Electronic Journal of Graph Theory and Applications **11** (2) (2023), 343–356

Electronic Journal of
Graph Theory and Applications

Graceful labeling construction for some special tree graph using adjacency matrix

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Abstract

In 1967, Rosa introduced $\beta - labeling$ which was then popularized by Golomb under the name graceful. Graceful labeling on a graph G is an injective function $f : V(G) \rightarrow \{0, 1, 2, \ldots, |E(G)|\}$ such that, when each edge $uv \in E(G)$ is assigned the label $|f(u) - f(v)|$ the resulting edge labels are distinct. If graph G has graceful labeling then G is called a graceful graph. Rosa also introduced $\alpha - labeling$ on graph G which is a graceful labeling f with an additional condition that there is $\lambda \in \{1, 2, \ldots, |E(G)|\}$ so that for every edge $uv \in E(G)$ where $f(u) < f(v)$ then $f(u) \leq \lambda < f(v)$. This paper gives a new approach to showing a graph is admitted $\alpha - labeling$ using an adjacency matrix. Then this construction will be used to construct graceful labeling for the superstar graph. Moreover, we give a graceful labeling construction for a super-rooted tree graph.

Keywords: graceful labeling, α-labeling, star graph, superstar graph, adjacency matrix
Mathematics Subject Classification : 05C78
DOI: 10.5614/ejgta.2023.11.2.1

Received: 4 July 2022, Revised: 6 January 2023, Accepted: 14 April 2023.

1. Introduction

A graph G consists of a finite nonempty set $V(G)$ of objects called vertices and a set $E(G)$ of objects called edges. The order of G is the number of vertices of graph G. The size of G is the number of edges of graph G [3]. In 1967, Rosa introduced $\beta - labeling$ which is more well known as graceful labeling after Golomb in 1972. Graceful labeling is an injective function $f : V(G) \to \{0, 1, 2, 3, \ldots, |E(G)|\}$ where $|E(G)|$ is the number of edges in the graph G such that, when each edge $uv \in E(G)$ is assigned the label $|f(u) - f(v)|$ the resulting edge labels are distinct. If f is a graceful labeling of a graph G of size m, then $g(v) = m - f(v)$ is graceful [3]. In 1967, Rosa also introduced labeling called $\alpha - labeling$. The $\alpha - labeling$ is a graceful labeling with an addition that there exists $\lambda \in \{1, 2, \ldots, |E(G)|\}$ such that for an arbitrary edge $uv \in E(G)$ where $f(u) < f(v)$ then $f(u) \leq \lambda < f(v)$ [8]. The value of λ is called the boundary value of the $\alpha - labeling$ f [4]. If graph G has $\alpha-$labeling then G is called an $\alpha-$labeling graph.

There have been many studies conducted on graceful labeling, where one of the famous conjectures regarding graceful labeling is the Ringel-Kotzig conjecture which says all tree graphs are graceful [4]. People try several constructions and methods to prove the conjecture, for example using computer search as in [1]. The new tree graph which is constructed from the known graph can be found in [9], and [10, 11]. In 2009, Cavalier [2] introduces a new way to show that a graph is graceful using the adjacency matrix. Ghosh [5] also shows several classes of lobster are graceful using an adjacency matrix. For more results in graceful labeling, the reader can see in [4].

Let G be a graph with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ then the matrix $A_G = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i v_j \in E(G), \\ 0, & \text{otherwise,} \end{cases}$$

is called the adjacency matrix of G. Let G be a graph with $|E(G)| = m$ and a valuation $f : V(G) \to \{0, 1, 2, \ldots, m\}$. Then the $(m + 1) \times (m + 1)$ matrix $A_G = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } xy \in E(G) \text{ for } f(x) = i \text{ and } f(y) = j, \\ 0, & \text{otherwise,} \end{cases}$$

is called the generalized adjacency matrix of G induced by f. The generalized adjacency matrix is an adjacency matrix where the vertex labels are used as the indices.

The graph used in this paper is simple and undirected, then its adjacency matrix is symmetric and entries on the main diagonal are 0. Let A be an $n \times n$ matrix, then the kth diagonal line of A is the collection of entries $D_k = \{a_{ij} | j - i = k\}$.

Observation 1.1. *Let G be a labeled graph and let A_G be the generalized adjacency matrix for G. Then A_G has exactly one entry 1 in each diagonal line, except the main diagonal of zeros, if and only if the valuation f on G that induces A_G is graceful.*

For any two graphs G and H, let u and v be fixed vertices of G and H, respectively. Then, the vertex amalgamation of G and H is the graph obtained from G and H by identifying G and H at the vertices u and v[6]. In this paper, we expand the construction given by Cavalier to show that a graph is α-labeling using the adjacency matrix. Then the construction will be used to make a bigger graph from two α-labeling graphs by amalgamating the vertex labeled 0.

In previous research, Pakpahan et al. [7] show that a supercaterpillar constructed from several caterpillar graphs of the same size with each caterpillar having uniform pairs is graceful. So, in this paper, we generalized the result to show that a supercaterpillar constructed from several star graphs of different sizes which is then called a superstar is graceful. Finally, we give a graceful labeling construction of a super-rooted tree graph.

2. Result

In this section, we introduce a new approach by using an adjacency matrix to show that a graph admits α-labeling by using the adjacency matrix, and then we give a graceful labeling construction for several families of trees.

