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Outer independent global dominating set of trees and unicyclic graphs

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Abstract

Let G be a graph. A set $D \subseteq V(G)$ is a global dominating set of G if D is a dominating set of G and $\overline{G}$. $\gamma_g(G)$ denotes global domination number of G. A set $D \subseteq V(G)$ is an outer independent global dominating set (OIGDS) of G if D is a global dominating set of G and $V(G) - D$ is an independent set of G. The cardinality of the smallest OIGDS of G, denoted by $\gamma_g^{oi}(G)$, is called the outer independent global domination number of G. An outer independent global dominating set of cardinality $\gamma_g^{oi}(G)$ is called a γ_g^{oi}-set of G. In this paper we characterize trees T for which $\gamma_g^{oi}(T) = \gamma(T)$ and trees T for which $\gamma_g^{oi}(T) = \gamma_g(T)$ and trees T for which $\gamma_g^{oi}(T) = \gamma^{oi}(T)$ and the unicyclic graphs G for which $\gamma_g^{oi}(G) = \gamma(G)$, and the unicyclic graphs G for which $\gamma_g^{oi}(G) = \gamma_g(G)$.

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1. Introduction

The usual graph theory notions not herein, refer to [15]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order of G is denoted by $n(G) = |V|$. A *unicyclic*

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graph is a connected graph with exactly one cycle. The *open neighborhood* of vertex u is denoted by $N(u) = \{v \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of vertex u is denoted by $N[u] = N(u) \cup \{u\}$. For $A \subseteq V(G)$, the open neighborhood and closed neighborhood of A are defined as $N(A) = \bigcup_{u \in A} N(u)$ and $N[A] = \bigcup_{u \in A} N[u]$. Let $u \in V(G)$ and $A \subseteq V(G)$, then $d(u, A) = min\{d(u, v) : v \in A\}$. A set $D \subseteq V(G)$ of a simple graph G is a *vertex cover* of G if every edge of G has at least one end in D. The *covering number* $\beta(G)$ is the minimum cardinality of a vertex cover in G. A set $B \subseteq V(G)$ is an *independent set* of G if for every edge $ab \in E(G)$, $a \notin B$ or $b \notin B$. The cardinality of the maximum independent set of graph G, denoted by $\alpha(G)$, is called *independence number* of G. The *diameter* of connected graph G is defined as $diam(G) = max\{d(u, v) : u, v \in V(G)\}$. For a vertex $u \in V(G)$, the *eccentricity* of u, defined as $\epsilon(u) = max\{d(u, v) : v \in V(G)\}$. The *radius* of a graph G defined as $R(G) = min\{\epsilon(u) : u \in V(G)\}$. The *center* of a graph G is defined as $C(G) = \{u \in V(G) : \epsilon(u) = R(G)\}$.
Let G be a graph and B be a subset of $V(G)$ and $u \in B$. We say that vertex v is a *private neighbor* of u respected to B if $N[v] \cap B = \{u\}$ and we say that u is an *isolated vertex* respected to B if $N(u) \cap B = \varnothing$ [15]. A vertex $v \in V(G)$ is called a *leaf*, if $d(v) = 1$. We denote the set of leaves of graph G by $L(G)$. A vertex $u \in V(G)$ that is adjacent to a leaf is called a *support vertex*. We denoted the set of support vertices of G by $S(G)$. A set $D \subseteq V(G)$ is a *dominating set* of G if every vertex of $V(G) - D$ is adjacent to at least one vertex of D. The cardinality of the smallest dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set of cardinality $\gamma(G)$ is called a γ-set of G [9]. For every $u \in S(G)$ delete all the leaves from $N(u)$ except one, then the remaining graph is called the *pruned* of G and denoted by G_p. Further about the pruned graphs and its application we refer to the reference [11]. A set $S \subseteq V(G)$ is a *global dominating set* of G if S is a dominating set of G and $\overline{G}$. The cardinality of the smallest global dominating set of G, denoted by $\gamma_g(G)$, is called the *global domination number* of G [3, 6]. A set $D \subseteq V(G)$ is an *outer independent dominating set* (OIDS) of G if D is a dominating set of G and $V(G) - D$ is an independent set of G. The cardinality of the smallest OIDS of G, denoted by $\gamma^{oi}(G)$, is called the *outer independent domination number* of G. An outer independent dominating set of cardinality $\gamma^{oi}(G)$ is called a γ^{oi}-set of G [10]. Also the global outer connected dominating set of graphs has already been studied in [1].

One of many applications of global domination have been given in [2], which relates to a communication network modeled by a graph G, where subnetworks are defined by some matching M_i of cardinality k. The necessity of these subnetworks could be due for reason of security, redundancy or limitation of recipients for different classes of messages. For this practical case, the global domination number represents the minimum number of master stations needed such that a message issued simultaneously from all masters reaches all desired recipients after traveling over only one communication link. We note that Carrington [4] gave two other applications of global dominating sets for graph partitioning commonly used in the implementation of parallel algorithms.
If $D \subseteq V(G)$ and $G - D$ is an independent set, then D is a vertex cover of G. For every graph G without isolate vertices we have $\beta(G) = \gamma^{oi}(G)$. All connected graphs G with $\gamma(G) = \beta(G)$ have been characterized in [12, 16]. Actually they characterized the connected graph G with equal domination number and outer independent domination number. Therefore we maybe use for a graph G, $\gamma^{oi}(G)$ instead of $\beta(G)$.

In this article we are going to define and study outer independent global dominating set of trees and unicycles.

2. Preliminaries results

Let τ denote the class of trees with $n \geq 2$ vertices and either radius one (that is, stars) or radius two having a vertex u with $d(u) \geq 2$ and $d(v) \geq 3$ for all $v \in N(u)$ [3].
Let M be the family of trees T with $diam(T) = 4$ and $C(T) \subseteq S(T)$ and N be the family of trees T with $diam(T) = 4$ and $C(T) \nsubseteq S(T)$.

Definition 2.1. *Let u and v be two distinct vertices of tree T. We define $D_T(u,v) = D(u,v)$ as follows: $D(u,v) = k$ if $d_T(u,v) = k$ and non of the internal vertices of the path between u and v is a support vertex, and $D(u,v) = 0$ if at least one of the internal vertices of the path between u, v is a support vertex.*

Stracke in [13] has shown the following result that has been stated in [14] too.

Proposition 2.1. ([14], Corollary 2.7) *For any tree T, $\gamma(T) = \beta(T)$ if and only if $T^* = T - N[L(T)] = \varnothing$ or each component of T^* is an isolated vertex or a star, where the center of these stars are not adjacent to a vertex of $S(T)$.*

We will prove an equivalent theorem (Theorem 2.2), using notation $D(u,v)$.

Proposition 2.2. [8] *For every nontrivial connected graph G, $\gamma^{oi}(G) = n(G) - \alpha(G)$.*

The following results has a straightforward proof and it is left.

Observation 2.1. *If T is a nontrivial tree then, $\gamma^{oi}(T) \leq \frac{n(T)}{2}$.*

The next result can be found in [9].

Theorem 2.1. ([9] Theorem 1.1) *A dominating set A is a minimal dominating set of G if and only if for each vertex $u \in A$, one of the following two conditions holds:*
(a) *u is an isolate vertex of A,*
(b) *u has a private neighbor with respect to A.*

Observation 2.2. *If A is a global dominating set of G, then $N[u] \bigcap A \neq \varnothing$ and $A - N(u) \neq \varnothing$, for every $u \in V(G)$.*

If A is a domination set (OIDS) of G, then every leaf or it's support vertex belongs to A, so we have the bellow observation.

Observation 2.3. *Let $T \neq P_2$ be a tree and A be a γ-set (γ^{oi}-set) of T. Then the set $(A \cup S(T)) - L(T)$ is a γ-set (γ^{oi}-set) of T, too.*

Lemma 2.1. *Let T be a tree and $\gamma^{oi}(T) = \gamma(T)$. Let A be a γ^{oi}-set of T and $D(u,v) > 1$, $u, v \in S(T)$. Let $P = ux_1x_2...x_{k-1}v$ be the path between u and v in T. If $ab \in E(P)$, then $a \notin A$ or $b \notin A$.*

Proof. On the contrary let $a, b \in A$. Since $D(u, v) > 1$, so $a \notin S(T)$ or $b \notin S(T)$. Without lose of generality let $b \notin S(T)$. A is a minimal dominating set of T too, therefore by Theorem 2.1, b has a private neighbor with respect to A like x. Since $b \notin S(T)$, so x is not a leaf, therefore x has a neighbor like y that $y \notin A$, thus the vertices x, y are two adjacent vertices in $V(T) - A$, therefore A is not an OIDS of T, that is a contradiction. □

The following result characterizes the tree T with $\gamma^{oi}(T) = \gamma(T)$ other than point of view of what Stracke do in Proposition 2.1.

