

Then we give the sufficient condition for a graph G that has SEMTL to have b-ECEMTL, which is bipartite graph with partite sets V_1 and V_2 and having SEMTL f that meets $f(u) \leq |V_1|$ for every $u \in V_1$ or $f(v) > |V_1|$ for every $v \in V_2$. As the result of this sufficient condition, it is shown that caterpillar, banana tree $BT(n_1, n_2, \ldots, n_k)$ with $n_1 = n_2 = \cdots = n_k = n$, $\lceil \frac{k}{2} \rceil \leq n \leq k-1$, generalized comb $Cb_k(n_1, n_2, \ldots, n_k)$ with $n_1 = n_2 = ... = n_k = n$, $n \geq 2$, and $k \geq 2$, as well as generalized comb $Cb_k(2, 3, \ldots, k+1)$ have b-ECEMTL. Likewise, the banana tree and firecracker that have previously been shown are SEMT graphs.

For further study we can find the SEMTL for banana tree and firecracker with less conditions than those given in this paper. For b-ECEMTL, we gave the sufficient condition for a graph G to have b-ECEMTL, then in future research the necessary condition for a graph G to have b-ECEMTL can be explored. The corollaries given in this study are only limited to connected graphs, in further research other corollaries for disconnected graph can be explored.

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