α-Labeling Using Adjacency Matrix

Theorem 2.1. *Let G be a graceful graph of size m and $A_G = [a_{ij}],\ i,j = 0,1,\ldots,m$ induces by graceful labeling f be the generalized adjacency matrix of graph G. Then the labeling f on G that induces A_G is $\alpha-$labeling if and only if there is $a_{\beta\lambda} = 1$ where $\beta - \lambda = 1$, $a_{ij} = 0$ for every $i < \beta, j < i$ and $a_{ij} = 0$ for every $j > \lambda, j < i$.*

Proof. Since G is a graceful graph and A_G is its generalized adjacency graph, then A_G is symmetric. So, we can only focus on the lower triangular part of A_G. That is for edges v_iv_j where $j < i$.

Suppose there is $a_{\beta_1\lambda_1} = 1$ where $\beta_1 - \lambda_1 = 1$, $a_{ij} = 0$ for every $i < \beta_1$, and $a_{ij} = 0$ for every $j > \lambda_1$. Then entry $a_{ij} = 1$ is only possible for $i \geq \beta_1$ and $j \leq \lambda_1$. Since $a_{ij} = 1$ if and only if $f(u) = i, f(v) = j$ and $uv \in E(G)$. Then there is $\lambda = \lambda_1 \in \{1, 2, \ldots, |E(G)|\}$ for an arbitrary edge $uv \in E(G)$ where $f(u) = j, f(v) = i$ and $j < i$, where $f(u) = j \leq \lambda < \lambda + 1 = \beta_1 \leq i = f(v)$. It is concluded that A_G induces $\alpha-$labeling on G.

Suppose f is $\alpha-$labeling on G then there is $\lambda \in \{1, 2, \ldots, |E(G)|$ such that for an arbitrary edge $uv \in E(G)$ where $f(u) < f(v)$ then $f(u) \leq \lambda < f(v)$. So, there is $\lambda_1 = \lambda$ and $\beta_1 = \lambda+1$ for every edge $uv \in E(G)$ where $f(u) = j, f(v) = i$ and $f(u) = j \leq \lambda_1 < \lambda_1 + 1 = \beta_1 \leq i = f(v)$. It means that entry $a_{ij} = 1$ is only possible for $i \geq \beta_1$ and $j \leq \lambda_1$. It is concluded that for every $i < \beta_1, j < i$ then $a_{ij} = 0$, and for every $j > \lambda_1, j < i$ then $a_{ij} = 0$. $\square$

Theorem 2.2. *Let G be an $\alpha - labeling$ graph of size m. Let $A_G = [a_{ij}],\ i,j = 0,1,\ldots,m$ be the generalized adjacency matrix induced by $\alpha - labeling$ f on G with the boundary value is λ. Then the generalized adjacency matrix $A^*_G = [a^*_{m-x\ y}]$ where*

$$a^*_{m-x\ y} = \begin{cases} 0, & \text{if } m-x < \lambda+1 \text{ and } y \leq \lambda, \\ 0, & \text{if } m-x \geq \lambda+1 \text{ and } y > \lambda, \\ a_{(\lambda+1+x) mod(m+1)\ (\lambda-y) mod(m+1)}, & \text{others}, \end{cases}$$

induces a new $\alpha-$labeling f^* on graph G.

Proof. Let G be an $\alpha-$labeling graph of size m. $\alpha-$labeling f on G induces A_G. labeling f is graceful since f is $\alpha-$labeling. It means there will be exactly one entry 1 on each diagonal $D_k, k = 1, 2, \ldots, m$. Let $a^*_{m-x\ y} = a_{ij}$ then $m - x = i \rightarrow x = m - i$ and $y = j$. Therefore

$$\begin{aligned}(m-x)-y &= ((\lambda+1+x) \bmod (m+1)) - ((\lambda-y) \bmod (m+1)) \\ &= (\lambda+1+x-\lambda+y) \bmod (m+1) \\ &= ((\lambda+1-\lambda)+(x+y)) \bmod (m+1) \\ &= (1+(m-i+j)) \bmod (m+1) \\ &= (1+m-(i-j)) \bmod (m+1) \\ &= -(i-j) \bmod (m+1).\end{aligned}$$

Notice that if $i - j = k$ then $-(i - j) \bmod (m + 1) = m - (k - 1)$. That means entry 1 on each diagonal D_k on A_G mapped to $D_{m-(k-1)}$ on A^*_G. Therefore, for each diagonal $D_l, l = (m-x)-y = m-(k-1) = 1, 2, \ldots, m$ on A^*_G has exactly one entry 1. Then we can conclude that the generalized adjacency matrix A^*_G induces graceful labeling f^* on graph G^*. The operation from matrix A_G to matrix A^*_G is a permutation $(0, 1, 2, \ldots, m) \mapsto (\lambda, \lambda-1, \ldots, 0, m, m-1, \ldots, \lambda+1)$. Permuting the indices of a graph produces an isomorphic graph, then graph G^* is isomorphic to G. Therefore f^* is graceful labeling on graph G.

