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Zonal graphs of small cycle rank

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Abstract

A zonal labeling of a plane graph G is an assignment of the two nonzero elements of the ring $\mathbb{Z}_3$ of integers modulo 3 to the vertices of G such that the sum of the labels of the vertices on the boundary of each region of G is the zero element of $\mathbb{Z}_3$. A plane graph possessing such a labeling is a zonal graph. There is a connection between zonal labelings of connected bridgeless cubic plane graphs and the Four Color Theorem. Zonal labelings of cycles play a role in this connection. The cycle rank of a connected graph of order n and size m is $m - n + 1$. Thus, cycles have cycle rank 1. All zonal connected graphs of cycle rank at most 2 are determined.

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1. Introduction

Let G be a connected plane graph each of whose vertices is labeled with one of the two nonzero elements 1 and 2 of the ring $\mathbb{Z}_3$ of integers modulo 3. The *value* of a region (zone) R of G is the sum in $\mathbb{Z}_3$ of the labels of the vertices on the boundary of R. Such a labeling of G is said to be a *zonal labeling* if the value of each zone in G is the zero element of $\mathbb{Z}_3$. If G admits a zonal labeling, then G is *zonal*. This concept was introduced by Cooroo Egan in 2014 (see [3]).

There is a close connection between the Four Color Theorem and zonal labelings of planar graphs. A connected bridgeless cubic plane graph (or multigraph) is referred to as a *cubic map*.

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Thus, every cubic map is a 3-regular 2-edge-connected plane graph. The following two results were established in [3].

Theorem 1.1. [3] *A connected cubic plane graph G is zonal if and only if G is bridgeless.*

Theorem 1.2. [3] *There exists a 4-coloring of the regions of a cubic map M if and only if M has a zonal labeling.*

Since it is known that there exists a 4-coloring of the regions of every plane map (the Four Color Theorem) if and only if there exists a 4-coloring of the regions of every cubic map (see [5], for example), it follows by Theorem 1.2 that showing every cubic map is zonal is equivalent to establishing the Four Color Theorem.

As described in [3], the argument given that establishes the truth of Theorems 1.1 and 1.2, however, makes use of the Four Color Theorem. It would be considerably more satisfying if it could be shown that every cubic map is zonal without using the Four Color Theorem (and without using computers). Thus, the following fundamental question was stated in [3].

Problem 1.3. *Can it be shown that every cubic map is zonal without using the Four Color Theorem or computers?*

Problem 1.3 brings up the natural question of determining the zonality of plane graphs and planar graphs in general. A planar graph G is *zonal* if there exists a zonal planar embedding of G. This concept has been studied in [1, 2, 3, 4]. Since the boundary of every region of a cubic map is a cycle, it is of value to know which cycles and related graphs are zonal. That every nontrivial tree and every cycle is zonal was established in [3].

Theorem 1.4. [3] *Every nontrivial tree and every cycle is zonal.*

The *cycle rank* of a connected graph of order n size m is the number $m - n + 1$. Thus, cycles have cycle rank 1. In general, the graphs of cycle rank 1 are connected graphs possessing exactly one cycle. The goal of this paper is to determine all zonal graphs of cycle rank 1 and 2. All such graphs are necessarily planar.

2. Zonal Unicyclic Graphs

A connected graph containing exactly one cycle is referred to as a *unicyclic graph*. Thus, the unicyclic graphs are precisely the graphs of cycle rank 1. In order to determine which unicyclic graphs are zonal, we first state some information on zonal labelings. Let ℓ be a labeling of the vertices of a graph G with the labels 1 and 2 of $\mathbb{Z}_3$. The vertex labeling $\overline{\ell}$ of G defined by $\overline{\ell}(v) = 3 - \ell(v)$ for each vertex v of G is called the *complementary labeling* of G.

Observation 2.1. [3] *If ℓ is a zonal labeling of a connected plane graph, then so too is its complementary labeling $\overline{\ell}$.*

The following notation will be useful to us. For a labeling $\ell : V(G) \to \{1, 2\} \subseteq \mathbb{Z}_3$ of a graph G and a subgraph H or a nonempty set X of vertices of G, let

$$\sum(\ell, H) = \sum_{x \in V(H)} \ell(x) \text{ in } \mathbb{Z}_3 \text{ and } \sum(\ell, X) = \sum_{x \in X} \ell(x) \text{ in } \mathbb{Z}_3.$$

In particular, if B is the boundary of a region R of a plane graph G and ℓ is a labeling of G, then $\sum(\ell, B)$ is the *value* of B (as well as the value of R).

Lemma 2.2. *Let X be a nonempty set of vertices of a graph.*

(1) *For each $i = 1, 2$, there is a labeling $\ell_i : X \to \{1, 2\} \subseteq \mathbb{Z}_3$ of X such that $\sum(\ell_i, X) = i$ in $\mathbb{Z}_3$.*
(2) *If $|X| \geq 2$, then there is a labeling $\ell_0 : X \to \{1, 2\} \subseteq \mathbb{Z}_3$ of X such that $\sum(\ell_0, X) = 0$ in $\mathbb{Z}_3$.*

Proof. First, we verify (1). For each $i = 1, 2$, define a labeling $\ell_i : X \to \{1, 2\}$ of X as follows.

- ⋆ If $|X| \geq 1$ is odd, then $|X| = 2k + 1$ for some nonnegative integer k. Let ℓ_1 assign the label 1 to $k + 1$ vertices of X and assign the label 2 to k vertices of X, giving the sum $\sum(\ell_1, X) = 1 \cdot (k + 1) + 2k = 1$ in $\mathbb{Z}_3$. Let ℓ_2 be the complementary labeling of ℓ_1, that is, let ℓ_2 assign the label 1 to k vertices of X and assign the label 2 to $k + 1$ vertices of X, giving the sum $\sum(\ell_2, X) = 1 \cdot k + 2(k + 1) = 2$ in $\mathbb{Z}_3$.
- ⋆ If $|X| \geq 2$ is even, then $|X| = 2k = (k - 1) + (k + 1)$ for some positive integer k. Let ℓ_1 assign the label 2 to $k + 1$ vertices of X and assign the label 1 to $k - 1$ vertices of X, giving the sum $\sum(\ell_1, X) = 1 \cdot (k - 1) + 2(k + 1) = 1$ in $\mathbb{Z}_3$. Let ℓ_2 be the complementary labeling of ℓ_1, that is, let ℓ_2 assign the label 1 to $k + 1$ vertices of X and assign the label 2 to $k - 1$ vertices of X, giving the sum $\sum(\ell_2, X) = 1 \cdot (k + 1) + 2(k - 1) = 2$ in $\mathbb{Z}_3$.

