



Complete multipartite graphs of non-QE class

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Abstract

We derive a formula for the QE constant of a complete multipartite graph and determine the complete multipartite graphs of non-QE class, namely, those which do not admit quadratic embeddings in Euclidean spaces. Moreover, we prove that there are exactly four primary non-QE graphs among the complete multipartite graphs.

Keywords: distance matrix, quadratic embedding, quadratic embedding constant (QEC), primary non-QE graph

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1. Introduction

Realization of a graph $G = (V, E)$ in a Euclidean space \mathbb{R}^N is of fundamental interest. In this paper, we focus on a particular realization called quadratic embedding, which traces back to the early works of Schoenberg [20, 21] and has been studied along with Euclidean distance geometry, see e.g., [1, 3, 6, 7, 10].

Let $G = (V, E)$ be a graph, which is always assumed to be finite and connected. A map $\varphi : V \rightarrow \mathbb{R}^N$ is called a *quadratic embedding* of G if

$$\|\varphi(x) - \varphi(y)\|^2 = d_G(x, y), \quad x, y \in V,$$

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where the left-hand side is the square of the Euclidean distance and the right-hand side is the graph distance. A graph G is called of *QE class* or of *non-QE class* according as it admits a quadratic embedding or not. It follows from Schoenberg [20, 21] that a graph G admits a quadratic embedding if and only if the distance matrix $D = [d_G(x, y)]$ is conditionally negative definite, i.e., $\langle f, Df \rangle \leq 0$ for all real column vectors f indexed by V with $\langle \mathbf{1}, f \rangle = 0$, where $\mathbf{1}$ denotes the column vector of which entries are all one and $\langle \cdot, \cdot \rangle$ is the canonical inner product. In this connection, a new numeric invariant of a graph was proposed in the recent papers [16, 18]. The *quadratic embedding constant (QE constant for short)* of a graph G is defined by

$$\text{QEC}(G) = \max\{\langle f, Df \rangle; \langle f, f \rangle = 1, \langle \mathbf{1}, f \rangle = 0\}, \tag{1}$$

where the right-hand side stands for the conditional maximum of the quadratic function $\langle f, Df \rangle$ with f running over a unit sphere determined by $\langle f, f \rangle = 1$ and $\langle \mathbf{1}, f \rangle = 0$. By definition, a graph G is of QE class if and only if $\text{QEC}(G) \leq 0$. An advantage of the QE constant is that (1) is determined explicitly or estimated finely by means of the method of Lagrange’s multipliers. The QE constants are explicitly known for some special series of graphs, see also [4, 5, 12, 14, 15, 17, 19].

There are interesting questions both on graphs of QE class and on those of non-QE class. One of the important questions on non-QE graphs is to obtain a sufficiently rich list of non-QE graphs. The main purpose of this paper is to determine all complete multipartite graphs of non-QE class. In this paper we first derive a general formula for the QE constant of a complete multipartite graph.

Theorem 1.1 (Theorem 3.1). *Let $k \geq 2$ and $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$. The QE constant of the complete k -partite graph K_{m_1, m_2, \dots, m_k} is given as follows.*

(1) *If $m_1 = m_2$, we have*

$$\text{QEC}(K_{m_1, m_2, \dots, m_k}) = -2 + m_1.$$

(2) *If $m_1 > m_2$, we have*

$$\text{QEC}(K_{m_1, m_2, \dots, m_k}) = -2 - \alpha^*,$$

where α^ is the minimal solution to the equation*

$$\sum_{i=1}^k \frac{m_i}{\alpha + m_i} = 0.$$

Moreover, $-m_1 < \alpha^ < -m_2$.*

Using the above explicit formula, we determine all complete multipartite graphs with positive QE constants, that is, of non-QE class.

Theorem 1.2 (Theorem 4.1). *Let $k \geq 2$ and $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$. Then the complete k -partite graph K_{m_1, m_2, \dots, m_k} is of non-QE class if and only if one of the following conditions is satisfied:*

- (i) $m_1 \geq 3$ and $m_2 \geq 2$,
- (ii) $k \geq 3$, $m_1 \geq 5$ and $m_2 = \dots = m_k = 1$,

- (iii) $k \geq 4$, $m_1 = 4$ and $m_2 = \cdots = m_k = 1$,
- (iv) $k \geq 5$, $m_1 = 3$ and $m_2 = \cdots = m_k = 1$.

If a graph H is isometrically embedded in a graph G , we have $\text{QEC}(H) \leq \text{QEC}(G)$. Hence, if G contains a non-QE subgraph H isometrically, then G is of non-QE class too. Thus, upon classifying graphs of non-QE class it is important to specify a *primary* non-QE graph, that is, a non-QE graph G which does not contain a non-QE graph H as an isometrically embedded proper subgraph.

By Theorem 1.2 there are four families of complete multipartite graphs of non-QE class. From each family a primary one is specified as follows.

Theorem 1.3 (Theorem 4.2). *Among the complete multipartite graphs there are four primary non-QE graphs which are listed with their QE constants as follows:*

$$\begin{aligned}\text{QEC}(K_{3,2}) &= 2/5, \\ \text{QEC}(K_{5,1,1}) &= 1/7, \\ \text{QEC}(K_{4,1,1,1}) &= 2/7, \\ \text{QEC}(K_{3,1,1,1,1}) &= 1/7.\end{aligned}$$

Moreover, any complete multipartite graph of non-QE class contains at least one of the above four primary non-QE graphs as an isometrically embedded subgraph.