Since f is $\alpha-$labeling then there is $a_{\beta\lambda} = 1$ where $\beta - \lambda = 1$ and $a_{ij} = 0$ for every $i < \beta, j \leq \lambda$,$j < i$ and $a_{ij} = 0$ for every $j > \lambda, i \geq \beta$, $j < i$. Let $\lambda^* = \lambda$ and $\beta^* = \lambda^* + 1 = \lambda + 1 = \beta$. If $\beta^* = m-x$ then $m-x = \lambda+1$. If $\lambda^* = y$ then $y = \lambda$. So, $a^*_{\beta^*\lambda^*} = a_{(\lambda+1+x) \bmod (m+1)\ (\lambda-y) \bmod (m+1)}$. If $\beta^* = m - x$ then $x = m - \beta^* = m - \beta = m - \lambda - 1$. If $\lambda^* = y$ then $y = \lambda$. Therefore,

$$\begin{aligned}a^*_{\beta^*\lambda^*} &= a_{(\lambda+1+x) \bmod (m+1)\ (\lambda-y) \bmod (m+1)}. \\ &= a_{(\lambda+1+m-\lambda-1) \bmod (m+1)\ (\lambda-\lambda) \bmod (m+1)} \\ &= a_{m0} \\ &= 1.\end{aligned}$$

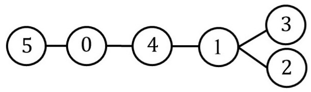
Since $\lambda^* = \lambda$ and $\beta^* = \beta$ then $a^*_{m-x\ y} = 0$ for every $m - x < \beta^*, y \leq \lambda^*$, $y < m - x$ and $a_{m-x\ y} = 0$ for every $y > \lambda^*, m - x \geq \beta^*$, $y < m - x$. So, by Theorem 2.1 we can conclude that A_G induces a new $\alpha-$labeling f^* on graph G. □

Consider entry 1 represented by $a^*_{\beta^*\lambda^*} = a_{m0}$. This operation indicates that the vertex labeled 0 on G with $\alpha - labeling\ f$ turns into a vertex labeled λ on G with the new $\alpha - labeling\ f^*$.

Example 1. Let T be a graph labeled by f. Graph T and its generalized adjacency matrix is shown in Figure 1.

From Figure 1b, we can see that A_T is induced by $\alpha-$labeling f where $\lambda = 1$. Then we can construct the new adjacency matrix $A^*_T = [a^*_{m-xy}]$ where

$$a^*_{m-xy} = \begin{cases} 0, & \text{if } m-x < 2 \text{ and } y \leq 1, \\ 0, & \text{if } m-x \geq 2 \text{ and } y > 1, \\ a_{(2+x) \bmod (m+1)\ (1-y) \bmod (m+1)}, & \text{others.} \end{cases}$$



(a) Graph T with labeling f.

$$A_T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

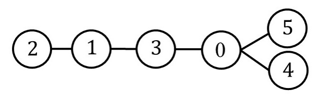
(b) The generalized adjacency matrix of T induced by labeling f.

Figure 1: Graph T and generalized adjacency matrix A_T.

We obtain the new adjacency matrix A_T^* and graph T with a new $\alpha-$labeling f^* as shown in Figure 2.

$$A_T^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Generalized adjacency matrix A_T^*.



(b) Graph T under the new $\alpha-$labeling f^*.

Figure 2: Matrix A_T^* and graph T under the new $\alpha-$labeling f^*.

We can see that the vertex labeled 0 on T with $\alpha-$labeling f turns into a vertex labeled $\lambda = 1$ on T with the new $\alpha-$labeling f^*.

Theorem 2.3. *Let G_1 and G_2 be an $\alpha-$labeling graph of size m_1 and m_2, respectively. Then graph G resulting from the amalgamation of graphs G_1 and G_2 at the vertex labeled 0 is an $\alpha-$labeling graph.*

Proof. Let A_1 and A_2 be the generalized adjacency matrix induced by $\alpha-$labeling f_1 and f_2 on G_1 and G_2, respectively. Then from Theorem 2.2, we can get a new generalized adjacency matrix A_1^* that induces a new $\alpha-$labeling f_1^* on G_1^* where the graph G_1^* is an isomorphic graph of graph G_1. Let λ be the boundary value of $\alpha-$labeling f_1. Then the vertex labeled 0 on G_1 with $\alpha-$labeling f_1 will turn into a vertex labeled λ on G_1 with $\alpha-$labeling f_1^*.

Construct a new generalized adjacency matrix of graph G by amalgamating the vertex labeled 0 on G_2 under the $\alpha-$labeling f_2 with the vertex labeled λ on G_1 under the $\alpha-$labeling f_1^* as shown in Figure 3.

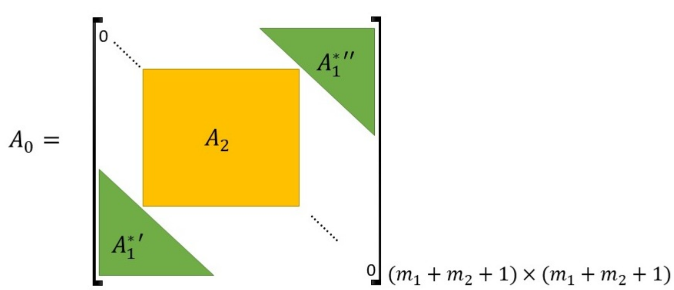


Figure 3: Generalized adjacency matrix of graph G.

Matrix A_2 is placed right on the main diagonal of submatrix $A_1^{*'}$and submatrix $A_1^{*'}$ so that the the first column of A_2 is in the column λ of A_0 and the first row of A_2 is in the row λ of A_0. Matrix A_0 is symmetric since A_1^* and A_2 is symmetric. Therefore we need only focus on the lower triangular part of A_0. That is for edges v_iv_j where $j < i$.