Next, we verify (2). For $|X| \geq 2$, let $x_0 \in X$ and $X' = X - \{x_0\}$. By (1), for each $i = 1, 2$, there is a labeling $\ell_i : X' \to \{1, 2\}$ of X such that $\sum(\ell_i, X') = i$ in $\mathbb{Z}_3$. Define the labeling $\ell_0 : X \to \{1, 2\}$ of X by $\ell_0(x) = \ell_i(x)$ if $x \in X'$ and $\ell_0(x_0) = 3 - i$. Thus, $\sum(\ell_0, X) = \sum(\ell_i, X') + \ell_0(x_0) = i + (3 - i) = 0$ in $\mathbb{Z}_3$. □

Let $C \star K_2$ be the unicyclic graph obtained by adding exactly one pendant edge at some vertex of a cycle C. We are now prepared to present the following result.

Theorem 2.3. *A unicyclic graph G is zonal if and only if $G \neq C \star K_2$ for any cycle C.*

Proof. Let G be a unicyclic graph containing the cycle C such that $G \neq C \star K_2$. We show that G is zonal. By Theorem 1.4, we may assume that $G \neq C$. Let G be embedded in the plane (resulting in two regions R_1 and R_2) such that the boundary of R_1 is C and that the boundary of R_2 is G. By Theorem 1.4, there is a zonal labeling ℓ_C of C. Let $U = V(G) - V(C)$ be the set of vertices of G that do not belong to C. Then $p = |U| \geq 2$. We now extend the labeling ℓ_C of C to a labeling ℓ of G. First, we define $\ell(v) = \ell_C(v)$ for all $v \in V(C)$. Thus, the sum of the labels of the vertices on C is $\sum(\ell, C) = \sum(\ell_C, C) = 0$. Next, we define the labels of vertices in U as follows. If $p \geq 2$ is even, then we assign the label 1 to half of the vertices of U and the label 2 to the other half, giving us a sum of 0 in $\mathbb{Z}_3$. If $p \geq 3$ is odd, then $p = 2k + 1$ for some positive integer k. Then $n = 2k + 1 = (k - 1) + (k + 2)$. If we assign the label 1 to $k + 2$

vertices of U, assign the label 2 to the other $k-1$ vertices of U, and add these labels, we have $1 \cdot (k+2) + 2(k-1) = (k+2) + 2k - 2 = 3k = 0$ in $\mathbb{Z}_3$. Consequently, the values of R_1 and R_2 are both 0 and so ℓ is a zonal labeling of G. Therefore, G is zonal.

For the converse, suppose that $G = C \star K_2$ for a cycle C, where u is the vertex of G that does not belong to C. Let there be given a planar embedding of G; that is, G is a plane graph. Assume, to the contrary, that G has a zonal labeling ℓ. Then $\ell(u) \in \{1, 2\}$. Since C is the boundary of some region R_1 of G, the value of C is 0. However then, the value of the boundary of the other region R_2 of G is the sum of the value of C and $\ell(u)$, that is, $0 + \ell(u) = \ell(u) \neq 0$ in $\mathbb{Z}_3$, which is a contradiction. □

Recall that a planar graph G is zonal if there exists a zonal planar embedding of G. It is possible that a planar graph has both a zonal planar embedding and a non-zonal planar embedding. For example, Figure 1 shows two distinct planar embeddings of a planar graph G. The planar embedding of G in Figure 1(a) is zonal and a zonal labeling is also shown in that figure, while the planar embedding of G in Figure 1(b) is not zonal. Next, we show that if G is a zonal graph of cycle rank 1 (namely, a unicyclic graph), then every planar embedding of G is zonal.

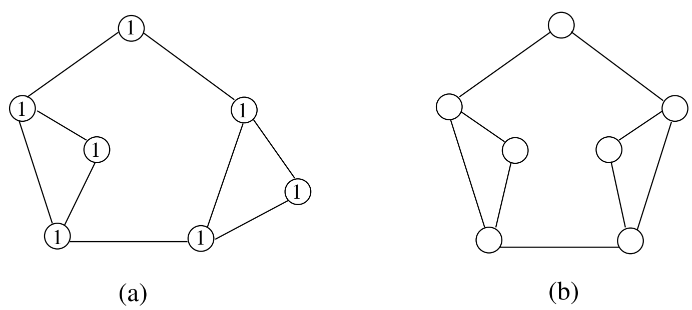

Figure 1. A graph with a zonal planar embedding
and a non-zonal planar embedding

Proposition 2.4. *Every planar embedding of a zonal graph of cycle rank* 1 *is zonal.*

Proof. Since there is only one planar embedding of a cycle, the statement is true trivially. Next, let G be a zonal unicyclic graph containing the cycle C. We may assume that $G \neq C$ and $G \neq C \star K_2$ by Theorem 2.3. Let $U = V(G) - V(C)$ be the set of vertices of G that do not belong to C. Then $p = |U| \geq 2$. Let G be embedded in the plane resulting two regions R_1 and R_2. We show that the resulting plane graph G has a zonal labeling. For $i = 1, 2$, let B_i be the boundary of R_i and so B_i contains C. If $\{B_1, B_2\} = \{C, G\}$, then G is zonal by the proof of Theorem 2.3. We may therefore assume that $\{B_1, B_2\} \neq \{C, G\}$ and so $\{B_1, B_2\} \cap \{C, G\} = \emptyset$. For $i = 1, 2$, let $U_i \subseteq U$ be the set of vertices belonging to B_i and so $|U_i| \geq 1$. Furthermore, $V(B_i) = V(C) \cup U_i$ for $i = 1, 2$. With the aid of Lemma 2.2, we can define a labeling $\ell : V(G) \to \{1, 2\}$ of G such that $\sum(\ell, C) = 1$ in $\mathbb{Z}_3$ and $\sum(\ell, U_i) = 2$ in $\mathbb{Z}_3$ for $i = 1, 2$. Hence, the value of B_i, $i = 1, 2$, is $\sum(\ell, C) + \sum(\ell, U_i) = 1 + 2 = 0$ in $\mathbb{Z}_3$ and so ℓ is a zonal labeling of G. Therefore, every planar embedding of G is zonal. □