This paper is organized as follows. In Section 2 we prepare basic notions and notations, for more details see [15, 18]. In Section 3 we prove the main formula for the QE constants of complete multipartite graphs (Theorem 1.1) and show some examples. In Section 4 we determine all complete multipartite graphs of non-QE class (Theorem 1.2) and specify primary ones (Theorem 1.3). All primary non-QE graphs on six or fewer vertices are already determined [17]. As a result, we find three new primary non-QE graphs on seven vertices.

The QE constant is not only useful for judging whether or not a graph G is of QE-class but also interesting as a possible scale of classifying graphs [4]. Moreover, the QE constant is interesting from the point of view of spectral analysis of distance matrices, for distance spectra see e.g., [2, 8, 9]. In fact, $\text{QEC}(G)$ lies between the largest and the second largest eigenvalues of the distance matrix D in such a way that $\lambda_2(D) \leq \text{QEC}(G) < \lambda_1(D)$. In this line, an interesting question is to characterize graphs such that $\lambda_2(D) = \text{QEC}(G)$. The work is now in progress.

2. Quadratic embedding of graphs

2.1. Distance matrices

A graph $G = (V, E)$ is a pair of a non-empty set V of vertices and a set E of edges, where V is assumed to be a finite set throughout the paper. For $x, y \in V$ we write $x \sim y$ if $\{x, y\} \in E$. A graph is called *connected* if any pair of vertices $x, y \in V$ there exists a finite sequence of vertices $x_0, x_1, \dots, x_m \in V$ such that $x = x_0 \sim x_1 \sim \cdots \sim x_m = y$. In that case the sequence of vertices is called a *walk* from x to y of length m . Unless otherwise stated, by a graph we mean a finite connected graph throughout this paper.

Let $G = (V, E)$ be a graph. For $x, y \in V$ with $x \neq y$ let $d_G(x, y)$ denote the length of a shortest walk connecting x and y . By definition we set $d_G(x, x) = 0$. Then $d_G(x, y)$ becomes a metric on V , which we call the *graph distance*. The *distance matrix* of G is defined by

$$D = [d_G(x, y)]_{x, y \in V},$$

which is a matrix with index set $V \times V$. For notational simplicity we sometimes write $d(x, y)$ for $d_G(x, y)$ when there is no danger of confusion.

Let $C(V)$ be the linear space of \mathbb{R} -valued functions f on V . We always identify $f \in C(V)$ with a column vector $[f_x]_{x \in V}$ through $f_x = f(x)$. The canonical inner product on $C(V)$ is defined

$$\langle f, g \rangle = \sum_{x \in V} f(x)g(x), \quad f, g \in C(V).$$

The distance matrix D induces a linear transformation on $C(V)$ by matrix multiplication as usual. Since D is symmetric, we have $\langle f, Dg \rangle = \langle Df, g \rangle$.

2.2. Quadratic embedding

A *quadratic embedding* of a graph $G = (V, E)$ in a Euclidean space \mathbb{R}^N is a map $\varphi : V \rightarrow \mathbb{R}^N$ satisfying

$$\|\varphi(x) - \varphi(y)\|^2 = d_G(x, y), \quad x, y \in V,$$

where the left-hand side is the square of the Euclidean distance. A graph G is called of *QE class* or of *non-QE class* according as it admits a quadratic embedding or not.

A graph $H = (V', E')$ is called a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. By our convention both G and H are assumed to be connected and hence admit graph distances of their own. We say that H is *isometrically embedded* in G if

$$d_H(x, y) = d_G(x, y), \quad x, y \in V'.$$

In that case we write $H \hookrightarrow G$.

Lemma 2.1. *Let $G = (V, E)$ and $H = (V', E')$ be graphs and assume that H is isometrically embedded in G , that is, $H \hookrightarrow G$.*

- (1) *If G is of QE class, so is H .*
- (2) *If H is of non-QE class, so is G .*

Proof. Obvious by definition. □

We say that a graph of non-QE class is *primary* if it contains no isometrically embedded proper subgraphs of non-QE class. In view of Lemma 2.1, for classifying graphs of non-QE class it is essential to explore primary non-QE graphs.

We mention a simple and useful criterion for isometric embedding. Recall that the *diameter* of a graph $G = (V, E)$ is defined by

$$\text{diam}(G) = \max\{d_G(x, y); x, y \in V\}.$$

Lemma 2.2. *Let $G = (V, E)$ be a graph and $H = (V', E')$ a subgraph.*

- (1) *If H is isometrically embedded in G , then H is an induced subgraph of G .*
- (2) *If H is an induced subgraph of G and $\text{diam}(H) \leq 2$, then H is isometrically embedded in G .*

Proof. (1) By assumption $d_H(x, y) = d_G(x, y)$ for all $x, y \in V'$. In particular, for $x, y \in V'$ $d_H(x, y) = 1$ if and only if $d_G(x, y) = 1$. Therefore, if two vertices $x, y \in V'$ are adjacent in G , so are in H . This means that H is an induced subgraph of G .