Theorem 2.2. *Let T be a tree. Then $\gamma^{oi}(T) = \gamma(T)$ if and only if for every vertices $u, v \in S(T)$, $D(u, v) \in \{0, 1, 2, 4\}$.*

Proof. Let $\gamma^{oi}(T) = \gamma(T)$ and A be a $\gamma^{oi}(T)$-set of T and $D(u, v) > 1$, $u, v \in S(T)$. Let $P = ux_1x_2...x_{k-1}v$ be the path between u and v in T. By Observation 2.3 without lose of generality we have $S(T) \subseteq A$. By Lemma 2.1, if k is odd then, it will be contradiction.
Now let $D(u, v) = k \geq 6$, and k is even.

By Lemma 2.1 we have $\{u, x_2, x_4, ..., x_{k-2}, v\} \subseteq A$. Let $A_1 = (A - \{x_2, x_4\}) \cup \{x_3\}$. We show that A_1 is a global dominating set of T at size $\gamma^{oi}(T) - 1$, that is contradiction. If $d_T(x_2) = d_T(x_4) = 2$, then it is clear that A_1 is a dominating set of T. If $d(x_2) > 2$ and $y \in N(x_2) - A$, then since $x_2 \notin S(T)$, so $d(y) \geq 2$. Let $z \in N(y) - \{x_2\}$. Since A is an OIDS of T, so $z \in A$, therefore y is dominated by A_1. If $d(x_4) > 2$ and $w \in N(x_4) - A$, then by a similar proof we find that w is dominated by A_1. Therefore A_1 is a dominating set of T.
Conversely, let T be a tree and for every vertices $u, v \in S(T)$ we have $D(u, v) = k$, $k \in \{0, 1, 2, 4\}$. Let A be a γ^{oi}-set of T. By Observation 2.3 we can consider $S(T) \subseteq A$. Let $H = \{u \in V(T) : d(u, S(T)) = 2\}$. By Lemma 2.1, $H \subseteq A$. Since $S(T) \cup H$ is an OIDS of T, so $A = S(T) \cup H$. Now let B be an arbitrary γ-set of T. By Observation 2.3 we can consider $S(T) \subseteq B$. For every disjoint vertices $c_1, c_2 \in H$ it is clear that $c_1, c_2 \notin N[S]$, $d(c_1, c_2) \geq 4$, so corresponding to every vertex $c \in H$ there exists a vertex $a_c \in B - S(T)$ that dominates c and does't dominate any vertex of $H - \{c\}$. So $|B| \geq |S(T) \cup H| = |A|$, thus $\gamma^{oi}(T) = \gamma(T)$. □

3. OIGDS of trees

We begin this section with a definition.

Definition 3.1. *A set $D \subseteq V(G)$ is an outer independent global dominating set (OIGDS) of G if D is a global dominating set of G and $V(G) - D$ is an independent set of G.*

The cardinality of the smallest OIGDS of G, denoted by $\gamma_g^{oi}(G)$, is called the *outer independent global domination number* OIGDN of G. An outer independent global dominating set of cardinality $\gamma_g^{oi}(G)$ is called a γ_g^{oi}-set of G.

Lemma 3.1. [7] *For any graph G, if $R(G) \geq 3$, then every dominating set of G is a dominating set of $\overline{G}$.*

Corollary 3.1. *For any graph G, if $R(G) \geq 3$, then*
a) $\gamma_g(G) = \gamma(G)$ *and*
b) $\gamma_g^{oi}(G) = \gamma^{oi}(G)$.

Proof. a) Let A be a γ-set of G. Then by Lemma 3.1 A is a dominating set of $\overline{G}$, too . So A is a global dominating set of G and therefore $\gamma_g(G) \leq \gamma(G)$. By inequality $\gamma(G) \leq \gamma_g(G)$ we have $\gamma_g(G) = \gamma(G)$.
b) Let A be a γ^{oi}-set of G. A is a dominating set of $\overline{G}$. Thus A is a global dominating set of G and therefore A is an OIGDS of G, so $\gamma_g^{oi}(G) \leq \gamma^{oi}(G)$. By inequality $\gamma^{oi}(G) \leq \gamma_g^{oi}(G)$ we have $\gamma_g^{oi}(G) = \gamma^{oi}(G)$.

□

Proposition 3.1. *For any tree T, $R(T) = \lceil \frac{diam(T)}{2} \rceil$.*

Proof. Let $diam(T) = k$ and $P = u_0u_1...u_k$ be a path at length k in T. $d(u_0, u_{\lceil \frac{k}{2} \rceil}) = \lceil \frac{k}{2} \rceil$. Since the path between two vertex of T is unique, if there exist a vertex v such that $d(v, u_{\lceil \frac{k}{2} \rceil}) > \lceil \frac{k}{2} \rceil$ then $d(u_0, v) > k$ or $d(u_k, v) > k$ that is contradiction. □

Corollary 3.2. *If T is a tree and $diam(T) \geq 5$, then $\gamma_g(T) = \gamma(T)$ and $\gamma_g^{oi}(T) = \gamma^{oi}(T)$.*

Theorem 3.1. *Let T be a tree. Then $\gamma_g^{oi}(T) = \gamma_g(T)$ if and only if one of the following conditions holds:*
a) $diam(T) \in \{0, 1, 2, 3\}$
b) $T \in M$
c) $T \in N \cap \tau$
d) $diam(T) \geq 5$ *and* $D(u, v) \in \{0, 1, 2, 4\}$, *for every* $u, v \in S(T)$.

Proof. If $diam(T) = 0$ or 1 or 2, then the proof is clear. If $diam(T) = 3$, then let us observe that $S(T)$ is a γ_g^{oi}-set and γ_g-set of T.
Now let $diam(T) = 4$. If $T \in M$, then let us observe that $S(T)$ is a γ_g^{oi}-set and γ_g-set of T and if $T \in N \cap \tau$, then let us observe that $S(T) \cup C(T)$ is a γ_g^{oi}-set and γ_g-set of T. If $T \in N$ but $T \notin \tau$, then T has a support vertex like x that $d(x) = 2$. Let y be the leaf that is adjacent to x. The set $(S(T) - \{x\}) \cup \{y\}$ is a γ_g-set of T but the set $S(T) \cup C(T)$ is a γ_g^{oi}-set of T, so $\gamma_g^{oi}(T) = \gamma_g(T) + 1$.
Now let $diam(T) \geq 5$. By Theorem 2.2, $\gamma^{oi}(T) = \gamma(T)$ if and only if $D(u, v) \in \{0, 1, 2, 4\}$, for every $u, v \in S(T)$. By Corollary 3.2 the proof is completed. □

Observation 3.1. *Let G be a graph. If $d_G(u) \neq \gamma^{oi}(G)$ for every vertex $u \in V(G)$, then $\gamma_g^{oi}(G) = \gamma^{oi}(G)$.*

Proof. Let $\gamma_g^{oi}(G) \neq \gamma^{oi}(G)$ and A be a γ^{oi}-set of G. Then there exist a vertex $x \in V(G) - A$ such that x is adjacent to all vertices of A and x is not adjacent to any vertex of $V(G) - A$, so $d_T(x) = |A| = \gamma^{oi}(G)$, that is contradiction. □

Theorem 3.2. *Let T be a tree.Then $\gamma_g^{oi}(T) = \gamma^{oi}(T)$ if and only if $T \neq P_2$ and $T \notin N$ and T is not a star.*

Proof. If $T = P_2$ or $T \in N$ or T is a star, then it is clear that $\gamma_g^{oi}(T) = \gamma^{oi}(T) + 1$. We show that for other trees T we have $\gamma_g^{oi}(T) = \gamma^{oi}(T)$. For $T = P_1$ we have $\gamma_g^{oi}(T) = \gamma^{oi}(T)$. If $T \in M$ or $diam(T) = 3$, then $S(T)$ is a γ_g^{oi}-set and γ^{oi}-set of T, so $\gamma_g^{oi}(T) = \gamma^{oi}(T)$. Now let

$\gamma_g^{oi}(T) \neq \gamma^{oi}(T)$ and $diam(T) \geq 5$. Let A be a γ^{oi}-set of T. Since A is not a γ_g^{oi}-set of T, so there exists a vertex $x \in V(T) - A$ such that x is adjacent to all vertices of A. Since T is not a star so $V(T) - A - \{x\} \neq \varnothing$. Since every vertex of $V(T) - A - \{x\}$ is adjacent to some vertices of A and all the vertices of A are adjacent to x, so $diam(T) = 3$ or 4. that is contradiction. □

Theorem 3.3. *Let T be a tree.Then $\gamma_g^{oi}(T) = \gamma(T)$ if and only if one of the following conditions holds.*
a) $diam(T) = 0$ *or* 3
b) $T \in M$
c) $diam(T) \geq 5$ *and* $D(u, v) \in \{0, 1, 2, 4\}$ *for every* $u, v \in S(T)$.