3. Zonal Graphs of Cycle Rank 2

We next determine those graphs of cycle rank 2 that are zonal. A graph G has cycle rank 2 if G contains a subgraph F where

(1) F is obtained from two cycles C and C' by identifying a vertex in C and a vertex in C', which is denoted by $F = C \star C'$, as shown in Figure 2(a),

(2) F is obtained from two disjoint cycles C and C' by adding a path P of length 1 or more and then by identifying an end-vertex of P with a vertex of C and identifying the other end-vertex of P with a vertex of C', which is denoted by $F = C \star P \star C'$, as shown in Figure 2(b), or

(3) F is a subdivision of $K_4 - e$, that is, G consists of three internally disjoint paths P_i $(1 \leq i \leq 3)$, where at least two paths P_i $(1 \leq i \leq 3)$ have length 2 or more, as shown in Figure 2(c).

Necessarily, every embedding of a graph of cycle rank 2 results in a plane graph with exactly three regions – namely one exterior region and two interior regions.

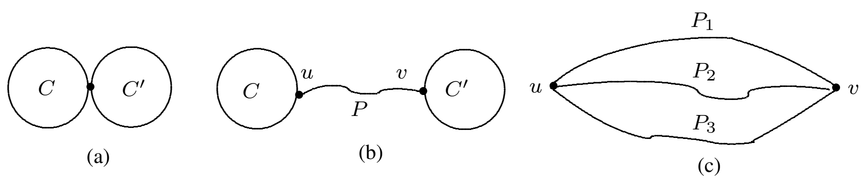

Figure 2. Three possible types of subgraphs

For $i = 1, 2, 3$, a graph G is referred to as a graph of cycle rank 2 and type (i) if G contains a subgraph F satisfing the properties described in (i) above. Furthermore, a graph G of cycle rank 2 is *minimal* if G is one of the graphs F described in (1), (2), or (3). Consequently, every minimal graph G of cycle rank 2 has minimum degree 2. The graph $K_4 - e$ is a minimal graph of cycle rank 2 and type 3 but it is not zonal. This gives rise to the more general question: Which graphs of cycle rank 2 are zonal?

First, we determine all zonal graphs of cycle rank 2 and type (1).

Theorem 3.1. *Let G be a graph of cycle rank* 2 *and type* (1). *Then G is zonal if and only if G is not minimal.*

Proof. Since G is a graph of cycle rank 2 and type (1), it follows that G contains a subgraph $C \star C'$ obtained from two cycles C and C' by identifying a vertex in C and a vertex in C', denoting this identified vertex by u.

First, suppose that G is minimal and so $G = C \star C'$. We show that G is not zonal. Assume, to the contrary, that G is zonal. Then there exists a planar embedding of G such that the resulting plane graph G has a zonal labeling ℓ. Since each of C and C' is the boundary of a region of G, it follows that $\sum(\ell, C) = \sum_{v \in V(C)} \ell(v) = 0$ and $\sum(\ell, C') = \sum_{v \in V(C')} \ell(v) = 0$. However then, the value of the boundary of the region R_3 is $[\sum(\ell, C) + \sum(\ell, C')] - \ell(u) = 0 + 0 - \ell(u) \neq 0$ in $\mathbb{Z}_3$, which is a contradiction.

For the converse, suppose that G is a graph of cycle rank 2 and type (1) such that $G \neq C \star C'$ where u is the vertex belonging to both C and C'. We show that G is zonal. We embed G in the plane resulting in three regions R_1, R_2, and R_3 such that the boundary of R_1 is C, the boundary of R_2 is C', and the boundary of R_3 is G. Since C and C' are zonal by Theorem 1.4, there are zonal labelings ℓ_C and $\ell_{C'}$ of C and C', respectively. We may assume that $\ell_C(u) = \ell_{C'}(u) = 1$ where u belongs to C and C'. Let $U = V(G) - V(C \star C')$ be the set of vertices of G that do not belong to $C \star C'$. Then $|U| \geq 1$. We now define a labeling ℓ of G. First, we define $\ell(v) = \ell_C(v)$ for all $v \in V(C)$ and $\ell(v) = \ell_{C'}(v)$ for all $v \in V(C')$. Thus, the value of the boundary C of R_1 is $\sum(\ell, C) = \sum(\ell_C, C) = 0$ and the value of the boundary C' of R_2 is $\sum(\ell, C') = \sum(\ell_{C'}, C') = 0$. Next, we can define the labels of vertices of U such that $\sum(\ell, U) = 1$ in $\mathbb{Z}_3$ by Lemma 2.2. Thus, the value of the boundary G of R_3 is $[\sum(\ell, C) + \sum(\ell, C') - \ell(u)] + \sum(\ell, U) = -1 + 1 = 0$ in $\mathbb{Z}_3$. Consequently, the values of R_1, R_2, and R_3 are all 0 and so ℓ is a zonal labeling of G. □

We now determine all zonal graphs of cycle rank 2 and type (2).

Theorem 3.2. *Let G be a graph of cycle rank 2 and type (2). Then G is zonal if and only if either every vertex of G belongs to a cycle of G or at least two vertices of G belong to no cycle of G.*

Proof. First, let G be a zonal graph of cycle rank 2 and type (2) such that exactly one vertex w of G belongs to no cycle of G. Therefore, either G is a minimal graph $C \star P \star C'$ where C and C' are two cycles and P is a path of length 2 with interior vertex w or G is obtained from a minimal graph $C \star P \star C'$ by adding a pendant edge $e = wv$ where $v \in V(C \star P \star C')$, C and C' are two cycles, and P is a path of length 1. Since G is zonal, there is a planar embedding of G, resulting in the three regions R_1, R_2, and R_3, and a zonal labeling ℓ of the resulting plane graph G.

