



# The dominating partition dimension and locating-chromatic number of graphs

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## Abstract

For every graph  $G$ , the dominating partition dimension of  $G$  is either the same as its partition dimension or one higher than its partition dimension. In this paper, we consider some general connections among these three graph parameters: partition dimension, locating-chromatic number, and dominating partition dimension. We will show that  $\beta_p(G) \leq \eta_p(G) \leq \chi_L(G)$  for any graph  $G$  with at least 3 vertices. Therefore, we will derive properties for which graphs  $G$  have  $\eta_p(G) = \beta_p(G)$  or  $\eta_p(G) = \beta_p(G) + 1$ .

*Keywords:* partition dimension, coloring, resolving, dominating, locating-chromatic

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## 1. Introduction

The study of determining all coordinates of vertices in a graph has received extensive attention. For example, Ore [8] defined dominating sets to determine the location of vertices in a graph by considering the neighbors of each member of a set of vertices. Slater [9] defined the locating set and locating number by assigning a unique distance coordinate for each vertex to a certain subset of vertices. On the other hand, independently, Harary and Melter [6] also studied the same concept

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but used different terms, namely resolving set and metric dimension. Chartrand, Salehi, and Zhang [5] introduced the resolving partition and partition dimension of a graph as a new perspective in determining the metric dimensions of graphs.

One of the studies to develop the concept of determining the coordinates of vertices in a graph is by combining two well-known concepts. Slater [10] has proposed the idea of combining the resolving set and the dominating set. Later, this study was enhanced by Brigham, et. al. [3] in 2003 and separately by Henning and Oellermann in 2004. Previously, Chartrand et. al. [4] defined a locating-chromatic number, combining the concept of vertex partitioning with vertex coloring. Following the idea of combining the concepts of resolving set and dominating set, Hernando, Mora, and Pelayo [7] expanded the concept of resolving partition by defining dominating partition and dominating partition dimension. They added a dominating condition to a resolving partition of a graph and then called it as the resolving dominating partition of the graph. In this paper, we consider some general connections among these three graph parameters: partition dimension, locating-chromatic number and dominating partition dimension.

Let  $G$  be a simple and connected graph with vertex set  $V(G)$  and edge set  $E(G)$ . The *distance* between vertices  $u$  and  $w$  in  $G$ , denoted by  $d(u, w)$ , is the length of a shortest path connecting  $u$  and  $w$  in  $G$ . The distance between a vertex  $u \in V(G)$  and a subset  $S \subseteq V(G)$  is the minimum of the distances between  $u$  and the vertices of  $S$ , that is,  $d(u, S) = \min\{d(u, w) : w \in S\}$ . Denote  $N_i(u)$  as the set of all vertices of  $G$  at distance  $i$  from  $u$ . The *open neighborhood* of vertex  $u$  is  $N(u) = \{w \in V(G) : uw \in E(G)\}$ , and the *closed neighborhood* of  $u$  is  $N[u] = N(u) \cup \{u\}$ . The *degree*  $d(u)$  of a vertex  $u$  is  $|N(u)|$ . If  $d(u) = 1$  then  $u$  is said to be a *leaf* of  $G$ . A vertex  $u$  of  $G$  is called an *earring* of size  $l$  if  $u$  is adjacent to exactly  $l$  leaves.

Let  $\Pi = \{S_1, S_2, \dots, S_k\}$  be the ordered partition of  $V(G)$ . The *representation* of a vertex  $u \in V(G)$  with respect to  $\Pi$  is the vector consisting all distances from  $u$  to all elements of  $\Pi$ , that is,  $r(u|\Pi) = (d(u, S_1), d(u, S_2), \dots, d(u, S_k))$ . The set  $S_i$  is called *resolves*  $u, w \in V(G)$ , if  $d(u, S_i) \neq d(w, S_i)$ . A partition  $\Pi = \{S_1, S_2, \dots, S_k\}$  is called a *resolving partition* of  $G$  if for any pair of distinct vertices  $u, v \in V(G)$ ,  $r(u|\Pi) \neq r(v|\Pi)$ , this is, if the set  $\{u, v\}$  is resolved by some  $S_i$  of  $\Pi$ . The *partition dimension*  $\beta_p(G)$  of  $G$  is the minimum cardinality of a resolving partition of  $G$ .

A partition  $\Pi = \{S_1, S_2, \dots, S_k\}$  is called a *dominating partition* of  $G$  if for every  $v \in V(G)$ ,  $d(v, S_j) = 1$ , for some  $j \in \{1, 2, \dots, k\}$ . The partition  $\Pi$  is called a *resolving dominating partition* of  $G$ , if it is both resolving partition and dominating partition. A resolving dominating partition of  $G$  with minimum cardinality is called a *minimum resolving dominating partition* of  $G$ . The cardinality of a minimum resolving dominating partition of  $G$  is called the *dominating partition dimension* of  $G$ , denoted by  $\eta_p(G)$ .

Let  $\sigma$  be a proper  $k$ -coloring of  $G$ , which mean that any two adjacent vertices in  $G$  have distinct colors. Recall that a proper  $k$ -coloring  $\sigma$  is equivalent to a partition  $\Pi = \{S_1, S_2, \dots, S_k\}$  of  $V(G)$  where  $S_i$  is the set of vertices receiving color  $i$  for  $1 \leq i \leq k$ . Let  $u \in V(G)$  and  $\Pi$  be a partition of  $V(G)$  induced by  $\sigma$ . The *color code*  $\sigma_\Pi(u)$  of a vertex  $u$  is defined as the ordered  $k$ -tuple  $(d(u, S_1), (u, S_2), \dots, (u, S_k))$ . The proper  $k$ -coloring  $\sigma$  (or partition  $\Pi$ ) is called a *locating-chromatic  $k$ -coloring* of  $G$ , *locating  $k$ -coloring* for short, if all vertices of  $G$  have distinct color codes. The *locating-chromatic number*  $\chi_L(G)$  of  $G$  is the smallest  $k$  such that  $G$  has a locating  $k$ -coloring, and this locating  $k$ -coloring is called a *minimum locating coloring* of  $G$ .