The first column of A_2 is in the column λ of A_0 then the vertex amalgamation occurs at the vertex labeled 0 of G_2 under the $\alpha-$labeling f_2 and the vertex labeled λ of G_1 under the $\alpha-$labeling f_1^*. Placing matrix A_2 right on the main diagonal of $A_1^{*'}$ since there is exactly one entry 1 on each diagonal $D_k, k = 1, 2, \ldots, m_1 + m_2$ and the main diagonal also other entries are 0. Therefore, we can conclude that graph G is graceful. Since Matrix A_0 is obtained from A_1^* and A_2 by permuting the indices of graph G_1^* using permutation $(\lambda_1, \lambda_1 - 1, \ldots, 0, m_1, m_1 - 1, \ldots, \lambda_1 + 1) \mapsto (\lambda_1, \lambda - 1, \ldots, 0, m_1 + m_2, m_1 + m_2 - 1, \ldots, \lambda_1 + m_2 + 1)$ and permuting the indices of graph G_2 using permutation $(0, 1, \ldots, m_2) \mapsto (\lambda_1, \lambda_1 + 1, \ldots, \lambda_1 + m_2)$ then graph G is obtained by vertex amalgamation from the isomorphic graph of graph G_1 and G_2 at vertex labeled 0.

Since G_1 is $\alpha-$labeling, then there is $a_{\beta_1\lambda_1} = 1$ where $\beta_1 - \lambda_1 = 1$ and $a_{ij} = 0$ for every $i < \beta_1, j \leq \lambda_1$, $i > j$ and $a_{ij} = 0$ for every $j > \lambda_1, i \geq \beta_1$, $i > j$ on A_1. Since G_2 is $\alpha-$labeling then there is $a_{\beta_2\lambda_2} = 1$ where $\beta_2 - \lambda_2 = 1$ and $a_{ij} = 0$ for every $i < \beta_2, j \leq \lambda_2$, $i > j$ and $a_{ij} = 0$ for every $j > \lambda_2, i \geq \beta_2$, $i > j$ on A_2. Therefore, there exists $a_{(\beta_1+\beta_2-1)\ (\lambda_1+\lambda_2)} = 1$ where $(\beta_1 + \beta_2) - (\lambda_1 + \lambda_2 - 1) = (\beta_1 - \lambda_1) + (\beta_2 - \lambda_2) - 1 = 1 + 1 - 1 = 1$ and $a_{ij} = 0$ for every $i < \beta_1 + \beta_2 - 1, j \leq \lambda_1 + \lambda_2$, $i > j$ and $a_{ij} = 0$ for every $j > \lambda_1 + \lambda_2, i \geq \beta_1 + \beta_2 - 1$, $i > j$ on A_0. Then it is concluded that G is $\alpha-$labeling. □

Some Special Graceful Tree Graph

Definition 2.1. *A superstar graph is a rooted tree graph constructed from several star graphs by connecting the leaves from each star graph to a root vertex r.*

An example of a superstar graph can be seen in Figure 4. This superstar graph is constructed from four stars S_3, S_4, S_5, S_6, and two isolated vertices (S_o).

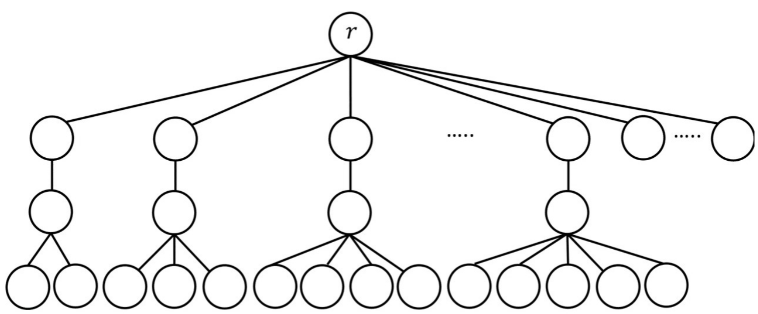


Figure 4: Example of superstar graph.

Theorem 2.4. *Superstar graph constructed by star graph $S_{m_i}, i = 1, 2, \ldots, p$ and q star graph S_0 where $|E(S_{m_i})| = m_i$ and $m_i \leq m_{i+1}$ is $\alpha - labeling$ graph if $i \leq m_i, i = 1, 2, \ldots, p - 1$ for all S_{m_i}.*

Proof. Let T be a tree graph constructed by star graph $S_{m_i}, i = 1, 2, \ldots, p$ and q star graph S_0 where $|E(S_{m_i})| = m_i, i = 1, 2, \ldots, p$ and $m_i \leq m_{i+1}$. Label the center vertex of each star graph S_{m_i} with 0 and the leaves with $1, 2, \ldots, m_i$. Let A_i be the generalized adjacency matrix of each star graph S_{m_i}. Then construct the generalized adjacency matrix of T as shown in Figure 5.