Proof. It is easy to see that $\gamma_g^{oi}(T) = \gamma(T)$ if $diam(T) = 0$ or 3 or $T \in M$ and $\gamma_g^{oi}(T) = \gamma(T) + 1$ if $diam(T) = 1$ or 2 or $T \in N$. If $diam(T) \geq 5$ and $D(u, v) \in \{0, 1, 2, 4\}$ for every $u, v \in S(T)$, then by Corollary 3.2, $\gamma_g^{oi}(T) = \gamma^{oi}(T)$ and by Theorem 2.2 we have $\gamma^{oi}(T) = \gamma(T)$, so $\gamma_g^{oi}(T) = \gamma(T)$. If $diam(T) \geq 5$ and $D(u, v) \notin \{0, 1, 2, 4\}$ for some $u, v \in S(T)$, then by Theorem 2.2 we have $\gamma^{oi}(T) \neq \gamma(T)$, and by Corollary 3.2 we have $\gamma_g^{oi}(T) = \gamma^{oi}(T)$, so $\gamma_g^{oi}(T) \neq \gamma(T)$. □

4. Unicyclic graphs G with $\gamma_g^{oi}(G) = \gamma(G)$

Observation 4.1. *For every graph G, $G - N[L(G)] = G_p - N[L(G_p)]$.*

Volkmann denoted by $c(x)$ the distance from x to cycle C.

Theorem 4.1. [14] *Let G be a unicyclic graph, $G^* = G - N[L(G)]$, and C the only cycle of G. Then $\beta(G) = \gamma(G)$ if and only if one of the following conditions holds:*
(1) $G = C_4$.
(2) *C is adjacent to an end vertex, and the graph $G^* = \varnothing$ or each component of G^* is an isolated vertex or a star, where the centers of these stars are not adjacent to a vertex of $N(L(G))$.*
(3) *$C = C_4$, $c(x) \geq 3$ for all $x \in L(G)$, $min\{(d(a), d(b)\} = 2$ for all pairs of adjacent vertices $a, b \in V(C)$, and all components $T_1, ..., T_k$ of the subgraph $G_0 = G - V(C)$ are trees with $\beta(T_i) = \gamma(T_i)$ for $i = 1, ...k$ such that no minimum dominating set of Go contains a vertex from $N(V(C)) \bigcap V(G_0)$.*

It is clear that if G is a unicyclic graph, then $\gamma^{oi}(G) = \beta(G)$.

Theorem 4.2. *Let G be a unicyclic graph, $G^* = G - N[L(G)]$, and C the only cycle of G. Then $\gamma_g^{oi}(G) = \gamma(G)$ if and only if one of the following conditions holds:*
a) *G satisfies the condition (3) of* Theorem 4.1.
b) *$R(G) \geq 3$ and G satisfies the condition (2) of* Theorem 4.1.
c) *G_p is one of the following graphs:*