**2. Main Results**

For any graph  $G$  of order  $n \geq 3$ , Hernando et al. [7] showed the dominating partition dimension of  $G$  is equal to either the partition dimension of  $G$  or the partition dimension of  $G$  plus one.

**Theorem 2.1.** [7] For any graph  $G$  of order  $n \geq 3$ ,  $\beta_p(G) \leq \eta_p(G) \leq \beta_p(G) + 1$ .

Based on Theorem 2.1, we can classify all graphs  $G$  depending on the value of its dominating partition dimension. A graph  $G$  is said to be of type DP1 if  $\eta_p(G) = \beta_p(G)$ , otherwise we call  $G$  as a graph of type DP2 (if  $\eta_p(G) = \beta_p(G) + 1$ ).

In this paper, we would like to classify which graphs  $G$  of type DP1 or type DP2. Before classifying these graphs, let us consider some general connections between these three graph parameters: partition dimension, locating-chromatic number, and dominating partition dimension.

**Theorem 2.2.** For any connected graph  $G$ , every locating coloring of  $G$  is also a resolving dominating partition of  $G$ .

*Proof.* Let  $G$  be a graph. Let  $\sigma$  be any locating coloring of  $G$  and  $\Pi := \{S_1, S_2, \dots, S_k\}$  be the partition of  $V(G)$  induced by  $\sigma$ . Since  $\sigma$  is a locating coloring of  $G$ , then for any two distinct vertices  $x$  and  $y$  in  $G$  there exists  $S_i$  for some  $i \in [1, k]$  such that  $d(x, S_i) \neq d(y, S_i)$ . So,  $r(x|\Pi) \neq r(y|\Pi)$  and  $\Pi$  is a resolving partition of  $G$ . Now, since every coloring of  $G$  is also a proper coloring, then for every two adjacent vertices  $x, y$  in  $G$ , we have  $\sigma(x) \neq \sigma(y)$ . So, this implies that  $x$  and  $y$  belong to different partition classes of  $\Pi$ . This fact yields that every vertex  $x$  is dominated by some partition class  $S_i$  for some  $i \in [1, k]$ . Thus,  $\Pi$  is a dominating partition of  $G$ . Therefore,  $\Pi$  is a resolving dominating partition of  $G$ . □

The converse of Theorem 2.2 is not always true. The following graph  $G$  in Figure 1 has  $\eta_p(G) = \beta_p(G) = 4$ , but  $\chi_L(G) = 5$ . A resolving partition as well as a resolving dominating partition of  $G$  with minimum cardinality is shown in Figure 1(a). A locating coloring of  $G$  with a minimum number of colors is shown in Figure 1(b).

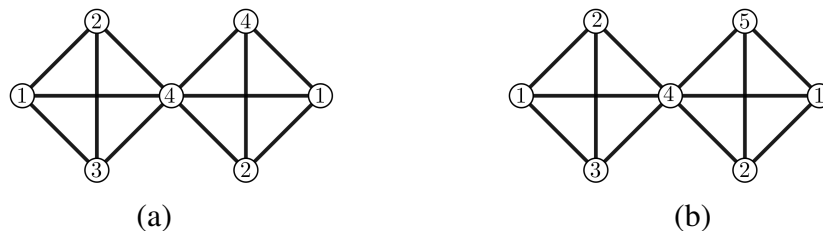


Figure 1. Graph  $G$  with  $\eta_p(G) = \beta_p(G) = 4$ , but  $\chi_L(G) = 5$ .

**Corollary 2.1.** For any graph  $G$  of order  $n \geq 3$ ,  $\beta_p(G) \leq \eta_p(G) \leq \chi_L(G)$ .

*Proof.* The first inequality ( $\beta_p(G) \leq \eta_p(G)$ ) follows directly from the definition of a resolving dominating partition of  $G$ . The second inequality ( $\eta_p(G) \leq \chi_L(G)$ ) follows from Theorem 2.2. □

The difference between the dominating partition dimension and the locating-chromatic number can vary. The following theorem shows the existence of a graph  $G$  with  $\eta_p(G) = a$  and  $\chi_L(G) = b$  for any integers  $a$  and  $b$  with  $3 \leq a < b \leq 2(a - 1)$ .

**Theorem 2.3.** For any integers  $a$  and  $b$  with  $3 \leq a < b \leq 2(a - 1)$ , there exists a graph  $G$  with  $\beta_p(G) = \eta_p(G) = a$  and  $\chi_L(G) = b$ .

*Proof.* Consider the graph  $G$  in Figure 2. A minimum locating coloring of  $G$  is shown in Figure 2(a). In Figure 2 (b), the partition gives a minimum resolving partition as well as a minimum resolving dominating partition of  $G$ . Hence, we have  $\beta_p(G) = \eta_p(G) = a$  and  $\chi_L(G) = b$ .  $\square$

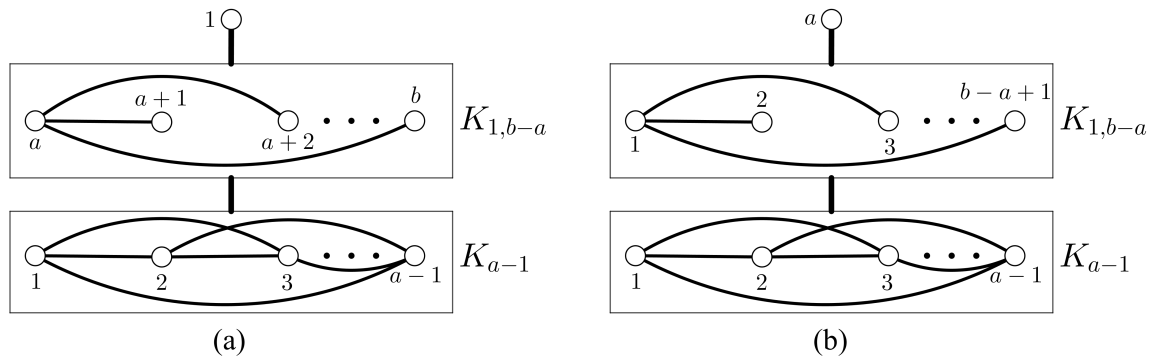


Figure 2. (a)  $\chi_L(G) = b$ , (b)  $\eta_p(G) = \beta_p(G) = a$ .