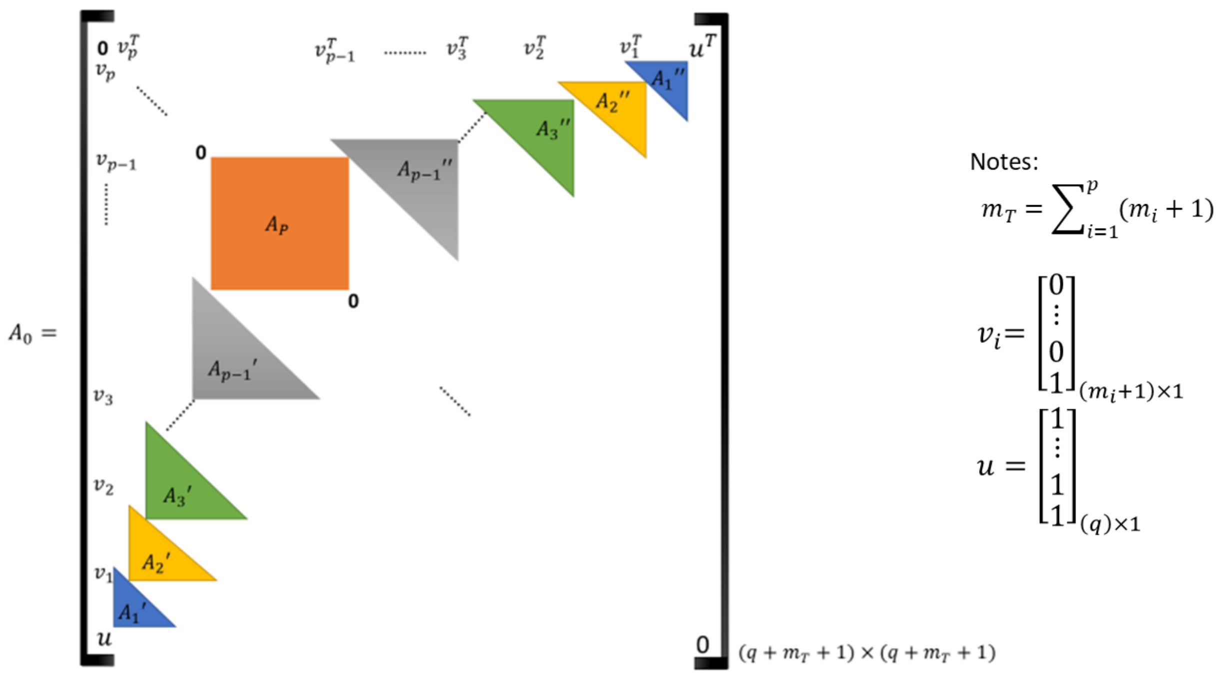


Figure 5: Generalized adjacency matrices of graph T.

The submatrix A_i' represents the entries of matrix A_i which lie on and below the main diagonal. The submatrix A_i'' represents the entries of matrix A_i which lie on and above the main diagonal. The matrix u^T and v_i^T are transposes of matrix u and v_i, respectively. Since A_i is a generalized adjacency matrix, it is symmetric. Thus, A_i' is the transpose of A_i''. We can see that the first column of A_0 is the transpose of the first row of A_0. Therefore, we can conclude that the A_0 matrix is symmetric. Then we only need to focus on the lower triangular part of A_0. Furthermore, the main diagonal of A_0 contains neither the entries of A_i' nor A_i'' and crosses the main diagonal of A_p, then all the entries on the main diagonal are 0.

The matrix u is placed in the first column since there is to be exactly one entry 1 on each diagonal $D_k, k = m_T + 1, m_T + 2, \ldots, m_T + q$. Meanwhile, placing submatrix A_{i+1}' above the main diagonal of submatrix A_i' since there to be exactly one entry 1 in each diagonal $D_k, k = 1, 2, 3, \ldots, m_T$, except for $k = m_p + 1, m_p + m_{p-1} + 2, m_p + m_{p-1} + m_{p-2} + 3, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p$. The remaining entry 1 on the diagonal D_k for $k = m_p + 1, m_p + m_{p-1} + 2, m_p + m_{p-1} + m_{p-2} + 3, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p$ is in the first column. So that all the diagonals D_k except the main diagonal have exactly one entry 1. Therefore, matrix A_0 that constructs the graph T is induced by graceful labeling.

We also have to prove graph T is a superstar graph. Placing submatrix A'_{i+1} in a_{01} of A'_i. Entry 1 in A_i is only on the first column of A'_i. Then there is no entry 1 in A'_i and A'_{i+1} lies in the same column. Therefore, there are no vertices in each star graph S_{m_i} has the same label. Notice that entry 1 in the first column denotes the label of the vertices connected to the root vertex r. Entries 1 are each located on the kth row, where $k = m_p + 1, m_p + m_{p-1} + 2, m_p + m_{p-1} + m_{p-2} + 3, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p + 1, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p + q$. Furthermore, there are as many as $p + q$ different values of k.

The vertices of each S_{m_i} and S_0 will have their respective labels in the range of values as follows.

- $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + m_1 + p + 1$ to $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + m_1 + p + q$ for as many as S_0
- $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + p + 1$ to $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + m_1 + p$ for S_{m_1}
- $m_p + m_{p-1} + m_{p-2} + \cdots + m_3 + p + 1$ to $m_p + m_{p-1} + m_{p-2} + \cdots + m_2 + p$ for S_{m_2}
- $m_p + m_{p-1} + m_{p-2} + \cdots + m_4 + p + 1$ to $m_p + m_{p-1} + m_{p-2} + \cdots + m_3 + p$ for S_{m_3}
  $\vdots$
- $m_p + m_{p-1} + p + 1$ to $m_p + m_{p-1} + m_{p-2} + p$ for $S_{m_{p-2}}$
- $m_p + p + 14$ to $m_p + m_{p-1} + p$ for $S_{m_{p-1}}$, and
- $p + 1$ to $m_p + p$ for S_{m_p}.

Since for each star graph $S_{m_i}, i \leq m_i$, then each different value of k is exactly on each leaf vertex label range of each star graph S_{m_i} and S_0. So, T is a superstar.