(2) We will prove that $d_H(x, y) = d_G(x, y)$ for all $x, y \in V'$. As $d_H(x, y) \leq 2$ for $x, y \in V'$ by assumption, we consider three cases. If $d_H(x, y) = 0$, obviously $d_G(x, y) = 0$. Suppose that $d_H(x, y) = 1$. Then x and y are adjacent in H , so are in G and $d_G(x, y) = 1$. Suppose that $d_H(x, y) = 2$. Since H is a subgraph of G , we have $d_G(x, y) \leq d_H(x, y) \leq 2$. If $d_G(x, y) = 0$, we have $x = y$ and $d_H(x, y) = 0$, which is contradiction. If $d_G(x, y) = 1$, then x and y are adjacent in G , so are in H since H is an induced subgraph of G . We then obtain $d_H(x, y) = 1$, which is again contradiction. Therefore, we have $d_G(x, y) = 2$ and $d_H(x, y) = d_G(x, y)$ holds. \square

2.3. Quadratic embedding constants

Let $G = (V, E)$ be a graph with $|V| \geq 2$. The *quadratic embedding constant* (QE constant for short) of G is defined by

$$\text{QEC}(G) = \max\{\langle f, Df \rangle; f \in C(V), \langle f, f \rangle = 1, \langle \mathbf{1}, f \rangle = 0\},$$

where $\mathbf{1} \in C(V)$ is defined by $\mathbf{1}(x) = 1$ for all $x \in V$.

It follows from Schoenberg [20, 21] that a graph G admits a quadratic embedding if and only if the distance matrix $D = [d_G(x, y)]$ is conditionally negative definite, i.e., $\langle f, Df \rangle \leq 0$ for all $f \in C(V)$ with $\langle \mathbf{1}, f \rangle = 0$. Hence a graph G is of QE class (resp. of non-QE class) if and only if $\text{QEC}(G) \leq 0$ (resp. $\text{QEC}(G) > 0$). The idea of QE constant was proposed in [18], where a standard method of computing the QE constants is established on the basis of Lagrange's multipliers.

Lemma 2.3. *Let $G = (V, E)$ and $H = (V', E')$ be two graphs with $|V| \geq 2$ and $|V'| \geq 2$. If H is isometrically embedded in G , we have*

$$\text{QEC}(H) \leq \text{QEC}(G).$$

As the distance matrix of H becomes a principal submatrix of the distance matrix of G , the proof is straightforward, see also [18]. Note also that Lemma 2.1 follows immediately from Lemma 2.3.

For later reference we recall some concrete values of the QE constants. Further examples are found in [5, 12, 15, 16, 19].

Example 2.1 ([18]). For the complete graph K_n with $n \geq 2$ we have

$$\text{QEC}(K_n) = -1.$$

Conversely, any graph G with $\text{QEC}(G) = -1$ is necessarily a complete graph [4]. Moreover, for any graph G on two or more vertices we have $\text{QEC}(G) \geq -1$.

Example 2.2 ([18]). For the cycles on odd number of vertices we have

$$\text{QEC}(C_{2n+1}) = -\left(4 \cos^2 \frac{\pi}{2n+1}\right)^{-1}, \quad n \geq 1,$$

and for those on even number of vertices we have

$$\text{QEC}(C_{2n}) = 0, \quad n \geq 2.$$

Example 2.3 ([14]). For the paths P_n with $n \geq 2$ we have

$$\text{QEC}(P_n) = -\left(1 + \cos \frac{\pi}{n}\right)^{-1}.$$

Example 2.4 ([18]). For the complete bipartite graph $K_{m,n}$ with $m \geq 1$ and $n \geq 1$ we have

$$\text{QEC}(K_{m,n}) = \frac{2(m-1)(n-1) - 2}{m+n}.$$

The above formula will be generalized to the complete multipartite graphs in Subsection 3.2, see Theorem 3.1.

Example 2.5 ([12]). For $m \geq 1$ and $n \geq 1$ the graph join $\bar{K}_m + K_n$ is called the *complete split graph*. We have

$$\text{QEC}(\bar{K}_m + K_n) = \frac{(m-2)(n-1) - 2}{m+n}.$$

For a more general result on the graph join of regular graphs, see [12].

Example 2.6. A table of the QE constants of graphs on $n \leq 5$ vertices is available [18]. We see directly from the table that all graphs on $n \leq 4$ vertices are of QE-class, and that there are 21 graphs on five vertices among which two are of non-QE class. Those graphs are shown in Figure 1, where G_{n-x} stands for the graph on n vertices with number x in the list of small graphs due to McKay [13]. Their QE constants are given by

$$\text{QEC}(G_{5-10}) = \frac{2}{5}, \quad \text{QEC}(G_{5-17}) = \frac{4}{11 + \sqrt{161}} \approx 0.1689.$$

Both are primary non-QE graphs since all graphs on four vertices are of QE class. Note that G_{5-10} is the complete bipartite graph $K_{3,2}$.

3. Complete multipartite graphs

3.1. Definition and basic properties

For $k \geq 2$ let V_1, V_2, \dots, V_k be mutually disjoint non-empty finite sets. Setting

$$V = \bigcup_{i=1}^k V_i, \quad E = \bigcup_{i \neq j} \{\{x, y\}; x \in V_i, y \in V_j\},$$

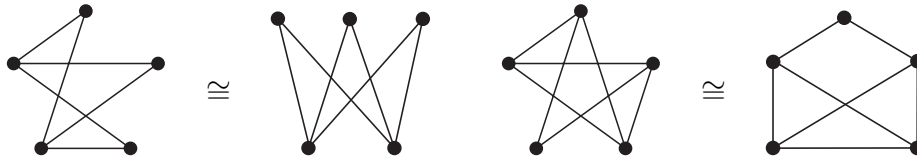


Figure 1. Non-QE graphs: G5-10 (left) and G5-17 (right)

we obtain a graph $G = (V, E)$, which is called a *complete k -partite graph* with parts V_1, V_2, \dots, V_k and is denoted by $KM(V_1, V_2, \dots, V_k)$. A *complete multipartite graph* is a complete k -partite graph for some $k \geq 2$.