There are many classes of graphs with these three parameters having the same values. An example of a graph  $G$  with a small order and  $\beta_p(G) = \chi_L(G) = \eta_p(G)$  is given in Figure 3. A resolving partition of  $G$  with minimum cardinality is shown in Figure 3(a). A resolving dominating partition of  $G$  with minimum cardinality is shown in Figure 3(b).

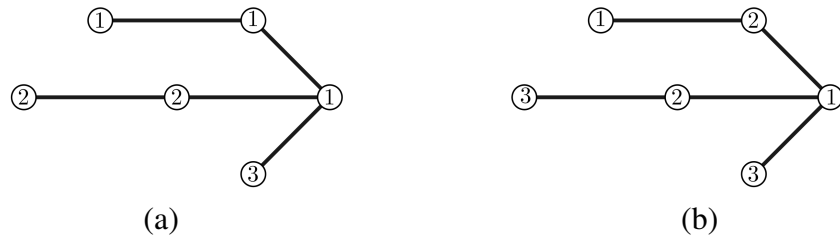


Figure 3. Graph  $G$  with  $\eta_p(G) = \beta_p(G) = \chi_L(G) = 3$ .

In the following two remaining sections, the graphs which are of type DP1 will be presented in section 3. In Section 4, we will further provide some classes of graphs of type DP2. In particular, we classify some class graphs of type DP1 or type DP2 with small parameters: partition dimension and dominating partition dimension. Before going to the next section, some known results regarding these parameters are shown.

**Theorem 2.4.** [5, 7] Let  $G$  be a graph on  $n$  vertices. Then,

- (a)  $\beta_p(G) = 2$  if and only if  $G$  is a path  $P_n$  of order  $n \geq 2$ .
- (b)  $\eta_p(G) = 2$  if and only if  $G$  is isomorphic to  $K_2$ .
- (c) If  $G$  is either a path or a cycle of order  $n \geq 3$ , then  $\eta_p(G) = 3$ .

### 3. Graphs of type DP1

In this section, we will further derive some graphs  $G$  of type DP1, namely graphs  $G$  with  $\eta_p(G) = \beta_p(G)$ . We begin by giving some graphs of type DP1 with a small dominating partition dimension. By Theorem 2.4 point (a) and (b), we have that if  $\eta_p(G) = 2$ , then the only graph  $G$  with  $\eta_p(G) = \beta_p(G) = 2$  is a complete graph  $K_2$ .

**Theorem 3.1.** Let  $C_n$  be a cycle on  $n \geq 3$  vertices then  $C_n$  is a graph of type DP1 with  $\eta_p(G) = \beta_p(G) = 3$ .

*Proof.* Let  $C_n$  be a cycle of order  $n \geq 3$  with  $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$  and  $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$ . Consider the sets of vertices:  $L_1 = \{v_1\}$ ,  $L'_1 = \{v_1, v_2\}$ ,  $L_2 = \{v_i \in V(C_n) | \text{even } i\}$ ,  $L'_2 = \{v_i \in V(C_n) | i \neq 2, \text{ even}\}$ , and  $L_3 = \{v_i \in V(C_n) | \text{odd } i\}$ .

It is easy to verify that  $\Pi = \{L_1, L_2, L_3\}$  is both resolving and dominating partition of  $C_n$  if  $n$  is odd and that  $\Pi' = \{L'_1, L'_2, L_3\}$  is both resolving and dominating partition of  $C_n$  if  $n$  is even. Thus,  $\beta_p(C_n) = \eta_p(G) \leq 3$ . According to Theorem 2.4 point (a) and (c),  $\beta_p(C_n) \geq 3$  and  $\eta_p(C_n) \geq 3$ . This implies that  $\eta_p(G) = \beta_p(G) = 3$ . Therefore,  $C_n$  is a graph of type DP1  $\square$

Now, in the following theorems, we will derive graphs  $G$  of type DP1 with  $\eta_p(G) = 3$ . This means that  $G$  has also  $\beta_p(G) = 3$ . To date, there is no complete characterization regarding all graphs with partition dimension three. However, there is a complete characterization on graphs with locating-chromatic three. Let  $\mathcal{T}$  be a set of all trees  $T$  on  $n$  vertices ( $n \geq 3$ ) with locating-chromatic number three. Baskoro and Asmiati (2013) characterized all the members of such a set  $\mathcal{T}$  as follows.

**Theorem 3.2.** [2] A tree  $T$  is in  $\mathcal{T}$  if and only if  $T$  is any subtree of one of the trees (A), (B) or (C) in Figure 4 containing vertices  $X, Y$  and  $Z$ , with  $a \geq 0, b \geq 0, c \geq 0, d \geq 0, e \geq 0, f \geq 0, k \geq 0, p \geq 0; g \geq 1, h \geq 1; e = f$  and  $k = p$ .

For graphs other than trees, Asmiati and Baskoro [1] have also characterized all such graphs  $G$  with  $\chi_L(G) = 3$ . Such graphs are stated in the following theorem.