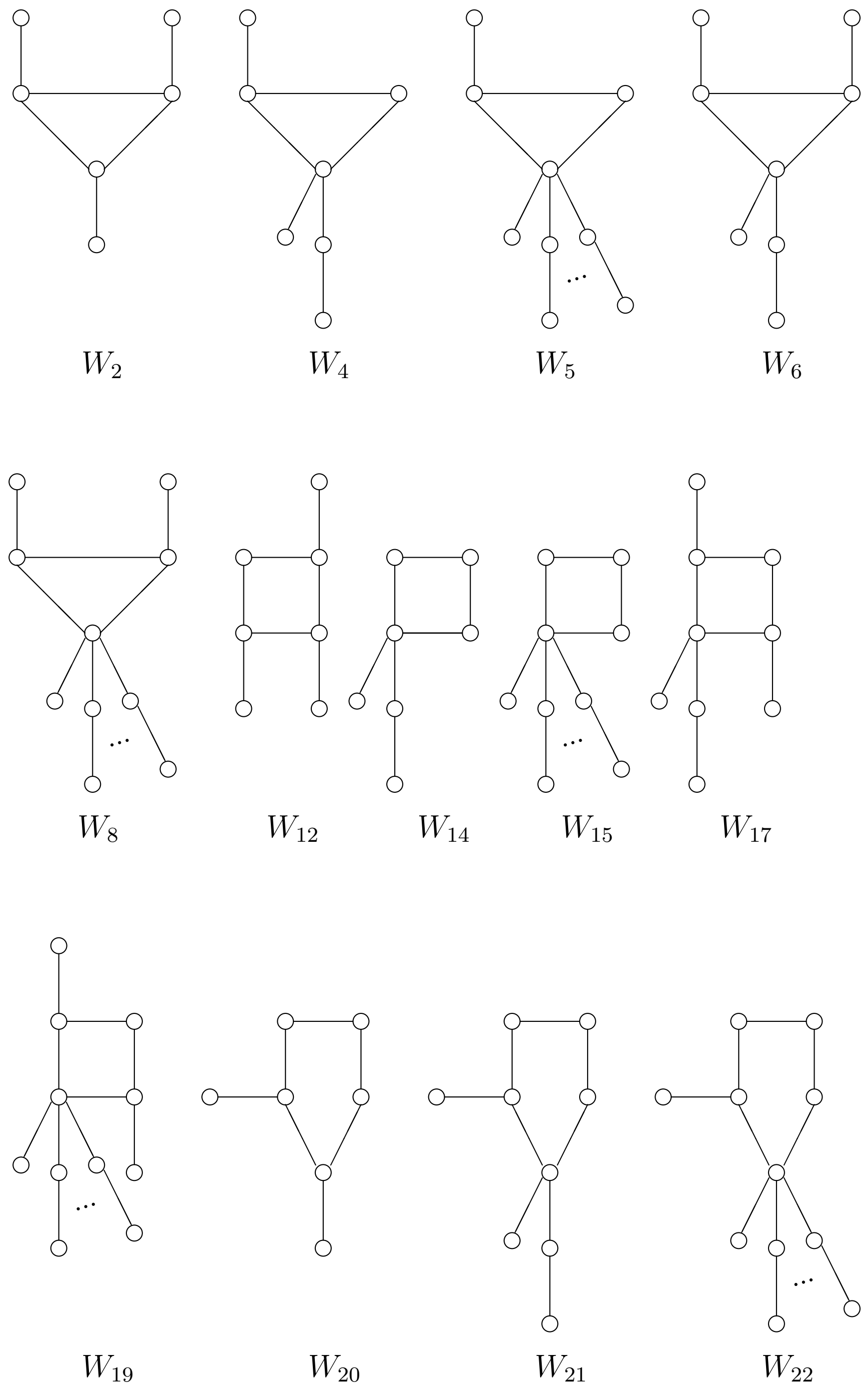


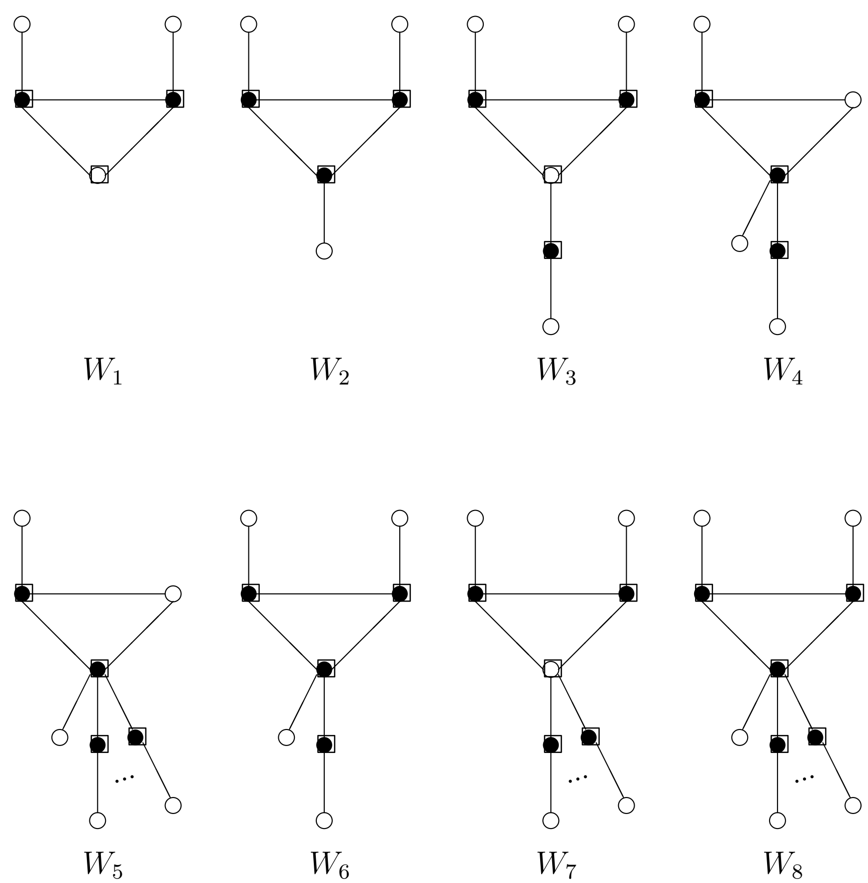
Figure 1. Pruned graphs of some unicyclic with equal OIGDN and DN.

Proof. In this proof we denote the vertex in γ-set and the vertex in γ_g^{oi}-set by bold circle $\bullet$ and empty square $\square$ respectively in the Figure 2. We show that any unicyclic graph G that satisfies condition (a) or (b) or (c) has equal $\gamma_g^{oi}(G)$ and $\gamma(G)$, and any unicyclic graph G that does not satisfy in the conditions (a), (b) and (c), $\gamma_g^{oi}(G) \neq \gamma(G)$

In Figure 2 we have the pruned of all unicycles G such that $R(G) \leq 2$ and G satisfies condition (2) of Theorem 20.

Let U be the family of all unicycles and V be the family of all unicycles satisfy in condition (1) or (2) or (3) of theorem 20 and W be the family of unicycles satisfy in condition (a) or (b) or (c). It is well known that $W \subseteq V \subseteq U$. Let $G \in W$. If G satisfies condition (a) or (b), then $R(G) \geq 3$ and $\gamma_g^{oi}(G) = \gamma^{oi}(G)$. Now $\gamma^{oi}(G) = \gamma(G)$ implies that $\gamma_g^{oi}(G) = \gamma(G)$. If G satisfies condition (c), then according to the γ-sets and γ_g^{oi}-sets of unicycles in Figure 1 that presented in Figure (2) we have $\gamma_g^{oi}(G) = \gamma(G)$.

Now let $G \in U - W$. If $G \in U - V$, then by Theorem 20, $\gamma^{oi}(G) > \gamma(G)$, so $\gamma_g^{oi}(G) \geq \gamma^{oi}(G) > \gamma(G)$, therefor $\gamma_g^{oi}(G) \neq \gamma(G)$. If $G \in V - W$, then $G = C_4$ or $G = W_i$, $i \in \{1, 3, 7, 9, 10, 11, 13, 16, 18\}$. For $G = C_4$ we have $\gamma_g^{oi}(G) \neq \gamma(G)$ and for $G = W_i$, $i \in \{1, 3, 7, 9, 10, 11, 13, 16, 18\}$ according to the γ-sets and γ_g^{oi}-sets presented in Figure (2) we have $\gamma_g^{oi}(G) \neq \gamma(G)$.



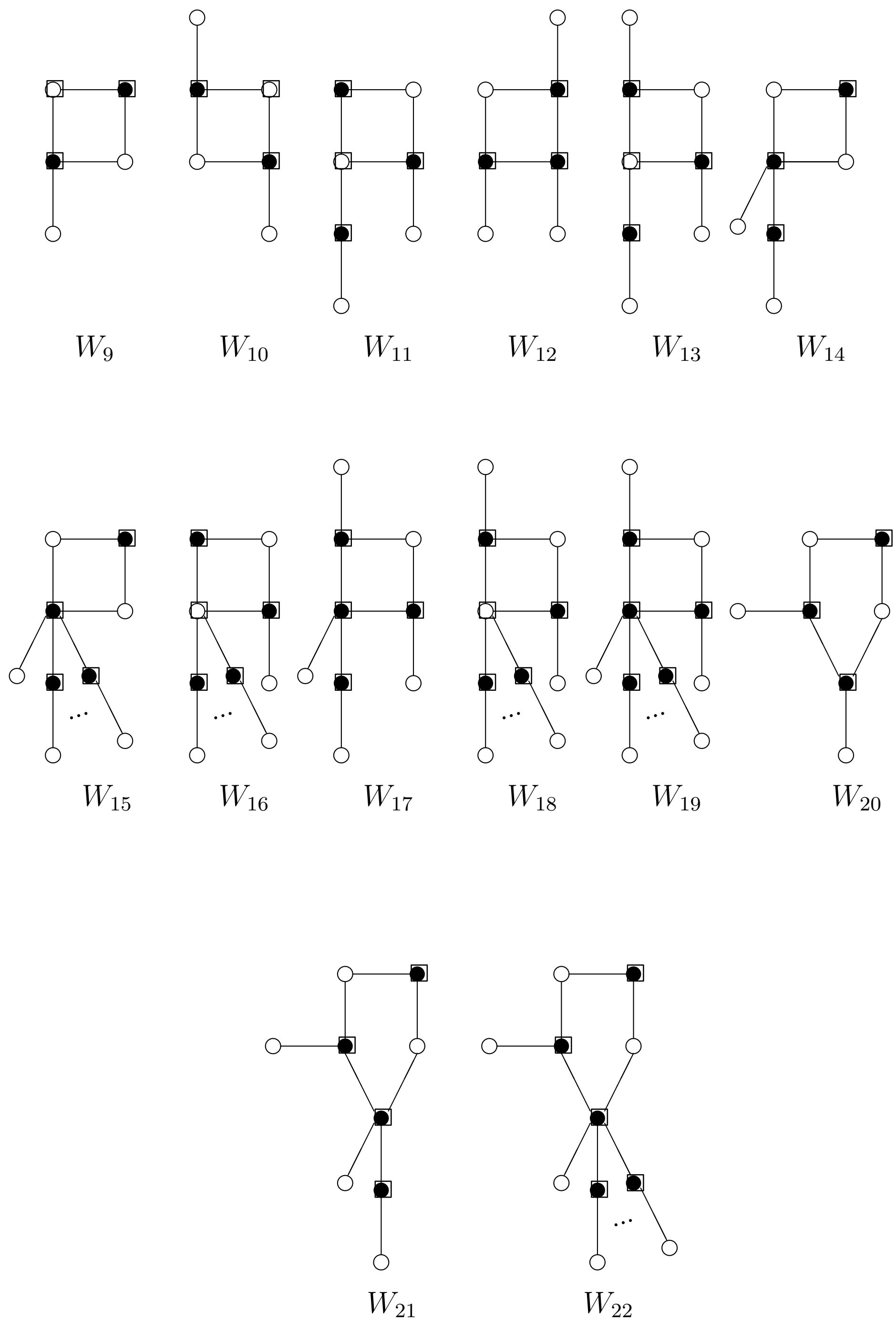


Figure 2. Pruned unicyclic graphs with $R(G) \leq 2$.