For $k \geq 2$ a complete k -partite graph is determined (up to graph isomorphisms) by the sequence m_1, m_2, \dots, m_k defined by $m_i = |V_i|$ and is denoted by

$$K_{m_1, m_2, \dots, m_k} = KM(V_1, V_2, \dots, V_k). \tag{2}$$

Without loss of generality we may assume that $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$. For simplicity we write $K_{1(k)}$ for $K_{1,1,\dots,1}$ (“1” appears k times). Obviously,

$$K_{1(k)} = K_{1,1,\dots,1} = K_k$$

which is nothing else but the complete graph on k vertices. Overusing our notation for the case of $k = 1$ we understand that $KM(V_1)$ is the empty graph on V_1 , that is the complement to the complete graph K_{m_1} so that

$$KM(V_1) = \bar{K}_{m_1}. \tag{3}$$

We note that (2) is not valid for $k = 1$.

For later use we list some basic properties of complete multipartite graphs, of which verification is straightforward and is omitted.

Lemma 3.1. *Let $k \geq 2$ and $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$. Then $\text{diam}(K_{m_1, m_2, \dots, m_k}) = 2$ unless $m_1 = m_2 = \dots = m_k = 1$.*

Lemma 3.2. *Let $k \geq 2$ and let V_1, V_2, \dots, V_k be mutually disjoint non-empty finite sets. Consider an arbitrary partition:*

$$\{V_1, V_2, \dots, V_k\} = \{U_1, \dots, U_p\} \cup \{U'_1, \dots, U'_q\},$$

where $p \geq 1, q \geq 1$ and $p + q = k$. Then we have

$$KM(V_1, V_2, \dots, V_k) = KM(U_1, U_2, \dots, U_p) + KM(U'_1, U'_2, \dots, U'_q),$$

where the right-hand side is the graph join.

For example, we have

$$\begin{aligned} K_{m_1, m_2} &= \bar{K}_{m_1} + \bar{K}_{m_2}, \\ K_{m_1, m_2, m_3} &= \bar{K}_{m_2} + K_{m_1, m_3}, \\ K_{m_1, m_2, m_3, m_4} &= K_{m_1, m_3} + K_{m_2, m_4}, \quad \text{etc.,} \end{aligned}$$

where (3) is taken into account.

Next we study a subgraph of $KM(V_1, V_2, \dots, V_k)$. For $2 \leq p \leq k$ let U_1, U_2, \dots, U_p be p parts chosen from V_1, V_2, \dots, V_k . For each $1 \leq i \leq p$ let $U'_i \subset U_i$ be a non-empty subset. It is then easy to see that

$$KM(U'_1, U'_2, \dots, U'_p) \hookrightarrow KM(V_1, V_2, \dots, V_k).$$

This isometric embedding is stated as follows.

Lemma 3.3. *Let $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$ and $n_1 \geq n_2 \geq \dots \geq n_p \geq 1$ be two sequences with $k \geq 2$ and $p \geq 2$. If there exist $1 \leq i_1 < i_2 < \dots < i_p \leq k$ such that*

$$n_1 \leq m_{i_1}, \quad n_2 \leq m_{i_2}, \quad \dots, \quad n_p \leq m_{i_p},$$

then we have

$$K_{n_1, n_2, \dots, n_p} \hookrightarrow K_{m_1, m_2, \dots, m_k}.$$

Obviously, any induced subgraph of $KM(V_1, V_2, \dots, V_k)$ is of the form $KM(U'_1, U'_2, \dots, U'_p)$ as mentioned above. With the help of Lemma 2.2 we come to the following

Lemma 3.4. *Let H be a subgraph of a complete k -partite graph K_{m_1, m_2, \dots, m_k} . If H is a graph on two or more vertices and is isometrically embedded in K_{m_1, m_2, \dots, m_k} , then H is a complete p -partite graph K_{n_1, n_2, \dots, n_p} of the form as in Lemma 3.3.*

3.2. Calculating QE constants

Throughout this subsection, letting $k \geq 2$ and $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$ be fixed, we set

$$G = K_{m_1, m_2, \dots, m_k}.$$

Our goal is to obtain an explicit formula for $\text{QEC}(G)$. Since $1 \leq \text{diam}(G) \leq 2$ the following general result is useful.

Proposition 3.1. [16, Proposition 2.1] *Let G be a graph with $1 \leq \text{diam}(G) \leq 2$. Let A be the adjacency matrix of G and set*

$$\alpha_{\min} = \min\{\langle f, Af \rangle; \langle f, f \rangle = 1, \langle \mathbf{1}, f \rangle = 0\}. \tag{4}$$

Then we have

$$\text{QEC}(G) = -2 - \alpha_{\min}.$$

Now let A be the adjacency matrix of $G = K_{m_1, m_2, \dots, m_k}$. Then A is written in a block matrix form:

$$A = \begin{bmatrix} 0 & J & J & \dots & J \\ J & 0 & J & \dots & J \\ \vdots & \vdots & \ddots & & \vdots \\ J & J & \dots & 0 & J \\ J & J & \dots & J & 0 \end{bmatrix} = J - \begin{bmatrix} J & 0 & 0 & \dots & 0 \\ 0 & J & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & J & 0 \\ 0 & 0 & \dots & 0 & J \end{bmatrix}, \tag{5}$$

where J is a matrix of which entries are all one and the size is understood in the context. For example, in the last block matrix in (5) J 's appear as diagonal entries. We understand naturally that their sizes are $m_1 \times m_1, \dots, m_k \times m_k$ from left top to right bottom.