**Theorem 3.3.** [1] Let  $G$  be a graph other than a tree with  $\chi_L(G) = 3$ . Then,

1. If  $G$  is bipartite then  $G$  is isomorphic to any subgraph of the graph in Figure 5 (A) containing at least all blue edges.
2. If  $G$  is not bipartite then  $G$  is isomorphic to any subgraph of either the graph (B), (C), (D), or (E) in Figure 5 containing the smallest odd blue cycle  $C_m$ .

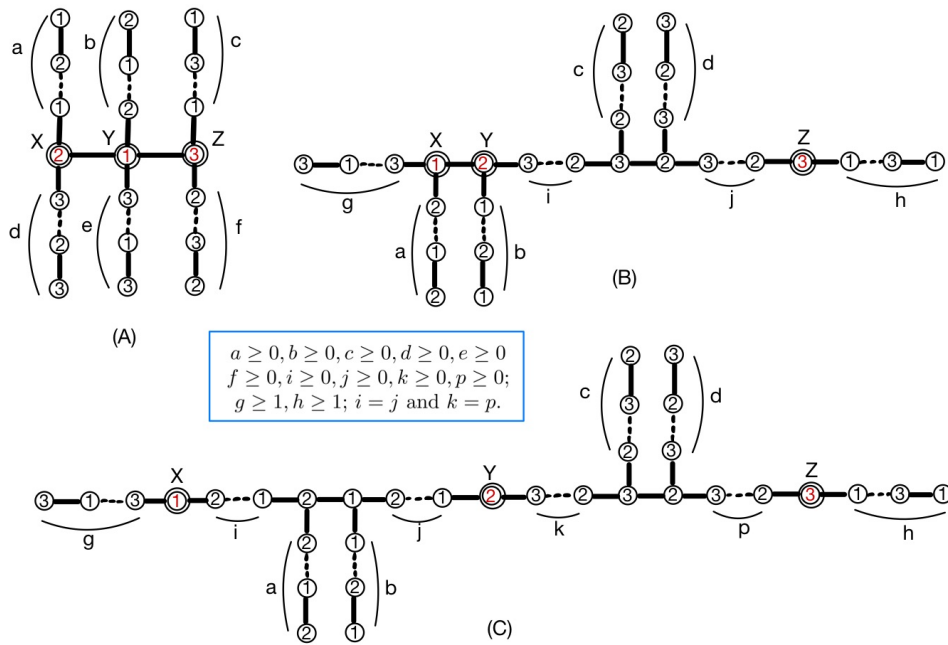


Figure 4. All trees  $T$  with  $\chi_L(T) = 3$  and with a minimum locating coloring.

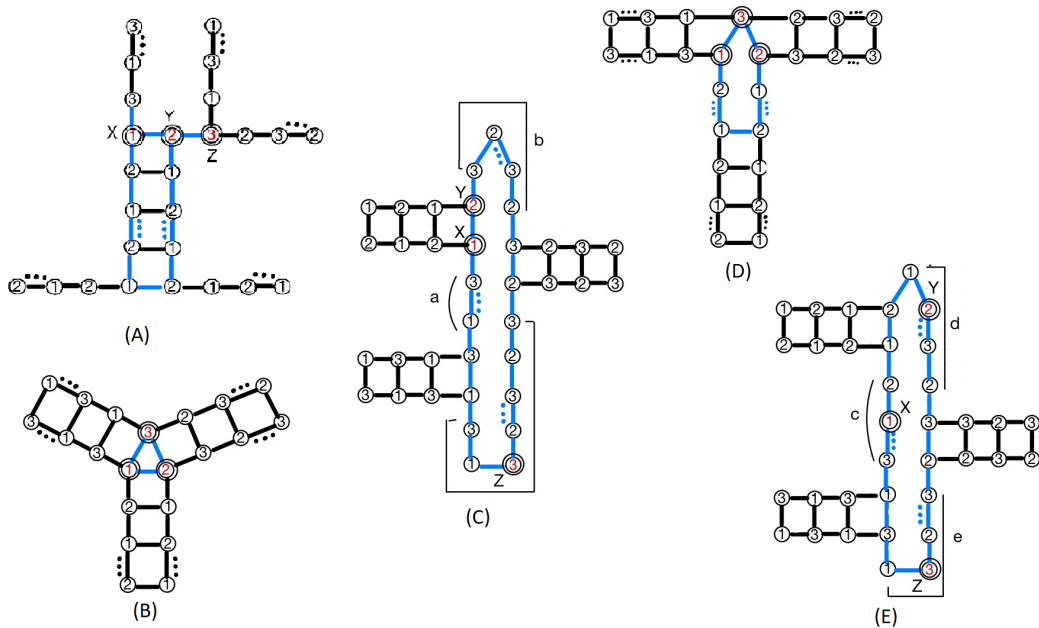


Figure 5. All graphs  $G$  other than trees with  $\chi_L(G) = 3$  and with a minimum locating coloring.

The next corollary gives all graphs of type DP1 with the locating-chromatic number three.

**Corollary 3.1.** *Let  $G$  be a graph other than a path with  $\chi_L(G) = 3$ . Then  $G$  belongs type DP1.*

*Proof.* Let  $G$  be a graph other than a path with  $\chi_L(G) = 3$ . Then,  $G$  must be isomorphic to one of the graphs characterized in Theorem 3.2 or Theorem 3.3. By Corollary 2.1,  $\beta_p(G) \leq \eta_p(G) \leq 3$ . Since the only graph with partition dimension two is a path, then  $\beta_p(G) \geq 3$ . This implies that  $\beta_p(G) = \eta_p(G) = 3$ . Thus,  $G$  is a graph of type DP1 with  $\chi_L(G) = 3$ .  $\square$

#### 4. Graphs of type DP2

In this section, we determine graphs on  $n$  vertices of type DP2, namely the graphs  $G$  with  $\eta_p(G) = \beta_p(G) + 1$ . We start this section with a corollary showing that any path with  $n \geq 3$  vertices is a graph of type DP2.