□

5. Unicyclic graphs G with $\gamma_g^{oi}(G) = \gamma_g(G)$

Observation 5.1. *Let G be a graph and A be a global dominating set of G and $A \cap L(G) = \varnothing$. Then A is a global dominating set of G_p.*

Proof. For every $u \in V(G_p)$ it is clear that $N_{G_p}[u] \cap A = N_G[u] \cap A \neq \varnothing$ and $A - N_{G_p}(u) = A - N_G(u) \neq \varnothing$. Therefore by Observation 2.2, A is a global dominating set of G_p, too. □

Observation 5.2. *Let G be a graph and A be a global dominating set of G_p and $A \cap L(G_p) = \varnothing$. Then A is a global dominating set of G.*

Proof. Let u be an arbitrary vertex of G. If $u \in V(G_p)$, then it is clear that $N_G[u] \cap A = N_{G_p}[u] \cap A \neq \varnothing$ and $A - N_G(u) = A - N_{G_p}(u) \neq \varnothing$. If $u \notin V(G_p)$, then $u \in L(G)$. Let w be the support vertex of u and $v \in N_{G_p}(w) \cap L(G_p)$. It is clear that $N_G[u] \cap A = N_{G_p}[v] \cap A \neq \varnothing$ and $A - N_G(u) = A - N_{G_p}(v) \neq \varnothing$. Therefore by Observation 2.2, A is a global dominating set of G, too. □

For every $v \in V(G)$ and $A \subseteq V(G)$ we define $L_v = L(G) \cap N(v)$ and $A_v = L_v \cap A$.

Theorem 5.1. *Let G be a graph. Then $\gamma_g(G_p) \leq \gamma_g(G)$.*

Proof. If $L(G) = \varnothing$, then $G_p = G$ and the result holds. Let $L(G) \neq \varnothing$ and A be a γ_g-set of G and $u \in S(G)$. If $|A_u| \geq 3$ and $x, y, z \in A_u$, then the set $(A \cup \{u\}) - \{y, z\}$ is a global dominating set of G that is contradiction, so $|A_u| \in \{0, 1, 2\}$. If $|A_u| = 2$ and $x, y \in A_u$, then the set $(A \cup \{u\}) - \{y\}$ is a global dominating set of G, too, therefore there exists a global dominating set of G like B, such that $|B_u| = 0$ or 1 for every $u \in S(G)$. We can construct G_p from G such that $B \subseteq V(G_p)$. Therefore for any $v \in V(G_p)$, $N_{G_p}[v] \cap A = N_G[v] \cap A \neq \varnothing$ and $A - N_{G_p}(v) = A - N_G(v) \neq \varnothing$. By Observation 2.2 A is a global dominating set of G_p and so $\gamma_g(G_p) \leq \gamma_g(G)$. □

Corollary 5.1. *Let G be a graph and G_p has a γ_g-set like A such that $A \cap L(G_p) = \varnothing$. Then $\gamma_g(G_p) = \gamma_g(G)$.*

Proof. By Observation 5.2, A is a global dominating set of G, so $\gamma_g(G) \leq \gamma_g(G_p)$. By Theorem 5.1 the result holds. □

Observation 5.3. *Let G be a graph and A be an OIGDS of G and $A \cap L(G) = \varnothing$. Then A is an OIGDS of G_p.*

Proof. Since A is a global dominating set of G, so by Observation 5.1, A is a global dominating set of G_p, too. Since $E(G_p) \subseteq E(G)$, if $E(\langle G_p - A \rangle) \neq \varnothing$ then $E(\langle G - A \rangle) \neq \varnothing$, that is contradiction. Therefore A is an outer independent set of G_p, too. □

Observation 5.4. *Let G be a graph and A be an OIGDS of G_p and $A \cap L(G_p) = \varnothing$. Then A is an OIGDS of G.*

Proof. Since A is a global dominating set of G_p, so by Observation 5.2, A is a global dominating set of G, too. Now we show that $E(\langle G-A\rangle)=\varnothing$. On the contrary, let $e\in E(\langle G-A\rangle)$ and $e=uv$.
If $u,v\in V(G_p)$, then $e\in E(\langle G_P-A\rangle)$ that is contradiction. If $u,v\notin V(G_p)$, then $u,v\in L(G)$ that is contradiction. If $u\notin V(G_p)$ and $v\in V(G_p)$, then $u\in L(G)$ and $v\in S(G)$, therefore $v\in S(G_p)$. Let w be the leaf of G_p that is adjacent to v. Since $A\cap L(G_p)=\varnothing$, so $vw\in E(\langle G_p-A\rangle)$, that is contradiction. □

Theorem 5.2. *Let G be a graph. Then $\gamma_g^{oi}(G_p)\leq\gamma_g^{oi}(G)$.*

Proof. If $L(G)=\varnothing$, then $G_p=G$ and the result holds. Let $L(G)\neq\varnothing$ and A be a γ_g^{oi}-set of G and $u\in S(G)$. If $|A_u|\geq 3$ and $x,y,z\in A_u$, then the set $(A\cup\{u\})-\{y,z\}$ is an OIGDS of G that is contradiction, so $|A_u|\in\{0,1,2\}$. If $|A_u|=2$ and $x,y\in A_u$, then the set $(A\cup\{u\})-\{y\}$ is an OIGDS of G, too, therefore there exists an OIGDS of G like B, such that $|B_u|=0$ or 1 for every $u\in S(G)$. We can construct G_p from G such that $B\subseteq V(G_p)$. Therefore for any $v\in V(G_p)$, $N_{G_p}[v]\cap A=N_G[v]\cap A\neq\varnothing$ and $A-N_{G_p}(v)=A-N_G(v)\neq\varnothing$. By Observation 2.2 A is a global dominating set of G_p. Since $E(V(G_p)-A)\subseteq E(V(G)-A)=\varnothing$, so A is an OIGDS of G_p, therefore $\gamma_g^{oi}(G_p)\leq\gamma_g^{oi}(G)$. □

Corollary 5.2. *Let G be a graph and G_p has a γ_g^{oi}-set like A such that $A\cap L(G_p)=\varnothing$. Then $\gamma_g^{oi}(G_p)=\gamma_g^{oi}(G)$.*

Proof. By Observation 5.4, A is an OIGDS of G, so $\gamma_g^{oi}(G)\leq\gamma_g^{oi}(G_p)$. By Theorem 5.2 the result holds. □

Theorem 5.3. *Let G be a unicyclic graph, $G^*=G-N[L(G)]$, and C the only cycle of G. Then $\gamma_g^{oi}(G)=\gamma_g(G)$ if and only if one of the following conditions holds:*
i) *G satisfies one of the conditions (a) or (b) or (c) of* Theorem 4.2.
ii) *G_p is one of the below graphs:*

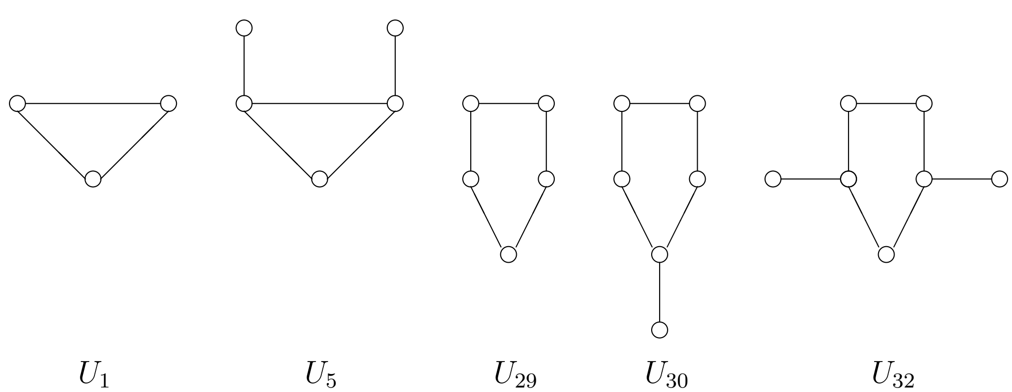
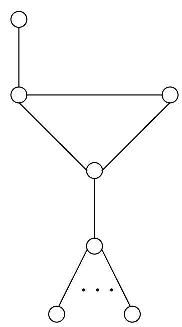
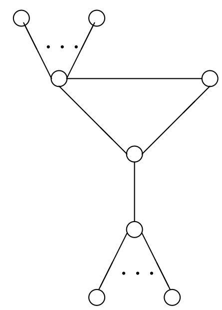


Figure 3. Pruned graphs of some unicyclic with equal OIGDN and GDN.