- ⋆ If $G = C \star P \star C'$, where P has length 2, then the boundaries of the three regions of G are C, C', and G. Thus, $\sum(\ell, C) = \sum(\ell, C') = 0$ and $\sum(\ell, G) = \sum(\ell, C) + \sum(\ell, C') + \ell(w) = \ell(w) = 0$ in $\mathbb{Z}_3$, which is impossible since $\ell(w) \in \{1, 2\}$.
- ⋆ If G consists of a minimal graph $C \star P \star C'$ and a pendant edge $e = wv$, where P is a path of length 1, then we may assume that the boundary of one region R_1 of G is C and so $\sum(\ell, C) = 0$ in $\mathbb{Z}_3$. The boundaries of the other two regions are either (a) C' and G or (b) $C \star P \star C'$ and C' with uv. If (a) occurs, then $\sum(\ell, C') = 0$ and $\sum(\ell, G) = \sum(\ell, C) + \sum(\ell, C') + \ell(w) = \ell(w) = 0$ in $\mathbb{Z}_3$, which is impossible. If (b) occurs, then $\sum(\ell, C \star P \star C') = \sum(\ell, C) + \sum(\ell, C') = 0$ and $\sum(\ell, C') + \ell(w) = 0$. Since $\sum(\ell, C) = 0$, it follows that $\sum(\ell, C') = 0$. However then, $\sum(\ell, C') + \ell(w) = 0 + \ell(w) = 0$, which is impossible.

For the converse, suppose that either every vertex of G belongs to a cycle of G or at least two vertices of G belong to no cycle of G. Thus, $p = 0$ or $p \geq 2$. We show that G is zonal. Embed G in the plane resulting in three regions R_1, R_2, and R_3 such that the boundary of R_1 is C, the boundary of R_2 is C', and the boundary of R_3 is G. Since C and C' are zonal, there are zonal labelings ℓ_C and $\ell_{C'}$ of C and C', respectively. We define a labeling ℓ of G as follows. First, we define $\ell(v) = \ell_C(v)$ for all $v \in V(C)$ and $\ell(v) = \ell_{C'}(v)$ for all $v \in V(C')$. Thus, the value of the boundary C of R_1 is $\sum(\ell, C) = \sum(\ell_C, C) = 0$ and the value of the boundary C' of R_2 is $\sum(\ell, C') = \sum(\ell_{C'}, C') = 0$. Next, we define the labels of vertices of U as follows.

If $p = 0$, then $\sum(\ell, U) = 0$ vacuously. If $p \geq 2$, then let ℓ assign the labels 1 and 2 to the vertices of U such that $\sum(\ell, U) = 0$ by Lemma 2.2. Hence, the value of the boundary G of R_3 is $\sum(\ell, C) + \sum(\ell, C') + \sum(\ell, U) = 0$ in $\mathbb{Z}_3$. Consequently, the values of R_1, R_2, and R_3 are all 0 and so ℓ is a zonal labeling of G. □

By Theorem 3.1, no minimal graph of cycle rank 2 and type (1) is zonal; while by Theorem 3.2, every minimal graph of cycle rank 2 and type (2) is zonal. For minimal graphs of cycle rank 2 and type (3), the situation is different, as we show next.

Theorem 3.3. *Let G be a minimal graph of cycle rank 2 and type (3) consisting of three internally disjoint $u - v$ paths P_i of order n_i for $i = 1, 2, 3$ where $2 \leq n_1 \leq n_2 \leq n_3$ and $n_2 \geq 3$. Then G is zonal if and only if $(n_1, n_2) \neq (2, 3)$.*

Proof. First, suppose that $(n_1, n_2) = (2, 3)$. We show that G is not zonal. Assume, to the contrary, that G is zonal. Then there exists a planar embedding of G, resulting in three regions R_1, R_2, and R_3, such that the resulting plane graph G has a zonal labeling ℓ. We may assume that the boundary of R_1 consists of P_1 and P_2, the boundary of R_2 consists of P_2 and P_3, and the boundary of R_3 consists of P_1 and P_3. Let w be the interior vertex of P_2. Since the value of the boundary of R_2 is $\sum(\ell, P_3) + \ell(w) = 0$ in $\mathbb{Z}_3$ and the value of the boundary of R_3 is $\sum(\ell, P_3) = 0$ in $\mathbb{Z}_3$, it follows that $\ell(w) = 0$ in $\mathbb{Z}_3$, which is impossible. Thus, G is not zonal.

For the converse, suppose that $(n_1, n_2) \neq (2, 3)$. Here, we show that G is zonal. We embed the graph G in the plane, resulting in three regions R_1, R_2, and R_3 where R_3 is the exterior region of G, where u and v are the two vertices of degree 3 in G. We show that the resulting plane graph G has a zonal labeling. For $i = 1, 2, 3$, let $Q_i = P_i - \{u, v\}$ and so Q_i is a path of order $n_i - 2$ (where there is no path Q_1 if $n_1 = 2$). We consider three cases, depending on whether $n_1 \geq 4$, $n_1 = 3$, or $n_1 = 2$.

Case 1. $n_1 \geq 4$. Since $n_3 \geq n_2 \geq n_1 \geq 4$, it follows by Theorem 1.4 that each path Q_i ($i = 1, 2, 3$) is zonal. For $i = 1, 2, 3$, let ℓ_i be a zonal labeling of Q_i and so $\sum(\ell_i, Q_i) = 0$ in $\mathbb{Z}_3$. We define a labeling ℓ of G by $\ell(u) = 2$, $\ell(v) = 1$, and $\ell(x) = \ell_i(x)$ if $x \in V(Q_i)$ for $i = 1, 2, 3$. Since the boundary of a region of G is $\sum(\ell_i, Q_i) + \sum(\ell_j, Q_j) + 1 + 2 = 0$ in $\mathbb{Z}_3$ for distinct integers $i, j \in \{1, 2, 3\}$, it follows that ℓ is a zonal labeling of G.