**Corollary 4.1.**  *$P_n$  on  $n \geq 3$  vertices is a graph of type DP2.*

*Proof.* From Theorem 2.4 part (a) we have that  $\beta_p(P_n) = 2$ . By Theorem 2.4 part (c), we conclude that  $\eta_p(P_n) = 3 = \beta_p(P_n) + 1$ . Therefore,  $P_n$  on  $n \geq 3$  vertices is a graph of type DP2.  $\square$

**Theorem 4.1.** *If  $G$  is a graph on  $n \geq 5$  vertices having a unique earring of size  $k$ , with  $\lceil \frac{n}{2} \rceil \leq k \leq n - 3$ , then  $G$  is of type DP2 with  $\eta_p(G) = \chi_L(G) = k + 1$  and  $\beta_p(G) = k$ .*

*Proof.* Let  $G$  be a graph on  $n \geq 5$  vertices with an earring  $x$  of size  $k$  ( $\lceil \frac{n}{2} \rceil \leq k \leq n - 3$ ). Let  $x_1, x_2, \dots, x_k$  be the leaves adjacent to earring  $x$  in  $G$ , and  $a_1, a_2, \dots, a_{n-k-1}$  are the remaining vertices of  $G$ . Assume that  $\beta_p(G) = k - 1$  and  $\Pi$  is a resolving partition of  $G$  with  $(k - 1)$ -classes. Since  $k$  leaves hanging from  $x$ , there are two leaves, w.l.o.g we may assume  $x_1$  and  $x_2$ , to be included in the same partition class of  $\Pi$ . Note that  $d(x_1, v) = d(x_2, v)$  for all  $v \in V(G) \setminus \{x_1, x_2\}$ , this implies that  $r(x_1|\Pi) = r(x_2|\Pi)$ , a contradiction. Therefore  $\beta_p(G) \geq k$ . Furthermore, let  $\sigma$  be a locating-coloring of  $G$ . Each locating-coloring  $\sigma$  of  $G$  assigns distinct colors to these  $k$  leaves. Since  $x$  is adjacent to these leaves, it must be colored a different color than the  $k$  colors that have been used on the leaves. Therefore,  $\chi_L(G) \geq k + 1$ .

Next, we show that  $\beta_p(G) = k$ ,  $\chi_L(G) = k + 1$ , and  $\eta_p(G) \leq k + 1$ . Now, define  $\Pi_1 := \{\{x, x_1\}, \{x_2, a_1\}, \{x_3, a_2\}, \dots, \{x_{n-k}, a_{n-k-1}\}, \{x_{n-k+1}\}, \dots, \{x_k\}\}$  and  $\Pi_2 := \{\{x\}, \{x_1\}, \{x_2, a_1\}, \{x_3, a_2\}, \dots, \{x_{n-k}, a_{n-k-1}\}, \{x_{n-k+1}\}, \dots, \{x_k\}\}$ . It is easy to verify that  $\Pi_1$  is a resolving partition of  $G$  with minimum cardinality and  $\Pi_2$  is a locating-chromatic of  $G$  with a minimum number of colors. It means that  $\beta_p(G) \leq k$  and  $\chi_L(G) \leq k + 1$ . Therefore,  $\beta_p(G) = k$  and  $\chi_L(G) = k + 1$ . It implies that  $\eta_p(G) \leq k + 1$  by Theorem 2.1 and Corollary 2.1.

Now, if  $\eta_p(G) = k$  then there exists a resolving partition  $\Pi$  of cardinality  $k$ . Since there are  $k$  leaves hanging from  $x$ , then each leaf hanging from  $x$  is in a different partition class of  $\Pi$ . Thus, there is a partition class of  $\Pi$  that contains both  $x$  and a leaf hanging from  $x$ . It implies that  $\Pi$  is not dominating partition. Hence,  $\eta_p(G) \geq k + 1$ , and then  $\eta_p(G) = k + 1$ .  $\square$

Let  $M_{t+1}$  be a tree of order  $t + 1$  with  $2 \leq t \leq \frac{n}{2}$  for any integers  $n$  and  $t$ . Let  $T_n$  be a tree of order  $n$  obtained by connecting  $n - t - 1$  new vertices to a vertex that is not an earring in the  $M_{t+1}$ . Syofyan et al. [11] characterized all trees of order  $n \geq 6$  with locating-chromatic number  $n - t$ . The characterization is as follows.

**Theorem 4.2.** [11] Let  $T_n$  be a tree of order  $n$  with  $n \geq 6$ . Then,  $\chi_L(T_n) = n - t$  where  $2 \leq t < \frac{n}{2}$  if and only if  $T_n$  has exactly one earring of size  $n - t - 1$ .

Now, we characterize all trees on  $n$  vertices with dominating partition dimension  $n - t$  for some  $t$ .

**Theorem 4.3.** Let  $T_n$  be a tree of order  $n$  with  $n \geq 6$ . Then,  $\eta_p(T_n) = n - t$  for some  $2 \leq t < \frac{n}{2}$  if and only if  $T_n$  has exactly one earring of size  $n - t - 1$ .

*Proof.* Let  $T_n$  be a tree of order  $n$  with  $n \geq 6$ . If  $T_n$  has exactly one earring of size  $n - t - 1$  for some fixed  $2 \leq t < \frac{n}{2}$ , then by Theorem 4.1,  $\eta_p(T_n) = n - t$ .