Since entry 1 in each adjacency matrix $A_i, i = 1, 2, \ldots, p$ is only on the first column, so there is no $a_{ij} = 1$ on A_0 where $j > p+1$ and $j < i$. Notice that entry 1 on the first column of A_0 is only on the row $k + 1$, where $k = m_p + 1, m_p + m_{p-1} + 2, m_p + m_{p-1} + m_{p-2} + 3, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p + 1, \ldots, m_p + m_{p-1} + m_{p-2} + \cdots + m_1 + p + q$. See that $k + 1 \geq m_p + 2$. So, there is no entry $a_{i0} = 1$ where $i < m_p + 2$ and $j < i$. Since A''_{i+1} is on the first row of A''_i and matrix v_i is on the first row of A_0, then there is no entry $a_{ij} = 1$ on A_0 where $i < p + 2$, $j \geq 1$ and $j < i$. Note that $i \leq m_i$ then $p + 2 \leq m_p + 2$. So, there is no entry $a_{ij} = 1$ where $i < p + 2$ and $j < i$. Thus, there exists $a_{\beta\lambda} = a_{p+2\ p+1} = 1$ where $\beta - \lambda = (p + 2) - (p + 1) = 1$ and $a_{ij} = 0$ for every $i < p + 1, j \leq p + 2$ and $j < i$ also $a_{ij} = 0$ for every $j > p + 2, i \geq p + 1$ and $j < i$. Therefore, T is $\alpha-$labeling. $\square$

Example 2. Let there be 6 star graphs as shown in Figure 6.
We have the generalized adjacency matrix of each graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$ as shown in Figure 7.

Using the construction as in Theorem 2.4, we get the generalized adjacency matrix of superstar graph T shown in Figure 8, and superstar graph T as shown in Figure 9.

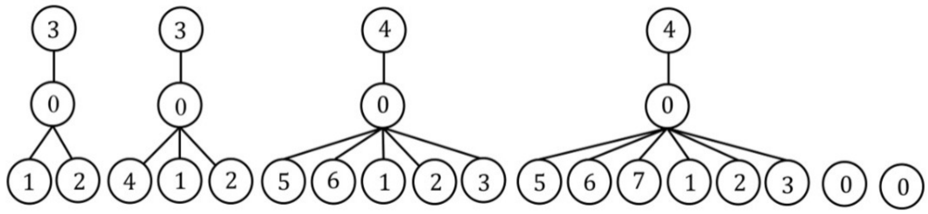


Figure 6: Star graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$, and 2 S_0.

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 7: Generalized adjacency matrix of each Star graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$.

$A_T =$

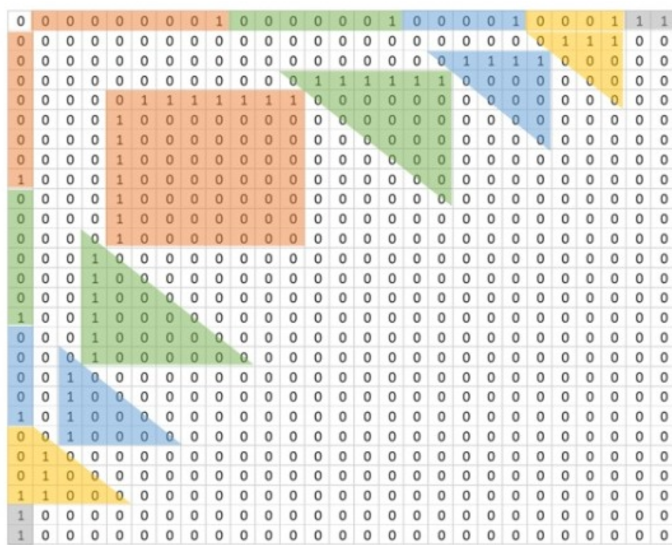


Figure 8: Generalized adjacency matrix of superstar graph constructed by star graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$ and 2 S_0.

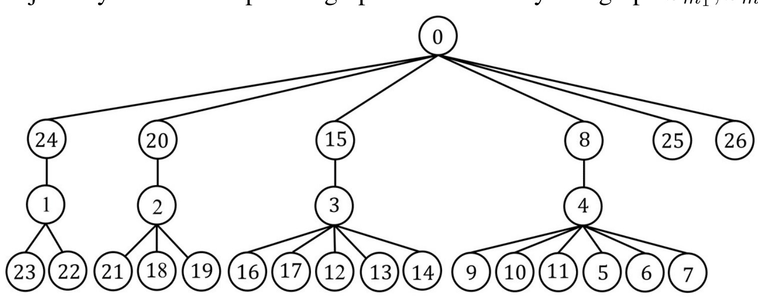


Figure 9: Superstar graph constructed by star graph $S_{m_1}, S_{m_2}, S_{m_3}, S_{m_4}$, and 2 S_0.

Corollary 2.1. *Let T_1 and T_2 be superstar graphs constructed as in Theorem 2.4, then superstar graph T constructed by amalgamating T_1 and T_2 on the root vertex is an $\alpha-$labeling graph.*

Proof. Since superstar graphs T_1 and T_2 are constructed as in Theorem 2.4, then it is an $\alpha-$labeling graph. Therefore, by Theorem 2.3, graph T, which is constructed by amalgamating superstar graphs T_1 and T_2 on the vertex labeled 0, is an $\alpha-$labeling graph. Notice that the root vertex of each superstar graph T_1 and T_2 is labeled 0. T constructed by amalgamating T_1 and T_2 on the root vertex. Then, T is a superstar. □

Using Corollary 2.1, we can construct a superstar graph that does not satisfy **Theorem 2.4**.