Case 2. $n_1 = 3$. We define a labeling ℓ of G such that $\ell(u) = \ell(v) = 2$ and $\sum(\ell, Q_i) = 1$ for $i = 1, 2, 3$ by Lemma 2.2. Since the boundary of the region R_i of G is $\sum(\ell, Q_i) + \sum(\ell, Q_j) + \ell(u) + \ell(v) = 1 + 1 + 2 + 2 = 0$ in $\mathbb{Z}_3$ for some $j \in \{1, 2, 3\} - \{i\}$, it follows that ℓ is a zonal labeling of G.

Case 3. $n_1 = 2$. Thus, there is no path Q_1. Since $(n_1, n_2) \neq (2, 3)$, it follows that $n_3 \geq n_2 \geq 4$. By Theorem 1.4, each path Q_i ($i = 2, 3$) is zonal and so has a zonal labeling ℓ_i. Thus, $\sum(\ell_i, Q_i) = 0$ in $\mathbb{Z}_3$ for $i = 2, 3$. As we saw in Case 1, we can define the labeling ℓ of G by $\ell(u) = 2$, $\ell(v) = 1$, and $\ell(x) = \ell_i(x)$ if $x \in V(Q_i)$ for $i = 2, 3$. Then ℓ is a zonal labeling of G. □

We now determine which graphs of cycle rank 2 are zonal if they contain a minimal non-zonal graph of cycle rank 2 and type (3) as a proper subgraph.

Theorem 3.4. *Let F be a minimal non-zonal graph of cycle rank 2 and type (3) and let G be a graph of cycle rank 2. If G contains F as a proper subgraph, then G is zonal.*

Proof. Let $U = V(G) - V(F)$ be the set of vertices of G that do not belong to F. Then $|U| \geq 1$. Since the subgraph F of G is non-zonal, it follows by Theorem 3.3 that F consists of three internally disjoint $u - v$ paths P_i of order n_i for $i = 1, 2, 3$ where $n_1 = 2$ and $3 = n_2 \leq n_3$. Let w be the interior vertex of P_2 and let $Q_3 = P_3 - \{u, v\}$ be a path of order $q_3 = n_3 - 2 \geq 1$. We embed the graph G in the plane in such a way that the boundary of R_1 is the triangle (u, w, v, u), the boundary of R_2 is $G - uv$, and the boundary of R_3 is (u, Q_3, v, u), as shown in Figure 3, for example.

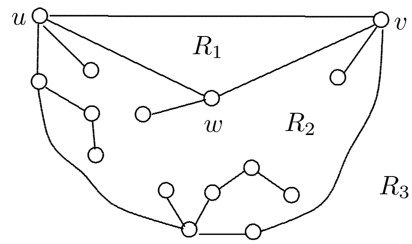

Figure 3. An embedding of the graph G

We define a labeling ℓ of G such that $\ell(u) = \ell(v) = \ell(w) = 1$, $\sum(\ell, Q_3) = 1$ in $\mathbb{Z}_3$, and $\sum(\ell, U) = 2$ in $\mathbb{Z}_3$. Then the value of the boundary of R_1 is $\ell(u) + \ell(w) + \ell(v) = 1 + 1 + 1 = 0$ in $\mathbb{Z}_3$, the value of the boundary of R_2 is $\ell(u)+\ell(w)+\ell(v)+\sum(\ell, Q_3)+\sum(\ell, U) = 1+1+1+1+2 = 0$ in $\mathbb{Z}_3$, and the value of the boundary of R_3 is $\ell(u) + \ell(v) + \sum(\ell, Q_3) = 1 + 1 + 1 = 0$ in $\mathbb{Z}_3$. Consequently, ℓ is a zonal labeling of G and so G is zonal. □

Next, we consider graphs of cycle rank 2 containing a proper minimal zonal subgraph of cycle rank 2 and type (3). For a minimal graph F of cycle rank 2 and type (3), let $F \star K_2$ be the graph obtained from F by adding a pendant edge at a vertex of F.

Theorem 3.5. *If F is a minimal zonal graph of cycle rank 2 and type (3), then $F \star K_2$ is zonal.*

Proof. Let F consist of three internally disjoint $u - v$ paths P_i of order n_i for $i = 1, 2, 3$ where $2 \leq n_1 \leq n_2 \leq n_3$ and $n_2 \geq 3$ and let $G = F \star K_2$. We consider two cases, according to whether $n_1 \geq 3$ or $n_1 = 2$.

Case 1. $n_1 \geq 3$. Then $n_3 \geq n_2 \geq n_1 \geq 3$. For $i = 1, 2, 3$, let $Q_i = P_i - \{u, v\}$ be the path of order $q_i \geq 1$. We may assume, without loss of generality, that G is obtained by adding a vertex w and joining w to a vertex z of P_3 (where it is possible that $z = u$ or $z = v$). Let G be embedded in the plane resulting in three regions R_1, R_2, and R_3 such that w belongs to the boundary of R_3. For $i = 1, 2, 3$, let B_i be the boundary of R_i. Define the labeling ℓ of G such that (i) $\ell(u) = \ell(w) = 1$ and $\ell(v) = 2$ and (ii) $\sum(\ell, Q_1) = \sum(\ell, Q_3) = 1$ and $\sum(\ell, Q_2) = 2$. Then the value of B_1 is $\ell(u) + \ell(v) + \sum(\ell, Q_1) + \sum(\ell, Q_2) = 1 + 2 + 1 + 2 = 0$ in $\mathbb{Z}_3$, the value of B_2 is $\ell(u) + \ell(v) + \sum(\ell, Q_2) + \sum(\ell, Q_3) = 1 + 2 + 2 + 1 = 0$ in $\mathbb{Z}_3$, the value of B_3 is $\ell(u) + \ell(v) + \ell(w) + \sum(\ell, Q_1) + \sum(\ell, Q_3) = 1 + 2 + 1 + 1 + 1 = 0$ in $\mathbb{Z}_3$. Thus, ℓ is a zonal labeling of G.