Now, conversely, assume that  $\eta_p(T_n) = n - t$  for some fixed  $2 \leq t < \frac{n}{2}$ . Then, by Corollary 2.1,  $\chi_L(T_n) \geq \eta_p(T_n) = n - t$ . Let  $\chi_L(T_n) = n - t' > n - t$  with  $t' < t$ . Then, by Theorem 4.2,  $T_n$  has exactly one earring of size  $n - t' - 1$ . Thus, by Theorem 4.1,  $\eta_p(T_n) = \chi_L(T_n) = n - t' > n - t$ , a contradiction. Therefore, the theorem follows.  $\square$

**Theorem 4.4.** For any integers  $n \geq 6$  and  $t$  with  $2 \leq t < \frac{n}{2}$ , the only trees  $T_n$  on  $n$  vertices of type DP2 with  $\eta_p(T_n) = n - t$  are the ones with exactly one earring of size  $n - t - 1$ .

*Proof.* This follows from Theorem 4.3 and the fact that  $\beta_p(T_n) = n - t - 1$ , by using a partition  $\Pi := \{\{x_2\}, \{x_3\}, \dots, \{x_{n-t-1}\}, \{x, x_1\} \cup B\}$ , where  $\{x_1, \dots, x_{n-t-1}\}$  are all the vertices of degree one adjacent to earring  $x$  and  $B$  is the set of all the remaining vertices in  $T_n$ .  $\square$

If  $G$  is not a tree and  $G$  is of type DP2 with  $\eta_p(G) = k$  for some integer  $k \geq 3$ , then  $G$  does not necessarily contain an earring of size  $k - 1$ . For any integers  $m, t \geq 3$ , let us consider a graph  $G$  having one earring of size 2 as depicted in Figure 6. We will show that this graph  $G$  is of type DP2 as stated in the following theorem. The number of these graphs  $G$  are infinite since  $m$  and  $t$  can be arbitrary integers greater than or equal to 3.

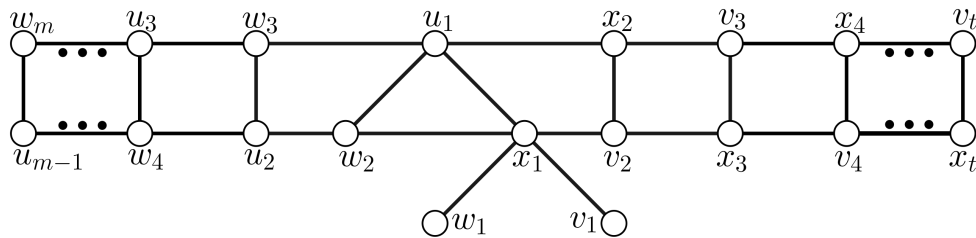


Figure 6. Graph  $G$  with  $\beta_p(G) = 3$  and  $\eta_p(G) = \chi_L(G) = 4$ .

**Theorem 4.5.** If  $G$  is the graph in Figure 6, then  $G$  is a graph of type DP2.

*Proof.* Let  $G$  be the graph in Figure 6 for some integers  $m \geq 3$  and  $t \geq 3$ . Notice that,  $G$  contains an earring  $x_1$  of size 2 with the vertices  $w_1$  and  $v_1$  as leaves hanging at  $x_1$ . To prove that  $G$  is a graph of type DP2, we must show first that  $\beta_p(G) = 3$ . Since  $G$  is not a path, then by Theorem 2.4 part (a) we have  $\beta_p(G) \geq 3$ . Now, take an ordered partition  $\Gamma = \{S_1, S_2, S_3\}$  where



$S_1 = \{v_1, x_1, x_2, \dots, x_t\}$ ,  $S_2 = \{w_1, w_2, \dots, w_m\}$ , and  $S_3 = \{u_1, u_2, \dots, u_{m-1}, v_2, v_3, \dots, v_t\}$ . Note that,  $S_3$  resolves  $v_1$  and  $x_i$ , for every  $i \in \{1, 2, \dots, t\}$ , and  $S_2$  resolves every pair of vertices of  $S_1 \setminus \{v_1\}$ . Now,  $S_3$  resolves the pair  $w_1, w_j$  for every  $j \in \{2, 3, \dots, m\}$ , and  $S_1$  resolves every pair of vertices of  $S_2 \setminus \{w_1\}$ . Next,  $S_1$  and  $S_2$  resolves every pair of vertices of  $\{u_1, u_2, \dots, u_{m-1}\}$  and  $\{u_1, v_2, v_3, \dots, v_t\}$ , respectively. For every  $i \in \{2, 3, \dots, m-1\}$  and  $j \in \{2, 3, \dots, t\}$ , the set  $\{u_i, v_j\}$  is resolved by both  $S_1$  and  $S_2$ . Hence, the partition  $\Gamma$  is a resolving partition of  $G$  and we conclude that  $\beta_p(G) = 3$ .

Second, we will show that  $\eta_p(G) = \chi_L(G) = 4$ . Notice that  $G$  is not isomorphic to any subgraph of a graph stated in Theorem 3.3, then  $\chi_L(G) \geq 4$ . Now, define a proper 4-coloring  $\sigma$  on the vertices of  $G$  such that  $\sigma(x_i) = 1$  for all  $i \in \{1, 2, \dots, t\}$ ,  $\sigma(w_j) = 2$  for all  $j \in \{1, 2, \dots, m\}$ ,  $\sigma(u_k) = \sigma(v_l) = 3$  for all  $k \in \{1, 2, \dots, m-1\}$  and  $l \in \{2, 3, \dots, t\}$ , and  $\sigma(v_1) = 4$ . Clearly,  $\sigma$  is a locating coloring of  $G$  and  $\chi_L(G) \leq 4$ . Therefore,  $\chi_L(G) = 4$ .