Example 3. Construction of a superstar graph from the star graphs in Figure 10.

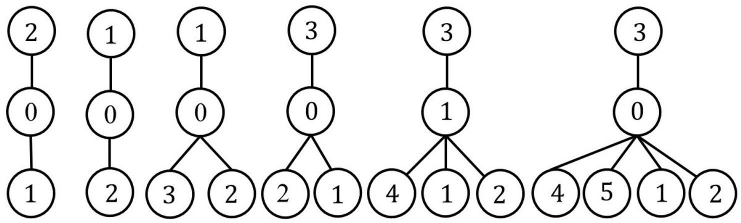


Figure 10: Star graph S_2, S_2, S_3, S_3, S_4 and S_5.

Since $m_4 = 3 < i = 4$, the construction on Theorem 2.4 cannot be used to construct a graceful superstar graph from star graphs S_2, S_2, S_3, S_3, S_4, and S_5. However, we can construct superstar graph T_1 from star graphs S_2, S_2, and S_3 and superstar graph T_2 from star graphs S_3, S_4, and S_5 so that T_1 and T_2 satisfy Theorem 2.4. Superstar Graphs T_1 and T_2 are shown in Figure 11.

(a) Superstar graph T_1. (b) Superstar graph T_2.

Figure 11: Superstar graph T_1 and T_2.

Next, we need to construct a new graceful labeling on superstar graph T_1 by using Theorem 2.2. Superstar graph T_1 with the new graceful labeling is shown in Figure 12.
Then, we can construct a graceful superstar graph T from star graphs S_2, S_2, S_3, S_3, S_4, and S_5 by amalgamating superstar graphs T_1 and T_2 on the root vertex by using Theorem 2.3. Superstar graph T is shown in Figure 13.

Definition 2.2. *A super-rooted tree is a rooted tree constructed from several rooted trees by connecting the root vertex of each rooted tree to a root vertex r.*

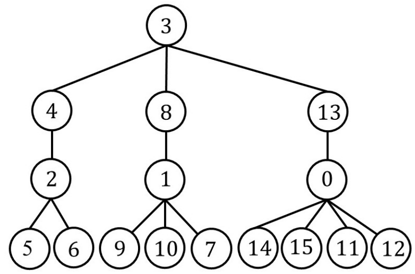


Figure 12: Superstar graph T_1 with the new graceful labeling.

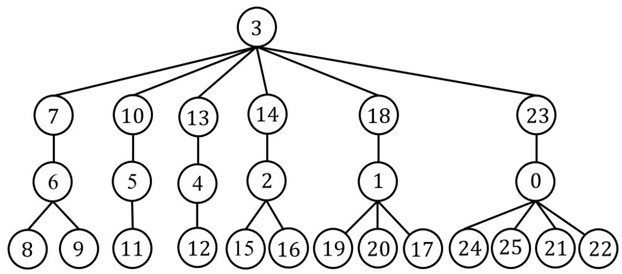


Figure 13: Superstar graph T.

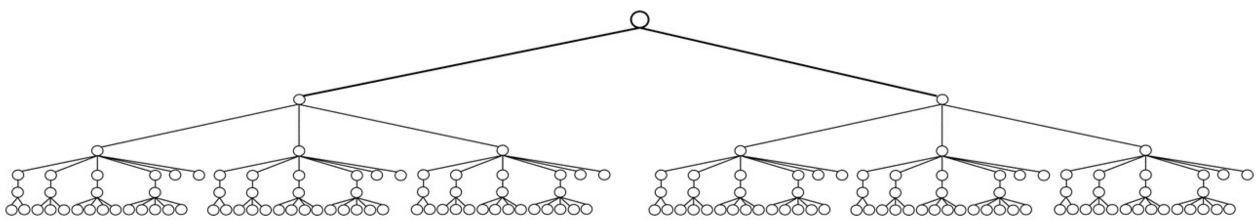


Figure 14: The example of super rooted tree graph.

Theorem 2.5. *Let T be a super rooted tree constructed by q uniform rooted tree T_1 of size m. If T_1 is a graceful graph with the root vertex of T_1 labeled by 0 or m, then T is graceful.*

Proof. We start by considering that tree T is constructed by q uniform tree T_1 of size m. Suppose T_1 has graceful labeling f. Let the root vertex of T_1 be r_1 and $f(r_1) = 0$ or $f(r_1) = m$.

Case 1. If $f(r_1) = 0$.

Let A_{T_1} be the generalized adjacency matrix of T_1 induced by graceful labeling f. Then construct the generalized adjacency matrix T as shown in Figure 15.

Matrix v^T is the transpose of matrix v, then the last row of A_T is the transpose of the last column of A_0. Matrix A_{T_1} is symmetric. Thus, matrix A_T is symmetric. So, we only need to focus on the lower triangular part of A_T. The root vertex of tree T is represented by the last row of A_T. Entry 1 on matrix v^T is only on the first column. Since the root vertex of T_1 is labeled by 0, then the root vertex of each tree graph T_1 will connect to the root vertex of T. Thus T is a super-rooted tree.

Since T_1 is graceful, then each diagonal $D_k, k = 1, 2, \ldots, m$ of A_{T_1} has exactly one entry 1,

$$A_T = \begin{bmatrix} 0 & \dots & 0 & A_{T_1} & v \\ \vdots & 0 & .\cdot^{\cdot} & 0 & v \\ 0 & A_{T_1} & 0 & 0 & \vdots \\ A_{T_1} & 0 & 0 & \ddots & v \\ v^T & v^T & \dots & v^T & 0 \end{bmatrix}$$

(a) Matrix A_T.