We will find the conditional minimum α_{\min} in (4). According to the block matrix form of the adjacency matrix A in (5), any $f \in C(V) \cong \mathbb{R}^{m_1+\dots+m_k}$ is written as

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{bmatrix}, \quad f_i = \begin{bmatrix} f_{i1} \\ f_{i2} \\ \vdots \\ f_{im_i} \end{bmatrix} \in \mathbb{R}^{m_i}, \quad 1 \leq i \leq k.$$

Then we have

$$\langle f, Af \rangle = \langle f, Jf \rangle - \sum_{i=1}^k \langle f_i, Jf_i \rangle = \langle \mathbf{1}, f \rangle^2 - \sum_{i=1}^k \langle \mathbf{1}, f_i \rangle^2. \tag{6}$$

Upon applying the method of Lagrange's multiplier we set

$$\varphi(f, \alpha, \beta) = \langle f, Af \rangle - \alpha(\langle f, f \rangle - 1) - \beta \langle \mathbf{1}, f \rangle.$$

By general theory we know that the conditional minimum (4) is attained at a certain stationary point. Let \mathcal{S} be the set of all stationary points (f, α, β) , that is, the set of solutions to

$$\frac{\partial \varphi}{\partial f_{ij}} = \frac{\partial \varphi}{\partial \alpha} = \frac{\partial \varphi}{\partial \beta} = 0, \quad 1 \leq i \leq k, \quad 1 \leq j \leq m_i. \tag{7}$$

After simple calculus (7) becomes the following system of equations:

$$(J + \alpha)f_i = -\frac{\beta}{2} \mathbf{1}, \quad 1 \leq i \leq k, \tag{8}$$

$$\langle f, f \rangle = \sum_{i=1}^k \langle f_i, f_i \rangle = 1, \tag{9}$$

$$\langle \mathbf{1}, f \rangle = \sum_{i=1}^k \langle \mathbf{1}, f_i \rangle = 0. \tag{10}$$

For $1 \leq s \leq m_i$ the s th entry of the left-hand side of (8) is given by

$$((J + \alpha)f_i)_s = \sum_{t=1}^{m_i} ((J)_{st} + \alpha \delta_{st}) f_{it} = \langle \mathbf{1}, f_i \rangle + \alpha f_{is}.$$

Thus (8) is equivalent to

$$\langle \mathbf{1}, f_i \rangle + \alpha f_{is} = -\frac{\beta}{2}, \quad 1 \leq s \leq m_i, \quad 1 \leq i \leq k. \tag{11}$$

As a result, \mathcal{S} is the set of all solutions $(f = [f_i], \alpha, \beta)$ to the equations (11), (9) and (10). For convenience we denote by \mathcal{S}_A the set of all $\alpha \in \mathbb{R}$ appearing in \mathcal{S} , that is, $(f, \alpha, \beta) \in \mathcal{S}$ for some f and β .

Lemma 3.5. For any $(f, \alpha, \beta) \in \mathcal{S}$ we have $\langle f, Af \rangle = \alpha$. Therefore,

$$\alpha_{\min} = \min \mathcal{S}_A.$$

Proof. Let $(f = [f_i], \alpha, \beta) \in \mathcal{S}$. Noting that the equations (8)–(10) are fulfilled, we have

$$\langle f_i, Jf_i \rangle = \left\langle f_i, -\alpha f_i - \frac{\beta}{2} \mathbf{1} \right\rangle = -\alpha \langle f_i, f_i \rangle - \frac{\beta}{2} \langle \mathbf{1}, f_i \rangle$$

and hence

$$\sum_{i=1}^k \langle f_i, Jf_i \rangle = -\alpha \sum_{i=1}^k \langle f_i, f_i \rangle - \frac{\beta}{2} \sum_{i=1}^k \langle \mathbf{1}, f_i \rangle = -\alpha.$$

On the other hand, by (10) we have

$$\langle f, Jf \rangle = \langle \mathbf{1}, f \rangle^2 = 0.$$

Consequently, we see from (6) that

$$\langle f, Af \rangle = \langle f, Jf \rangle - \sum_{i=1}^k \langle f_i, Jf_i \rangle = 0 - (-\alpha) = \alpha,$$

as desired. The assertion in the second half is then apparent. □

In order to study \mathcal{S}_A we come back to the equations (8)–(10), where (8) is equivalent to (11).

Lemma 3.6. If $(\xi_1, \dots, \xi_k, \alpha, \beta) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}$ satisfies

$$(m_i + \alpha)\xi_i = -\frac{\beta}{2}, \quad 1 \leq i \leq k, \tag{12}$$

$$\sum_{i=1}^k m_i \xi_i^2 = 1, \tag{13}$$

$$\sum_{i=1}^k m_i \xi_i = 0, \tag{14}$$

then, setting $f_i = \xi_i \mathbf{1}$, we have $(f = [f_i], \alpha, \beta) \in \mathcal{S}$. Conversely, for any $(f = [f_i], \alpha, \beta) \in \mathcal{S}$ with $\alpha \neq 0$ there exist $\xi_1, \dots, \xi_k \in \mathbb{R}$ such that $f_i = \xi_i \mathbf{1}$ and (12)–(14) are satisfied.