Based on the above results and Corollary 2.1, we have that  $3 \leq \eta_p(G) \leq 4$ . Next, we will show that  $\eta_p(G) \geq 4$ . For a contradiction, let  $\Pi = \{C_1, C_2, C_3\}$  be a resolving dominating partition of  $V(G)$  induced by a 3-coloring  $c$ . Note that  $c$  must be not a proper coloring, since otherwise  $\chi_L(G) = 3$ . Therefore,  $r(y|\Pi)$  must have at least one ordinate '1' for any  $y \in V(G)$ . Since  $d(w_1, y) = d(v_1, y)$  for every  $y \in V(G) \setminus \{w_1, v_1\}$  and  $d(w_1, z) = d(v_1, z) \geq 2$  for every  $z \in V(G) \setminus \{x_1\}$ , it follows that  $c(x_1)$ ,  $c(w_1)$ , and  $c(v_1)$  must be distinct. We may assume that  $x_1 \in C_1$ ,  $w_1 \in C_2$  and  $v_1 \in C_3$ . Thus,  $r(x_1|\Pi) = (0, 1, 1)$ ,  $r(w_1|\Pi) = (1, 0, 2)$ , and  $r(v_1|\Pi) = (1, 2, 0)$ .

Next, let  $M = \{u_1, v_2, w_2\}$  and define the multiset  $c(M) = \{c(u_1), c(v_2), c(w_2)\}$ . Since  $\Pi$  is a resolving dominating partition, then  $c(M) \neq \{1, 1, 1\}$ ,  $c(M) \not\supseteq \{2, 2\}$  or  $c(M) \not\supseteq \{3, 3\}$ . This implies that  $c(M)$  contains exactly one or two '1's.

**Case 1.**  $c(M)$  contains exactly one '1'. Then,  $c(M) = \{1, 2, 3\}$ . This forces that the representations of all members of  $M$  must be  $(1, 0, 1)$ ,  $(1, 1, 0)$  and one of  $\{(0, 1, 2), (0, 2, 1)\}$ . This implies that:

- (i)  $c(v_2) = 1$  and  $\{c(u_1), c(w_2)\} = \{2, 3\}$ , or
- (ii)  $c(u_1) = 1$  and  $\{c(w_2), c(v_2)\} = \{2, 3\}$ , or
- (iii)  $c(w_2) = 1$  and  $\{c(u_1), c(v_2)\} = \{2, 3\}$ .

If (i) holds and let  $c(w_2) = 3$  then  $c(x_2) \notin \{2, 3\}$ . Since otherwise  $r(x_2|\Pi) = r(w_1|\Pi)$  or  $r(x_2|\Pi) = r(w_2|\Pi)$ , a contradiction. Hence  $c(x_2) = 1$ . On other hand, we must have  $c(x_3) = 3$ , since otherwise  $v_2$  is not dominated or  $r(v_2|\Pi) = r(x_2|\Pi)$ . Next, let us consider  $c(v_3)$ . If  $c(v_3) \in \{2, 3\}$  then  $r(v_3|\Pi) = r(u_1|\Pi)$  or  $r(x_2|\Pi) = r(x_1|\Pi)$ , a contradiction. Thus  $c(v_3) = 1$ . But, this implies that  $r(v_3|\Pi) = r(v_2|\Pi)$  or  $r(v_3|\Pi) = r(x_1|\Pi)$ , a contradiction. It is similar if  $c(w_2) = 2$ .

If (ii) holds and let  $c(w_2) = 2$ , then  $c(u_2) = 3$  (to get  $r(w_2|\Pi) = (1, 0, 1)$ ),  $c(x_3) = 2$  (to get  $r(v_2|\Pi) = (1, 1, 0)$ ) and  $c(x_2)$  must be 1 (to get all distinct representations). This forces that  $c(w_3)$  must be 3. But, now  $r(w_3|\Pi) = r(v_1|\Pi)$  or  $r(w_3|\Pi) = r(v_2|\Pi)$ , a contradiction. It is similar if  $c(w_2) = 3$ .

If (iii) holds and  $c(u_1) = 2$  then  $r(w_2|\Pi) = (0, 1, 2)$  and  $c(x_2) \neq 1$ . Now, consider  $c(x_2) \in \{2, 3\}$ . Let  $c(x_2) = 3$ , then  $c(x_3) = 2$  (to get  $r(v_2|\Pi) = (1, 1, 0)$ ). On other hand, since  $r(w_2|\Pi) = (0, 1, 2)$  then  $c(u_2) \neq 3$ . Next, consider  $c(u_2) \in \{1, 2\}$ . If  $c(u_2) = 2$  then  $c(w_3) \notin \{1, 3\}$ ,

since otherwise  $r(w_3|\Pi) = r(w_2|\Pi)$  or  $r(w_3|\Pi) = r(x_2|\Pi)$  or  $r(u_2|\Pi) = r(u_1|\Pi)$ . But now, if  $c(w_3) = 2$  then  $w_3$  is not dominated,  $r(w_3|\Pi) = r(w_1|\Pi)$  or  $r(w_3|\Pi) = r(x_3|\Pi)$ , a contradiction. Next, if  $c(u_2) = 1$  then  $c(w_3) \notin \{1, 2, 3\}$ , a contradiction. Hence,  $c(x_2) \neq 3$ . Now, let  $c(x_2) = 2$ . Then,  $c(w_3)$  must be 3 (to get  $r(u_1|\Pi) = (1, 0, 1)$ ). But, now  $c(u_2) \notin \{1, 3, 2\}$ , since otherwise  $r(w_3|\Pi) = r(v_2|\Pi)$  or  $r(u_2|\Pi) = r(u_1|\Pi)$  or  $r(u_2|\Pi) = r(v_1|\Pi)$ , a contradiction. It is similar if  $c(u_1) = 3$  and  $c(v_2) = 2$ .

**Case 2.**  $c(M)$  contains exactly two '1's. There are 3 subcases to be considered: (i)  $c(u_1) = c(w_2) = 1$ , (ii)  $c(u_1) = c(v_2) = 1$ , and (iii)  $c(w_2) = c(v_2) = 1$ .