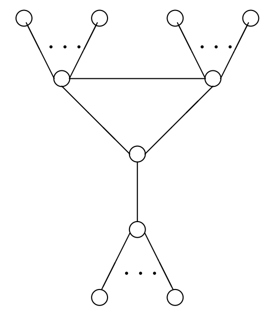
iii) G *is one of the below graphs:*



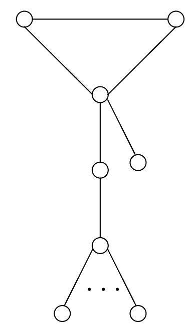
H_2



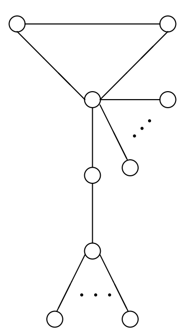
H_4



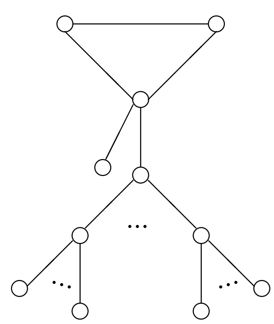
H_5



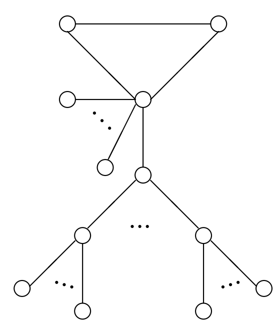
H_7



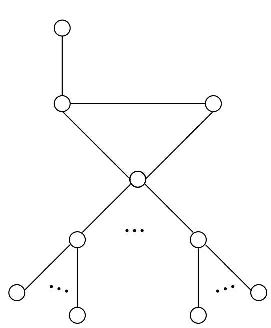
H_8



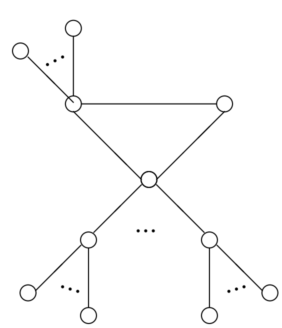
H_9



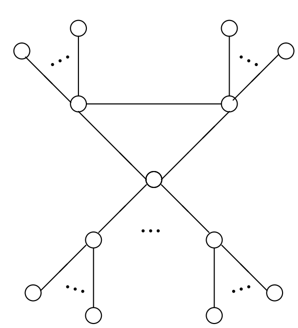
H_{10}



H_{11}



H_{12}



H_{13}

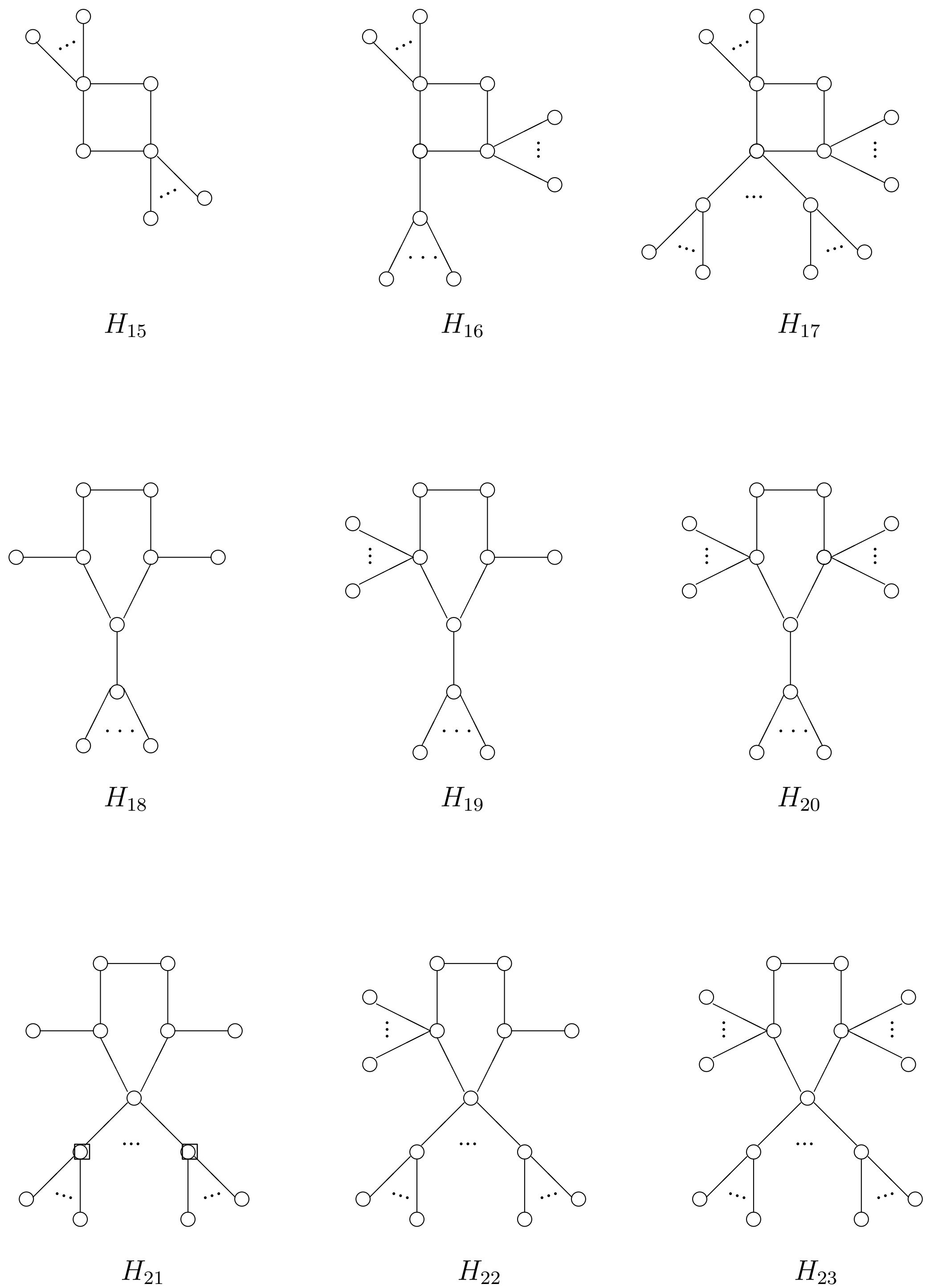
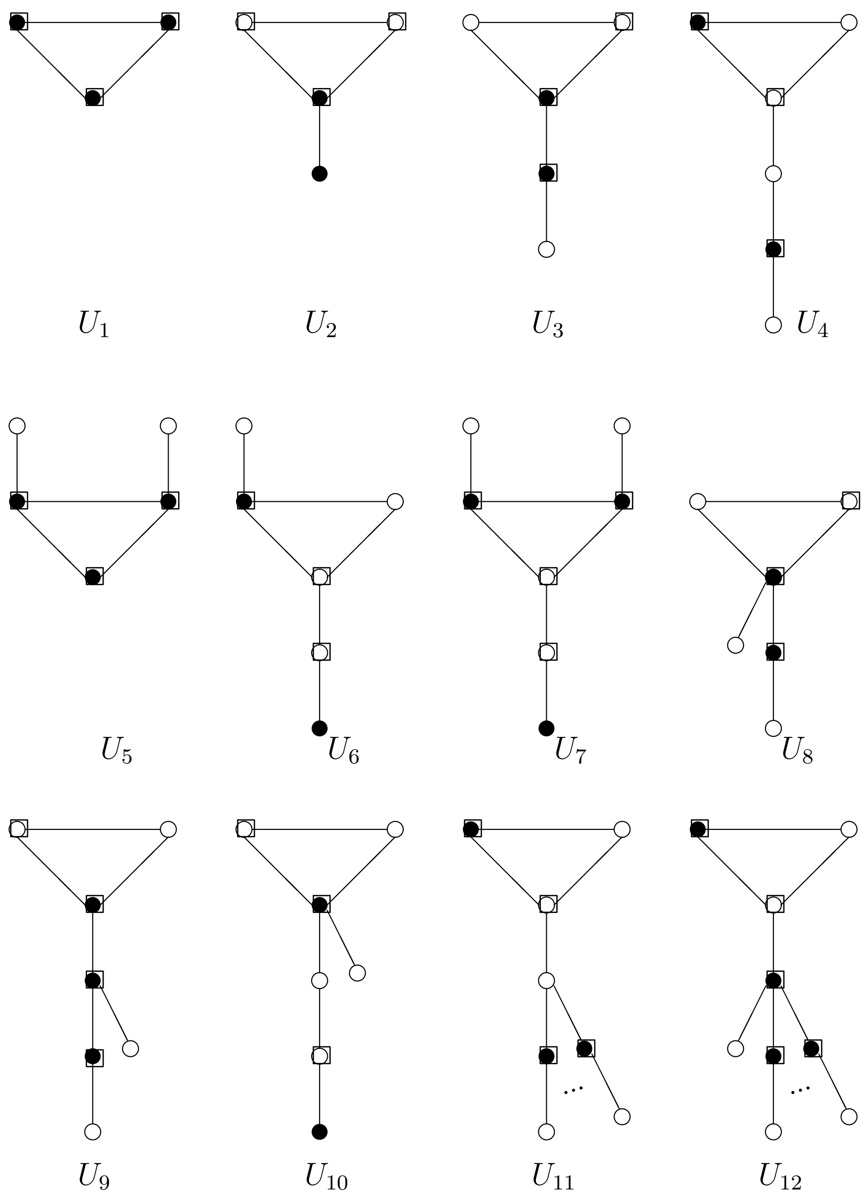
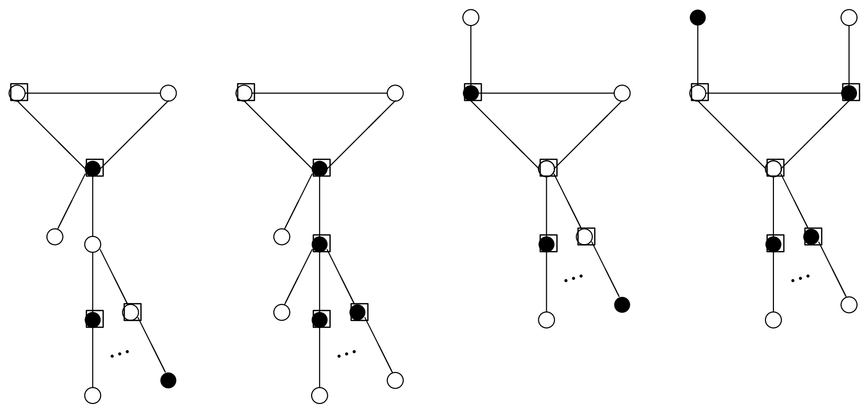