Proof. It is straightforward to show that $(f = [f_i], \alpha, \beta)$ with $f_i = \xi_i \mathbf{1}$ satisfies (11), (9) and (10) under conditions (12)–(14). We prove the converse. Suppose that $\alpha \neq 0$ and $(f = [f_i], \alpha, \beta)$ is a solution to (11), (9) and (10). In view of $\alpha \neq 0$ we see from (11) that f_{is} is constant independently of $1 \leq s \leq m_i$. In other words, f_i is a constant multiple of $\mathbf{1}$, say $f_i = \xi_i \mathbf{1}$ with $\xi_i \in \mathbb{R}$. Then equations (11), (9) and (10) are reduced to the equations (12), (13) and (14), respectively. □

Lemma 3.7. *If $\alpha \in \mathcal{S}_A$ and $\alpha \notin \{0, -m_1, -m_2, \dots, -m_k\}$, then α satisfies*

$$\sum_{i=1}^k \frac{m_i}{\alpha + m_i} = 0. \tag{15}$$

Conversely, if $\alpha \in \mathbb{R}$ satisfies (15), then $\alpha \in \mathcal{S}_A$.

Proof. Let $\alpha \in \mathcal{S}_A$ with $\alpha \notin \{0, -m_1, -m_2, \dots, -m_k\}$. Then $(f, \alpha, \beta) \in \mathcal{S}$ for some f and β so that the equations (11), (9) and (10) are satisfied. By Lemma 3.6 there exist $\xi_1, \dots, \xi_k \in \mathbb{R}$ such that $f_i = \xi_i \mathbf{1}$ such that $(f = [f_i], \alpha, \beta)$ fulfills (12)–(14). From (12) we have

$$\xi_i = -\frac{\beta}{2} \frac{1}{\alpha + m_i}, \quad 1 \leq i \leq k. \tag{16}$$

Then (13) and (14) become

$$\frac{\beta^2}{4} \sum_{i=1}^k \frac{m_i}{(\alpha + m_i)^2} = 1, \tag{17}$$

$$-\frac{\beta}{2} \sum_{i=1}^k \frac{m_i}{\alpha + m_i} = 0, \tag{18}$$

respectively. We see from (17) that $\beta \neq 0$ and then from (18) we obtain (15).

For the converse assertion let α be a solution to (15). Noting that $\alpha \notin \{-m_1, -m_2, \dots, -m_k\}$, we define ξ_i by (16) and set $f_i = \xi_i \mathbf{1}$. Choosing β satisfying (17), we see that $(f = [f_i], \alpha, \beta) \in \mathcal{S}$ and hence $\alpha \in \mathcal{S}_A$. □

Lemma 3.8. *If $m_1 = m_2$, then $-m_1 \in \mathcal{S}_A$.*

Proof. It is straightforward to see that equations (12)–(14) are fulfilled by

$$\begin{aligned} \xi_1 &= \frac{1}{\sqrt{2m_1}}, & \xi_2 &= -\frac{1}{\sqrt{2m_1}}, & \xi_i &= 0, & 3 \leq i \leq k, \\ \alpha &= -m_1, & \beta &= 0. \end{aligned}$$

It then follows from Lemma 3.6 that $(f = [f_i], \alpha, \beta)$ with $f_i = \xi_i \mathbf{1}$, $\alpha = -m_1$ and $\beta = 0$ belongs to \mathcal{S} . Hence $-m_1 \in \mathcal{S}_A$. □

Lemma 3.9. *If $m_1 > m_2$, then $-m_1 \notin \mathcal{S}_A$.*

Proof. Suppose that $-m_1 \in \mathcal{S}_A$. Then $(f, -m_1, \beta) \in \mathcal{S}$ for some f and β . Since $\alpha = -m_1 \neq 0$, by Lemma 3.6 there exist $\xi_1, \dots, \xi_k \in \mathbb{R}$ satisfying (12)–(14). Note that (12) becomes

$$(m_i - m_1)\xi_i = -\frac{\beta}{2}, \quad 1 \leq i \leq k. \tag{19}$$

Putting $i = 1$, we obtain $\beta = 0$ and (19) becomes

$$(m_i - m_1)\xi_i = 0, \quad 1 \leq i \leq k. \tag{20}$$

Since $m_1 > m_2 \geq m_3 \geq \dots \geq m_k \geq 1$ by assumption, we see from (20) that $\xi_i = 0$ for $2 \leq i \leq k$. It then follows from (14) that $\xi_1 = 0$ too. Thus we obtain $\xi_i = 0$ for all $1 \leq i \leq k$, which contradicts (13). Consequently, $-m_1 \notin \mathcal{S}_A$. \square

We are now in a position to sum up the above argument.

Proposition 3.2. *Let $k \geq 2$ and $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$. Let A be the adjacency matrix of the complete k -partite graph K_{m_1, m_2, \dots, m_k} and set*

$$\alpha_{\min} = \min\{\langle f, Af \rangle; \langle f, f \rangle = 1, \langle \mathbf{1}, f \rangle = 0\}.$$

- (1) *If $m_1 = m_2$, then $\alpha_{\min} = -m_1$.*
- (2) *If $m_1 > m_2$, then $\alpha_{\min} = \alpha^*$, where α^* is the minimal solution to*

$$\sum_{i=1}^k \frac{m_i}{\alpha + m_i} = 0. \tag{21}$$

Moreover, $-m_1 < \alpha^ < -m_2$.*

Proof. We set

$$\psi(\alpha) = \sum_{i=1}^k \frac{m_i}{\alpha + m_i}.$$

It follows from Lemma 3.7 that

$$\mathcal{S}_A \subset \{0, -m_1, \dots, -m_k\} \cup \{\alpha \in \mathbb{R}; \psi(\alpha) = 0\}, \tag{22}$$

where $\{\alpha \in \mathbb{R}; \psi(\alpha) = 0\} = \emptyset$ may occur. Since $\psi(\alpha) < 0$ for all $\alpha < -m_1$, we have $\{\alpha \in \mathbb{R}; \psi(\alpha) = 0\} \subset (-m_1, +\infty)$. Therefore, the minimum of the right-hand side of (22) is $-m_1$.