Consider the first subcase, without loss of generality, we may assume  $c(v_2) = 2$ . Note that, either  $c(x_2) = 3$  or  $c(x_3) = 3$  to get  $r(v_2|\Pi) = (1, 0, 1)$ . Now, if  $c(x_2) = 3$  then  $c(u_2) = 2$ , since otherwise  $r(w_2|\Pi) = r(u_1|\Pi)$  or  $w_2$  is not dominated. But, now  $c(w_3) \notin \{1, 2, 3\}$ , a contradiction. Hence,  $c(x_3) = 3$ . Now, let us consider  $c(x_2) \in \{1, 2\}$ . If  $c(x_2) = 1$  then  $c(v_3) = 1$ , since otherwise  $r(v_3|\Pi) = r(v_2|\Pi)$  or  $r(x_2|\Pi) = r(x_1|\Pi)$ . On other hand  $c(w_3) \notin \{1, 2, 3\}$  since otherwise  $u_1$  is not dominated,  $r(u_1|\Pi) = r(x_2|\Pi)$ , or  $r(u_1|\Pi) = r(v_3|\Pi)$ , a contradiction. But now, if  $c(x_2) = 2$  then  $r(x_2|\Pi) = r(w_1|\Pi)$ , a contradiction.

For the second subcase, without loss of generality, we can assume  $c(u_1) = c(v_2) = 1$  and  $c(w_2) = 2$ . This implies that  $c(u_2) = 3$  (to get  $r(w_2|\Pi) = (1, 0, 1)$ ). Since  $r(u_1|\Pi) \neq (0, 1, 1)$  then  $c(w_3) \neq 3$  and  $c(x_2) \neq 3$ . Furthermore,  $c(w_3) \neq 2$  since otherwise  $r(w_3|\Pi) = r(w_2|\Pi)$ . Therefore,  $c(w_3) = 1$ . Now, observe  $c(x_2)$ . Note that  $c(x_2) \notin 2$ , since otherwise  $r(v_2|\Pi) = r(u_1|\Pi)$ . Thus,  $c(x_2) = 1$ . This implies that  $c(x_3) \neq 1$  (to get  $v_2$  dominated). But,  $c(x_3)$  is neither 2 nor 3, since otherwise  $r(v_2|\Pi) = r(u_1|\Pi)$  or  $r(v_2|\Pi) = r(w_3|\Pi)$ , a contradiction.

Now, consider the third subcase, namely  $c(w_2) = c(v_2) = 1$ . Without loss of generality, we may assume  $c(u_1) = 2$ . Then,  $r(u_1|\Pi)$  must be  $(1, 0, 1)$ . This implies that either  $c(w_3) = 3$  or  $c(x_2) = 3$ . Now, let  $c(w_3) = 3$ . Then,  $c(u_2) = 1$ . Further,  $c(x_2) \notin \{1, 2, 3\}$  to get a distinct representation, a contradiction. Next, let  $c(x_2) = 3$ . Then  $c(x_3) \neq 2$  (since otherwise  $r(v_2|\Pi) = r(x_1|\Pi)$ ). If  $c(x_3) = 1$  then whatever color of  $v_3$  yields a duplication of the representations of vertices. Thus,  $c(x_3) = 3$ . Next, let us consider  $c(v_3)$ . If  $c(v_3) \neq 3$  then  $r(v_3|\Pi) = r(v_2|\Pi)$ ,  $r(v_3|\Pi) = r(x_1|\Pi)$ , or  $r(x_3|\Pi) = r(x_2|\Pi)$ , a contradiction. But, if  $c(v_3) = 3$  then  $v_3$  is not dominated or  $r(v_3|\Pi) = r(v_1|\Pi)$ , or  $r(x_3|\Pi) = r(v_1|\Pi)$ , a contradiction.

To conclude this proof, we obtain that  $\eta_p(G) = 4$  and  $\beta_p(G) = 3$ . Then  $G$  is a graph of type DP2. □

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### References

- [1] Asmiati and E.T. Baskoro, Characterizing all graphs containing cycles with locating-chromatic number 3, *AIP Conf. Proc.*, **1450** (2012), 351.

- [2] E.T. Baskoro and Asmiati, Characterizing all trees with locating-chromatic number 3, *Electron. J. Graph Theory Appl.*, **1** (2) (2013), 109–117.
- [3] R.C. Brigham, G. Chartrand, R.D. Dutton and P. Zang, Resolving dominating in graphs, *Math. Bohem.*, **128** (2003), 25–36
- [4] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater and P. Zhang, The locating-chromatic number of a graph, *Bull. Inst. Combin. Appl.*, **36** (2002), 101
- [5] G. Chartrand, E. Salehi and P. Zhang, The partition dimension of a graph, *Aequationes Math.*, **59** (2000), 45–54
- [6] F. Harary and R.A. Melter, On the metric dimension of graph, *Ars. Combin.*, **2** (1976), 191–195.
- [7] C. Hernando, M. Mora and I.M. Pelayo, Resolving dominating partitions in graphs, *Discrete Appl. Math.* **266** (2019), 237–251.
- [8] O. Ore, Theory of graphs, American Mathematical Society Colloquium Publications, *Amer. Math. Soc., Providence*, **38** (1962).
- [9] P.J. Slater, Leaves of trees, *Proc. 6th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Congr. Numer.*, **14** (1975), 549–559.
- [10] P.J. Slater, Domination and location in acyclic graphs, *Networks*, **17** (1987), 55–64
- [11] D.K. Syofyan, E.T. Baskoro and H. Assiyatun, Trees with certain locating-chromatic number, *J. Math. Fund. Sci.*, **48** (1) (2016), 39–47.