Case 2. $n_1 = 2$. Since F is zonal, it follows by Theorem 3.3 that $n_3 \geq n_2 \geq 4$. For $i = 2, 3$, let $Q_i = P_i - \{u, v\}$ be the path of order $q_i \geq 2$. We may assume, without loss of generality, that G is obtained by adding a vertex w and joining w to a vertex z of P_3 (where it is possible that $z = u$ or $z = v$). Let G be embedded in the plane resulting in three regions R_1, R_2, and R_3 such that w belongs to the boundary of R_3. For $i = 1, 2, 3$, let B_i be the boundary of R_i. Define a labeling ℓ of G such that (i) $\ell(u) = \ell(v) = \ell(w) = 1$ and (ii) $\sum(\ell, Q_2) = 1$ and $\sum(\ell, Q_3) = 0$. Hence, the value of B_1 is $\ell(u) + \ell(v) + \sum(\ell, Q_2) = 1 + 1 + 1 = 0$ in $\mathbb{Z}_3$, the value of B_2 is $\ell(u) + \ell(v) + \sum(\ell, Q_2) + \sum(\ell, Q_3) = 1 + 1 + 1 + 0 = 0$ in $\mathbb{Z}_3$, and the value of B_3 is $\ell(u) + \ell(v) + \ell(w) + \sum(\ell, Q_3) = 1 + 1 + 1 + 0 = 0$ in $\mathbb{Z}_3$. Consequently, ℓ is a zonal labeling of G. □

Theorem 3.6. *Let F be a minimal zonal graph of cycle rank 2 and type (3) and let G be a graph of cycle rank 2. If G contains F as a proper subgraph and $G \neq F \star K_2$, then G is zonal.*

Proof. Since F is a minimal zonal graph of cycle rank 2 and type (3), it follows that F consists of three internally disjoint $u - v$ paths P_i of order n_i for $i = 1, 2, 3$ where $2 \leq n_1 \leq n_2 \leq n_3$, $n_2 \geq 3$ and $(n_1, n_2) \neq (2, 3)$ by Theorem 3.4. For $i = 1, 2, 3$, let $Q_i = P_i - \{u, v\}$ and so Q_i is a path of order $q_i = n_i - 2$ (where there is no path Q_1 if $n_1 = 2$). Let $U = V(G) - V(F)$ be the set of vertices of G that do not belong to F and let $p = |U|$. Since $G \neq F \star K_2$, it follows that $p \geq 2$. We consider two cases.

Case 1. *There is some path $P \in \{P_1, P_2, P_3\}$ such that every interior vertex of P has degree 2 in G.* There are two subcases, according to whether $n_1 = 2$ or $n_1 \geq 3$.

Subcase 1.1. $n_1 = 2$. Since $(n_1, n_2) \neq (2, 3)$, it follows that $n_3 \geq n_2 \geq 4$ and so $q_3 \geq q_2 \geq 2$. Let G be embedded in the plane in such a way that the boundary of R_1 is $P_2 + uv$, the boundary of R_2 is $G - uv$, and the boundary of R_3 is $P_3 + uv$ (see Figure 4, for example).

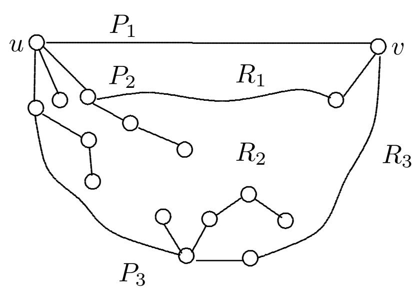

Figure 4. An embedding of the graph G in Subcase 1.1

We define a labeling ℓ of G such that $\ell(u) = \ell(v) = 1$, $\sum(\ell, Q_i) = 1$ in $\mathbb{Z}_3$ for $i = 2, 3$, and $\sum(\ell, U) = 2$ in $\mathbb{Z}_3$. Then the value of the boundary of R_1 is $\ell(u) + \ell(v) + \sum(\ell, Q_2) = 1 + 1 + 1 = 0$ in $\mathbb{Z}_3$, the value of the boundary of R_2 is $\ell(u) + \ell(v) + \sum(\ell, Q_2) + \sum(\ell, Q_3) + \sum(\ell, U) = 1 + 1 + 1 + 1 + 2 = 0$ in $\mathbb{Z}_3$, and the value of the boundary of R_3 is $\ell(u) + \ell(v) + \sum(\ell, Q_3) = 1 + 1 + 1 = 0$ in $\mathbb{Z}_3$. Consequently, ℓ is a zonal labeling of G.

Subcase 1.2. $n_1 \geq 3$. Thus, $n_3 \geq n_2 \geq n_1 \geq 3$. We may assume that every interior vertex of P_1 has degree 2. We embed the graph G in the plane in such a way that the boundary B_1 of R_1 consists of P_1 and P_2, the boundary B_2 of R_2 is $G - V(Q_1)$, and the boundary B_3 of R_3 consists of P_1 and P_3 (see Figure 5, for example). Thus, all vertices of U belong to the boundary B_2 of R_2. The planar embedding of G Figure 5 gives rise to a planar embedding of F such that $B_1^* = B_1$,

$B_2^* = B_2 - U$, and $B_3^* = B_3$ are the boundaries of the three regions $R_1^* = R_1$, R_2^*, and $R_3^* = R_3$ of F. Since $F = G - U$ is zonal, there is a zonal labeling ℓ_F of F.

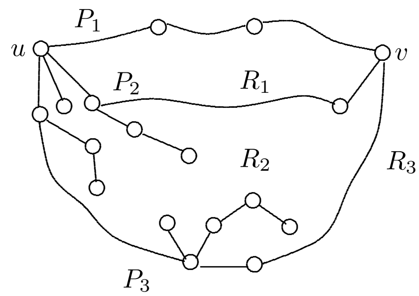

Figure 5. An embedding of the graph G in Subcase 1.2

We define a labeling ℓ of G such that $\ell(x) = \ell_F(x)$ if $x \in V(F)$ and $\sum(\ell, U) = 0$ in $\mathbb{Z}_3$ (by Lemma 2.2). Since B_2^* is the boundary of the region R_2^* of F, it follows that

$$\begin{aligned}\sum(\ell_F, B_2^*) &= \ell_F(u) + \ell_F(v) + \sum(\ell_F, Q_2) + \sum(\ell_F, Q_3)\\ &= \ell(u) + \ell(v) + \sum(\ell, Q_2) + \sum(\ell, Q_3) = 0 \text{ in } \mathbb{Z}_3.\end{aligned}$$

In the graph G, the value of the boundary B_1 of R_1 is $\sum(\ell, B_1) = \sum(\ell_F, B_1^*) = 0$ in $\mathbb{Z}_3$, the value of the boundary of R_2 is

$$\begin{aligned}\sum(\ell, B_2) &= \ell(u) + \ell(v) + \sum(\ell, Q_2) + \sum(\ell, Q_3) + \sum(\ell, U)\\ &= \sum(\ell_F, B_2^*) + \sum(\ell, U) = 0 + 0 = 0 \text{ in } \mathbb{Z}_3,\end{aligned}$$

and the value of the boundary of R_3 is $\sum(\ell, B_3) = \sum(\ell_F, B_3^*) = 0$ in $\mathbb{Z}_3$. Consequently, ℓ is a zonal labeling of G.