(1) By Lemma 3.8 we know that $-m_1 \in \mathcal{S}_A$. We then see from (22) that $\min \mathcal{S}_A = -m_1$ so that $\alpha_{\min} = -m_1$.

(2) We first note from $m_1 > m_2$ that $\psi(\alpha) = 0$ has a solution. Hence the minimal solution α^* certainly exists and, as is easily verified by simple calculus, we have $-m_1 < \alpha^* < -m_2$. Moreover, $\alpha^* \in \mathcal{S}_A$ by Lemma 3.8. On the other hand, $-m_1 \notin \mathcal{S}_A$ by Lemma 3.9. Therefore, $\min \mathcal{S}_A = \alpha^*$ so that $\alpha_{\min} = \alpha^*$. \square

We now come to the first main result.

Theorem 3.1 (Theorem 1.1). *Let $k \geq 2$ and $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$.*

- (1) *If $m_1 = m_2$, we have $\text{QEC}(K_{m_1, m_2, \dots, m_k}) = -2 + m_1$.*
- (2) *If $m_1 > m_2$, we have $\text{QEC}(K_{m_1, m_2, \dots, m_k}) = -2 - \alpha^*$, where α^* is the minimal solution to (21). Moreover, $-m_1 < \alpha^* < -m_2$.*

Proof. Straightforward from Propositions 3.1 and 3.2. \square

3.3. Some special cases

We apply Theorem 3.1 to some special cases.

Example 3.1 (See Example 2.4). The QE constant of a bipartite graph $K_{m,n}$ is obtained by simple algebra. If $m = n \geq 1$, we have

$$\text{QEC}(K_{m,m}) = -2 + m. \tag{23}$$

If $m > n$, we have

$$\text{QEC}(K_{m,n}) = \frac{2(m-1)(n-1) - 2}{m+n}. \tag{24}$$

The former formula (23) is reproduced by setting $m = n$ in the latter (24). Moreover, since the right-hand side of (24) is symmetric in m and n , we see that (24) is valid for any $m \geq 1$ and $n \geq 1$.

Example 3.2. The QE constant of a tripartite graph $K_{l,m,n}$ is obtained by simple algebra. If $l = m \geq n \geq 1$, we have

$$\text{QEC}(K_{l,l,n}) = -2 + l. \tag{25}$$

Consider the case of $l > m \geq n \geq 1$. The equation (21) becomes

$$\frac{l}{\alpha+l} + \frac{m}{\alpha+m} + \frac{n}{\alpha+n} = 0$$

and the minimal solution is given by

$$\alpha^* = -\frac{lm + mn + nl + \sqrt{(lm + mn + nl)^2 - 3lmn(l + m + n)}}{l + m + n}.$$

Then we have

$$\text{QEC}(K_{l,m,n}) = -2 + \frac{lm + mn + nl + \sqrt{(lm + mn + nl)^2 - 3lmn(l + m + n)}}{l + m + n}. \tag{26}$$

By simple algebra we see that (25) is reproduced from (26) by setting $l = m$. Moreover, since the right-hand side of (26) is symmetric in l, m and n , we conclude that the formula (26) is valid for any $l \geq 1, m \geq 1$ and $n \geq 1$.

Example 3.3 (see Example 2.5). Let $m \geq 1$ and $n \geq 1$. The $(n+1)$ -partite graph $K_{m,1(n)}$ coincides with the graph join $\bar{K}_m + K_n$, which is known as a *complete split graph*. The QE constant is obtained by simple algebra. If $m = 1$, we have

$$\text{QEC}(K_{1,1(n)}) = -2 + 1 = -1. \tag{27}$$

In fact, $K_{1,1(n)} = K_{n+1}$ is the complete graph and the result is mentioned also in Example 2.1. Assume that $m > 1$. The minimal solution to

$$\frac{m}{\alpha+m} + \frac{n}{\alpha+1} = 0$$

is given by $\alpha^* = -m(n + 1)/(m + n)$ and we then come to

$$\text{QEC}(K_{m,1(n)}) = -2 - \alpha^* = -2 + \frac{m(n + 1)}{m + n} = \frac{(m - 2)(n - 1) - 2}{m + n}. \tag{28}$$

Note that (27) is reproduced from (28) by setting $m = 1$. Hence the formula (28) is valid for any $m \geq 1$ and $n \geq 1$.

Example 3.4. For $n \geq 2$ the complete multipartite graph $K_{2(n)}$ is called a *cocktail party graph* or a *hyperoctahedral graph*. A straightforward application of Theorem 3.1 yields

$$\text{QEC}(K_{2(n)}) = 0.$$

Note that $K_{2(n)}$ is obtained by deleting n disjoint edges from the complete graph K_{2n} . It is known [18] that the QE constant of a graph obtained by deleting two or more disjoint edges from a complete graph is zero.

4. Primary non-QE graphs

4.1. Complete multipartite graphs of non-QE class

With the help of Theorem 3.1 (Theorem 1.1 in Introduction) all complete multipartite graphs of non-QE class are determined.