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(m+1)\times 1}$$

(b) Matrix v.

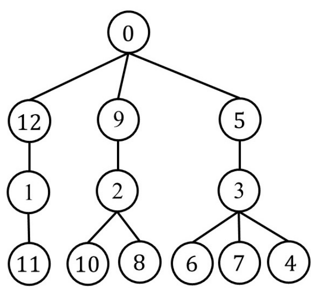
Figure 15: Matrix A_T and Matrix v.

except the main diagonal and other entries are 0. Thus each diagonal $D_l, l = 1, 2, \dots, (m+1)q$, except $l = (m+1)n, n = 1, 2, \dots, q$ has exactly one entry 1. Matrix v_T is $(m+1) \times 1$ matrix. Entry 1 in v_T is only on the first column. Then, entry 1 of each v^T is on the diagonal $D_l, l = (m+1)n, n = 1, 2, \dots, q$. Thus each diagonal line of matrix A_T has exactly one entry 1 except the main diagonal and the other entries are 0. Therefore, T is graceful.

Case 2. If $f(r_1) = m$.

From [3], we know that if $f(v)$ is graceful labeling on graph G of size m then, $g(v) = m - f(v)$ is also graceful labeling on graph G. Since f is graceful labeling on graph T and $f(r_1) = m$, then $g = m - f$ is graceful labeling on graph T and $g(r_1) = m - f(r_1) = m - m = 0$. Thus, we can use the graceful labeling g with $g(r_1) = 0$ to construct a super-rooted tree graph as in **Case 1.** □

Example 4. Let T_1 be the superstar graph as shown in Figure 16.



(a) Superstar T_1.

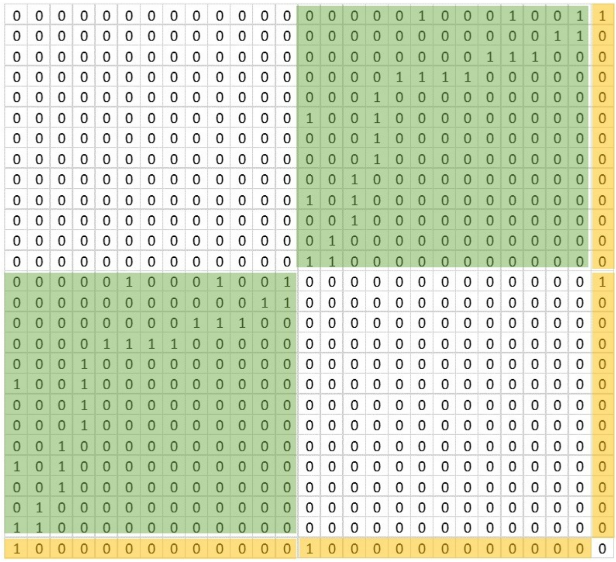
0	0	0	0	0	1	0	0	0	1	0	0	1
0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	1	1	1	0	0
0	0	0	0	1	1	1	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0

(b) The generalized adjacency matrix of superstar T.

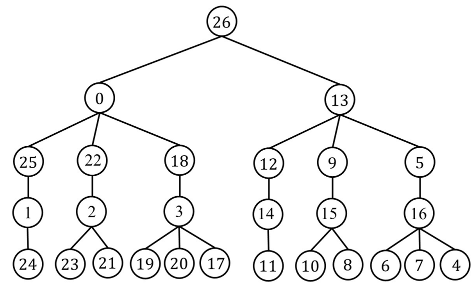
Figure 16: Superstar T_1 and generalized adjacency matrix of T_1.

Then, we can construct the generalized adjacency matrix A_T to get the super-rooted tree T constructed by two superstar graphs T_1 as shown in Figure 17.

Since the root vertex of super-rooted tree T is labeled by m_T. Then, following the construction from Theorem 2.5, we also can make a graceful super-rooted tree G from super-rooted tree T as shown in Figure 18. This process can be repeated several times to obtain a bigger tree.



(a) Matrix A_T.



(b) Super-rooted tree T constructed by 2 superstar graph T_1.

Figure 17: Super-rooted tree T and generalized adjacency matrix A_T.

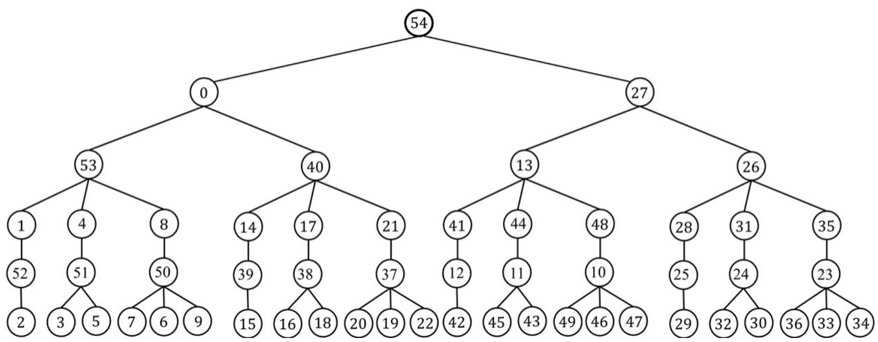


Figure 18: Super-rooted tree G constructed by 2 super-rooted tree T.

Acknowledgement

Part of this research is funded by Universitas Indonesia Research Grant PUTI Q2 No NKB-654/UN2.RST/HKP.05.00/2022.

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