Case 2. *There is no path P_i, $i = 1, 2, 3$, every interior vertex of which has degree* 2 *in* G. Hence, $n_i \geq 3$ for $i = 1, 2, 3$. In this case, we don't need the condition that $n_1 \leq n_2 \leq n_3$. Let $U = V(G) - V(F)$. For each integer i with $1 \leq i \leq 3$, let $U_i \subseteq U$ consist of those vertices belonging to any branch at an *interior* vertex of P_i. In addition, let $U_0 \subseteq U$ consist of those vertices belonging to any branch at u or v. Thus, $|U_i| \geq 1$ for $i = 1, 2, 3$ and $|U_0| \geq 0$. We consider two subcases.

Subcase 2.1. *There is an integer $i \in \{1, 2, 3\}$ such that* $|U_0 \cup U_i| \geq 2$. Since we don't use the condition that $n_1 \leq n_2 \leq n_3$, we may assume that $|U_0 \cup U_1| \geq 2$. We embed G in the plane in such a way that the boundary of R_1 consists of P_1 and P_2, the boundary of R_2 consists of P_2, P_3, and every branch at any *interior* vertex of P_2 and P_3, and the boundary of R_3 consists of P_1, P_3, and every branch at any vertex of P_1 (including u and v) (see Figure 6, for example, where $|U_0| = 1$, $|U_1| = 3$, $|U_2| = 1$, and $|U_3| = 4$). Since $n_i \geq 3$ for $i = 1, 2, 3$, it follows that $G_1 = G - (U_0 \cup U_1)$ is a graph of cycle rank 2 and type (3) that satisfies the conditions of Subcase 1.2. Thus, G_1 is zonal. Let ℓ_1 be a zonal labeling of G_1. We then extend the labeling ℓ_1 of G_1 to a zonal labeling ℓ of G by defining $\ell(x) = \ell_1(x)$ if $x \in V(G_1)$ such that $\sum(\ell, U_0 \cup U_1) = 0$ by Lemma 2.2.

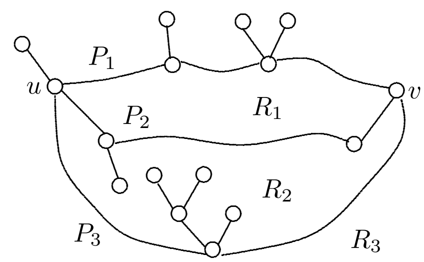

Figure 6. An embedding of the graph G in Subcase 2.1

Subcase 2.2. *For each* $i \in \{1, 2, 3\}$, $|U_0 \cup U_i| = 1$. Since $|U_i| \geq 1$ for $i = 1, 2, 3$, it follows that $U_0 = \emptyset$ and $|U_1| = |U_2| = |U_3| = 1$. For $i = 1, 2, 3$, let $U_i = \{w_i\}$ and so w_i is adjacent to an interior vertex of P_i. We embed the graph G in the plane in such a way that w_i belongs to R_i for $i = 1, 2, 3$. We now define a labeling ℓ of G by letting $\ell(u) = \ell(v) = 1$ and $\ell(w_i) = 2$ for $i = 1, 2, 3$ such that $\sum(\ell, Q_i) = 1$ in $\mathbb{Z}_3$ for $i = 1, 2, 3$ by Lemma 2.2. The value of the boundary of R_i ($i = 1, 2, 3$) is $\ell(u) + \ell(v) + \ell(w_i) + \sum(\ell, Q_i) + \sum(\ell, Q_j) = 1 + 1 + 2 + 1 + 1 = 0$ in $\mathbb{Z}_3$, where $j \in \{1, 2, 3\} - \{i\}$. Consequently, ℓ is a zonal labeling of G. □

Combining the results on graphs of a cycle rank 2 and type (3), we have the following characterization of zonal graphs of of a cycle rank 2 and type (3).

Corollary 3.7. *Let G be a graph of a cycle rank 2 and type (3) containing three internally disjoint paths of order n_i for $i = 1, 2, 3$ where $2 \leq n_1 \leq n_2 \leq n_3$ and $n_2 \geq 3$. Then G is zonal if and only if G is not minimal or G is minimal and $(n_1, n_2) \neq (2, 3)$.*

As a consequence of Theorem 3.1 and 3.2, and Corollary 3.7, we present the following characterization of those graphs of a cycle rank 2 that are zonal.

Theorem 3.8. *Let G be a graph of a cycle rank 2.*

(1) *If G is of type (1), then G is zonal if and only if G is not minimal.*
(2) *If G is of type (2), then G is zonal if and only if either every vertex of G belongs to a cycle of G or at least two vertices of G belong to no cycle of G.*
(3) *If G is of type (3) with three internally disjoint paths of order n_i for $i = 1, 2, 3$ where $2 \leq n_1 \leq n_2 \leq n_3$ and $n_2 \geq 3$, then G is zonal if and only if G is not minimal or G is minimal and $(n_1, n_2) \neq (2, 3)$.*

While every planar embedding of a zonal graph of cycle rank 0 or 1 is zonal, this is not case for zonal graphs of cycle rank 2. For example, consider the two zonal graphs H_1 and H_2 of cycle rank 2 in Figure 7. Each of H_1 and H_2 has exactly two distinct planar embeddings, as shown in Figure 7. Thus, every planar embedding of the zonal graph H_1 is zonal (where a zonal labeling of each planar embedding of H_1 is also shown in that figure); while this is not true for the zonal graph H_2 (since one of the two planar embeddings of H_2 is not zonal). Observe that each of H_1 and H_2 has the form $F \star K_2$ for some minimal graph F of cycle rank 2. We close with the following result dealing with minimal graphs of cycle rank 2, which we state without proof.