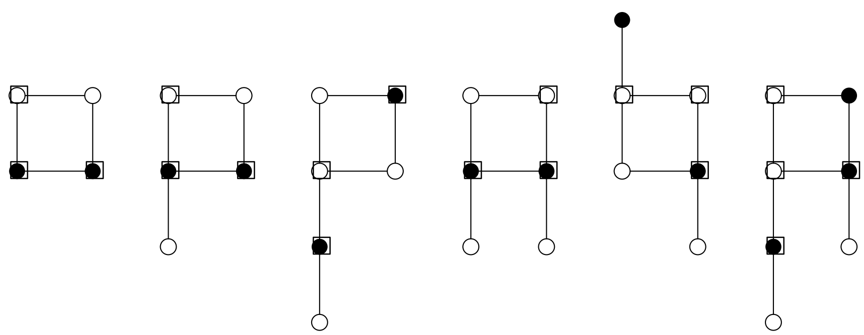
Figure 4. Some unicyclic graphs with equal OIGDN and GDN.

Proof. In this proof we denote the vertex in γ_g-set and the vertex in γ_g^{oi}-set by bold circle $\bullet$ and empty square $\square$ respectively in the Figures 5,6,...,21. Let U be the set of all unicyclic graphs and A, B and C are the set of all unicyclic graphs satisfying conditions (i), (ii) and (iii), respectively. It is clear that A, B and C are disjoint sets. For every $G \in U$ we will show that $\gamma_g^{oi}(G) = \gamma_g(G)$ if $G \in A \cup B \cup C$ and $\gamma_g^{oi}(G) \neq \gamma_g(G)$ if $G \in U - (A \cup B \cup C)$. Let $G \in A$. Then by Theorem 4.2, $\gamma_g^{oi}(G) = \gamma(G)$. Because of the inequality $\gamma(G) \leq \gamma_g(G) \leq \gamma_g^{oi}(G)$ we have $\gamma_g^{oi}(G) = \gamma_g(G)$. Now let $G \in U - A$. Then by Theorem 4.2, $\gamma_g^{oi}(G) \neq \gamma(G)$. If $R(G) \geq 3$, then $\gamma_g(G) = \gamma(G)$, therefore $\gamma_g^{oi}(G) \neq \gamma_g(G)$. If $R(G) \leq 2$, then G_p is one of the graphs in Figure 5. The pruned of all unicyclic graphs G with $R(G) \leq 2$ which are not in Figure 1 are presented in Figure 5.