Let $k \geq 2$ and $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$. If $m_1 \geq 3$ and $m_2 \geq 2$, then K_{m_1, m_2, \dots, m_k} is of non-QE class because $K_{3,2} \hookrightarrow K_{m_1, m_2, \dots, m_k}$ and $K_{3,2}$ is of non-QE class. We will examine the rest cases.

(Case 1) $m_1 \geq 3$ and $m_2 = 1$. In that case we have $m_1 > m_2 = \dots = m_k = 1$ so that $K_{m_1, m_2, \dots, m_k} = K_{m_1, 1(k-1)}$ becomes a complete split graph. Using the formula (28) in Example 3.3, we obtain

$$\text{QEC}(K_{m_1, 1(k-1)}) = \frac{(m_1 - 2)(k - 2) - 2}{m_1 + k - 1}.$$

Hence $\text{QEC}(K_{m_1, 1(k-1)}) > 0$ if and only if one of the following three conditions are fulfilled: (i) $k \geq 3$ and $m_1 \geq 5$; (ii) $k \geq 4$ and $m_1 \geq 4$; (iii) $k \geq 5$ and $m_1 \geq 3$.

(Case 2) $m_1 = 2$. If $m_1 = m_2$, we see from Theorem 3.1 that

$$\text{QEC}(K_{2,2,m_3,\dots,m_k}) = -2 + m_1 = 0.$$

If $m_1 > m_2$, we have $m_2 = \dots = m_k = 1$. Applying the formula (28), we have

$$\text{QEC}(K_{2,1(k-1)}) = -\frac{2}{k + 1} < 0.$$

(Case 3) $m_1 = 1$. In that case we have $m_2 = \dots = m_k = 1$ so that $K_{1(k)} = K_k$ becomes a complete graph. We know that $\text{QEC}(K_{1(k)}) = -1$.

Thus, there are no non-QE graphs in Cases 2 and 3. Summing up the above argument, we claim the following

Theorem 4.1 (Theorem 1.2). *Let $m_1 \geq m_2 \geq \dots \geq m_k \geq 1$ with $k \geq 2$. Then the complete k -partite graph K_{m_1, m_2, \dots, m_k} is of non-QE class if and only if one of the following conditions is satisfied:*

- (i) $m_1 \geq 3$ and $m_2 \geq 2$,
- (ii) $k \geq 3$, $m_1 \geq 5$ and $m_2 = \dots = m_k = 1$,
- (iii) $k \geq 4$, $m_1 = 4$ and $m_2 = \dots = m_k = 1$,
- (iv) $k \geq 5$, $m_1 = 3$ and $m_2 = \dots = m_k = 1$.

The four families of complete multipartite graphs in Theorem 4.1 are mutually exclusive. With the help of Lemma 3.4, from each family we find a primary non-QE graph. The result is stated in the following

Theorem 4.2 (Theorem 1.3). *Among the complete multipartite graphs there are four primary non-QE graphs, which are listed with their QE constants as follows:*

$$\begin{aligned} \text{QEC}(K_{3,2}) &= 2/5, \\ \text{QEC}(K_{5,1,1}) &= 1/7, \\ \text{QEC}(K_{4,1,1,1}) &= 2/7, \\ \text{QEC}(K_{3,1,1,1,1}) &= 1/7. \end{aligned}$$

Moreover, any complete multipartite graph of non-QE class contains at least one of the above four primary non-QE graphs as an isometrically embedded subgraph.

4.2. Exploring primary non-QE graphs

In Theorem 4.2 we find three primary non-QE graphs on seven vertices. Note also that they are all complete split graphs:

$$\begin{aligned} K_{5,1,1} &= \bar{K}_5 + K_2 = \text{G7-351}, \\ K_{4,1,1,1} &= \bar{K}_4 + K_3 = \text{G7-774}, \\ K_{3,1,1,1,1} &= \bar{K}_3 + K_4 = \text{G7-845}, \end{aligned}$$

where G_n-x stands for the graph on n vertices with number x in the list of small graphs due to McKay [13].

On the other hand, as is mentioned in Example 2.6, there are exactly two primary non-QE graphs on five vertices. Moreover, it is proved [17] that there are exactly three primary non-QE graphs on six vertices. They are the graphs G6-30, G6-60 and G6-84, see Figure 2. For the readers' convenience, we record their QE constants:

$$\begin{aligned} \text{QEC}(\text{G6-30}) &= \frac{-4 + \sqrt{19}}{3} \approx 0.1196, \\ \text{QEC}(\text{G6-60}) &= \lambda_{60}^* \approx 0.2034, \\ \text{QEC}(\text{G6-84}) &= \lambda_{84}^* \approx 0.1313, \end{aligned}$$

where λ_{60}^* is the unique positive root of $5\lambda^3 + 26\lambda^2 + 24\lambda - 6 = 0$, and λ_{84}^* is the unique positive root of $3\lambda^4 + 14\lambda^3 + 18\lambda^2 + 5\lambda - 1 = 0$.

In this line an interesting question is to systematically construct primary non-QE graphs and to study the distribution of their QE constants.

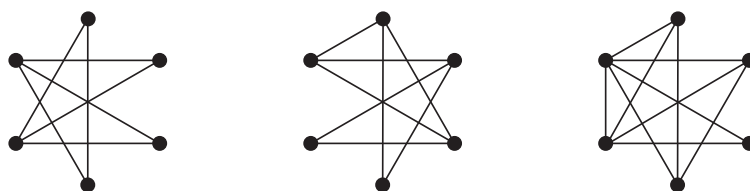


Figure 2. Primary non-QE graphs: G6-30, G6-60 and G6-84 (from left to right)

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