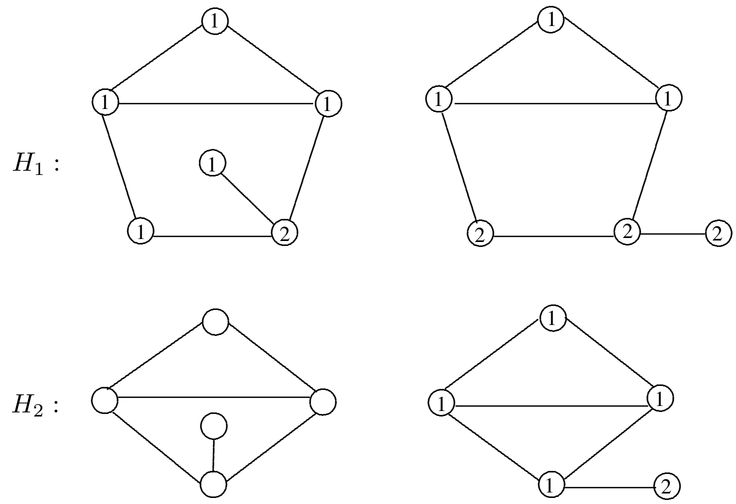

Figure 7. Two planar embeddings of two zonal graphs

Theorem 3.9. *Let F be a minimal graph of cycle rank 2.*

(1) *If F is of type (1), then every planar embedding of $F \star K_2$ is zonal.*
(2) *If F is of type (2) such that at least one vertex of F belongs to no cycle in F, then every planar embedding of $F \star K_2$ is zonal.*
(3) *If F is of type (3), then the following hold:*
(3.1) *If F is zonal, then every planar embedding of $F \star K_2$ is zonal.*
(3.2) *If F is not zonal and w is the only vertex of $F \star K_2$ that does not belong to F, then every planar embedding of $F \star K_2$ is zonal if and only if w is adjacent to a vertex not on any triangle of F.*

4. Closing Remarks on Graphs of Cycle Rank 3

Recall for a connected graph G of order n and size m that the number $m - n + 1$ is called the *cycle rank* of G. Since $K_{3,3}$ is a graph of cycle rank 4 and K_5 is a graph of cycle rank 6, it follows by Kuratowski's theorem [6] that every graph of cycle rank 3 is planar. Results that have been obtained on graphs of cycle rank 0, 1, 2 give rise to the following question: *Which graphs of cycle rank k, $k \geq 3$, are zonal?* The following result was proved in [1].

Proposition 4.1. *Every 2-connected bipartite plane graph is zonal.*

We now present some observations on 2-connected graphs of cycle rank 3.

Proposition 4.2. *There are infinitely many 2-connected zonal graphs of cycle rank 3.*

Proof. Let H be a 2-connected bipartite graph of cycle rank 3 and let E_0 be a set of edges of H. If G is a graph obtained by subdividing each edge of E_0 an even number of times, then G is a 2-connected bipartite graph of cycle rank 3. Thus, G is zonal by Proposition 4.1. In fact, if H is any 2-connected zonal graph of cycle rank 3 and a set E_0 be a set of edges of H, then the graph obtained by subdividing each edge of E_0 at least twice is a 2-connected zonal graph of cycle rank 3 by Lemma 2.2. Hence, there are infinitely many 2-connected zonal bipartite or non-bipartite graphs of cycle rank 3. □

If all edges of a 2-connected bipartite graph of cycle rank 3 are subdivided a number of times of the same parity, then the resulting graph is a 2-connected bipartite graph of cycle rank 3 and so is zonal Proposition 4.1. Thus, we have the following result.

Proposition 4.3. *Let H be a 2-connected bipartite graph of cycle rank 3. If all edges of H are subdivided a number of times of the same parity, then the resulting graph is zonal.*

Proposition 4.4. *There are infinitely many 2-connected non-zonal graphs of cycle rank 3.*

Proof. The 2-connected graph G of Figure 8 has cycle rank 3 and is non-zonal.

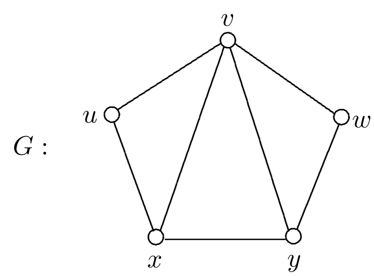

Figure 8. A 2-connected non-zonal graph of cycle rank 3

Any graph obtaining by subdividing the edge xy of the graph G of Figure 8 any number of times is a 2-connected graph of cycle rank 3. We show that each such graph is non-zonal as well. To see this, let H be a graph obtained by subdividing the edge xy of G one or more times. Consequently, H is obtained from G by replacing the edge xy of G by an $x - y$ path P of length 2 or more. Assume, to the contrary, that H has a zonal labeling ℓ. Since three vertices in a triangle of H must be labeled the same, we may assume $\ell(x) = \ell(u) = \ell(v) = \ell(w) = \ell(y) = 1$ by Observation 2.1. Let $X = V(H) - V(G)$. Since the 3-path (x, v, y) and P form the boundary of an interior region of H, it follows that $\ell(x) + \ell(v) + \ell(y) + \sum(\ell, X) = 3 + \sum(\ell, X) = 0$ in $\mathbb{Z}_3$ and so $\sum(\ell, X) = 0$. On the other hand, the 5-path (x, u, v, w, y) and P form the boundary of the exterior region of H and so $\ell(x) + \ell(u) + \ell(v) + \ell(w) + \ell(y) + \sum(\ell, X) = 5 + \sum(\ell, X) = 2 \neq 0$ in $\mathbb{Z}_3$, which is impossible. Therefore, H is not zonal. □

We conclude with the following problem.

Problem 4.5. *Characterize the zonal planar graphs of cycle rank 3.*

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