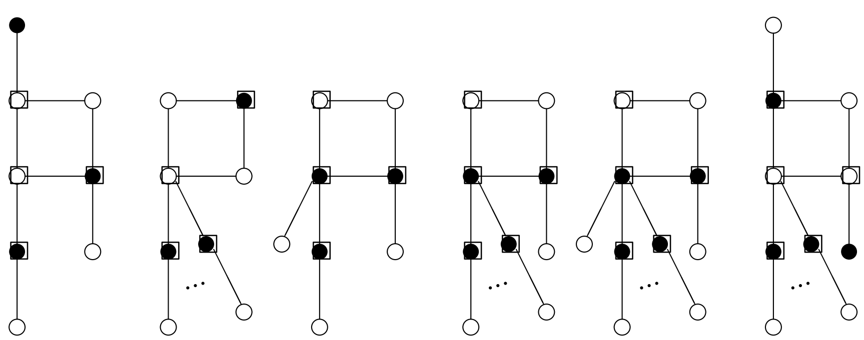




U_{13} U_{14} U_{15} U_{16}



U_{17} U_{18} U_{19} U_{20} U_{21} U_{22}



U_{23} U_{24} U_{25} U_{26} U_{27} U_{28}

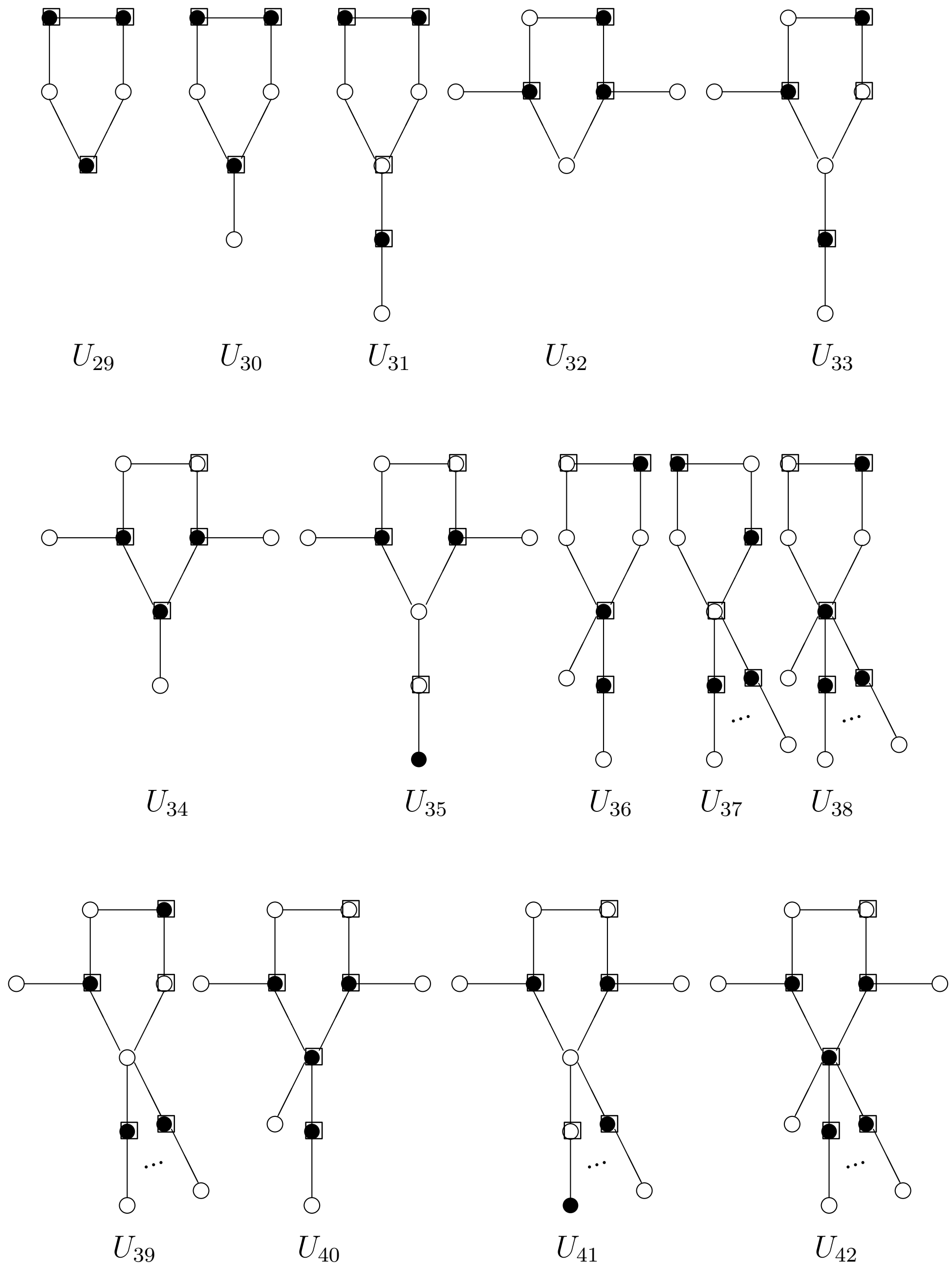


Figure 5. Unicyclic graphs with $R(G) \leq 2$ which are not in Figure 1.

By Corollary 5.1, if $G_p = U_i, i \in \{1, 2, ..., 42\} - \{2, 6, 7, 10, 13, 15, 16, 21, 23, 28, 35, 41\}$, then $\gamma_g(G_p) = \gamma_g(G)$ and by Corollary 5.2, if $G_p = U_i$, $i \in \{1, 2, ..., 42\}$, then $\gamma_g^{oi}(G_P) = \gamma_g^{oi}(G)$.

According to the presented γ_g-sets and γ_g^{oi}-sets in Figure 5, $\gamma_g^{oi}(G) = \gamma_g(G)$ if $G_p = U_i$, $i \in \{1, 5, 29, 30, 32\}$ ($G \in B$) and $\gamma_g^{oi}(G) \neq \gamma_g(G)$ if $G_p = U_i$, for $i \in \{3, 4, 8, 9, 11, 12, 14, 17, 18, 19, 20, 22, 24, 25, 26, 27, 31, 33, 34, 36, 37, 38, 39, 40, 42\}$. For $G_p = U_i$, $i \in \{2, 6, 7, 10, 13, 15, 16, 21, 23, 28, 35, 41\}$ we verify $\gamma_g(G)$ and $\gamma_g^{oi}(G)$ in all possible cases.

If $G_p = U_2$, then $G = U_2$ or G is the graph bellow:

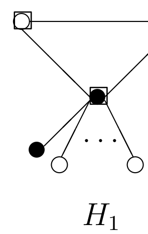


Figure 6. Graphs G with $G_p = U_2$.

If $G_p = U_6$, then $G = U_6$ or G is one of the below graphs:

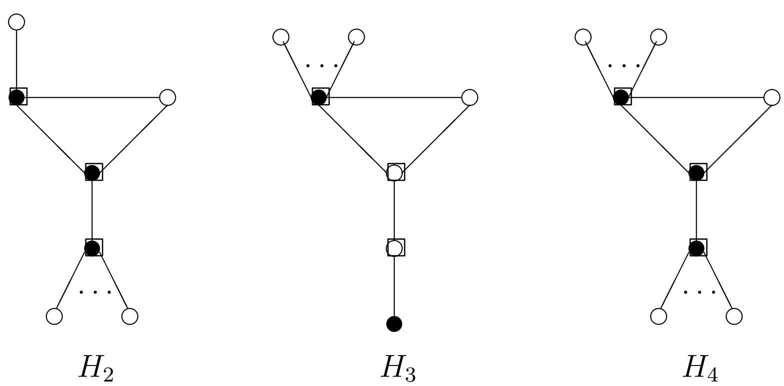
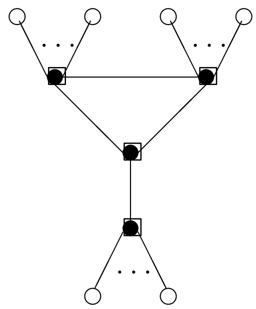


Figure 7. Graphs G with $G_p = U_6$.

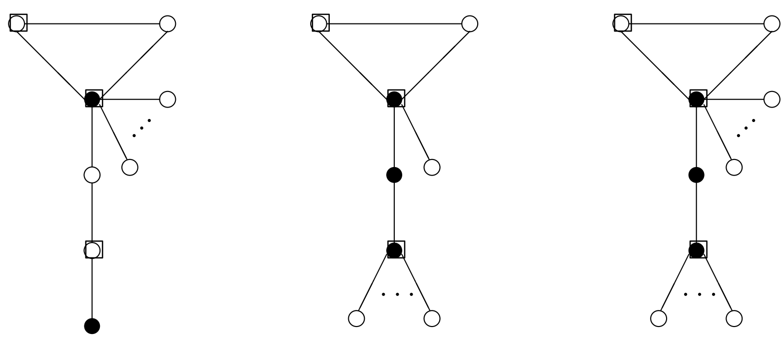
If $G_p = U_7$, then the set $S(G) \cup C(G)$ is a γ_g^{oi}-set of G. If G has a vertex $u \in S(G)$ such that there is only one leaf, x, adjacent to u, then the set $(S(G) \cup \{x\}) - \{u\}$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G) + 1$. If every vertex $u \in S(G)$ is adjacent to at least two leaves (Figure 8), then the set $S(G) \cup C(G)$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G)$.



H_5

Figure 8. Graphs G with $G_p = U_7$ and every support vertex has at least two leaves

If $G_p = U_{10}$, then $G = U_{10}$ or G is one of the below graphs:



H_6 H_7 H_8

Figure 9. Graphs G with $G_p = U_{10}$.

If $G_p = U_{13}$ (Figure 10), then the set $S(G) \cup \{a\}$ is a γ_g^{oi}-set of G.

If there exists $i \in \{1, 2, ..., k\}$ such that $d_G(x_i) = 2$ and y_i is the leaf adjacent to x_i, then the set $(S(G) \cup \{y_i\}) - \{x_i\}$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G) + 1$. If for every $i \in \{1, 2, ..., k\}$, $d_G(x_i) > 2$, then the set $S(G) \cup C(G)$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G)$ (Figure 11).

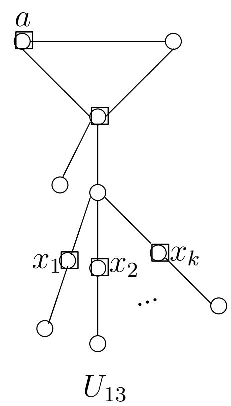


Figure 10

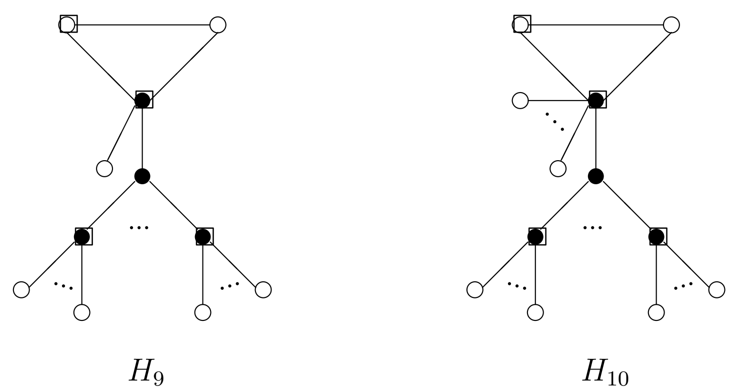


Figure 11. Graphs G with $G_p = U_{13}$.

If $G_p = U_{15}$ (Figure 12), then the set $S(G) \cup C(G)$ is a γ_g^{oi}-set of G.

If there exists $i \in \{1, 2, ..., k\}$ such that $d_G(x_i) = 2$ and y_i is the leaf adjacent to x_i, then the set $(S(G) \cup \{y_i\}) - \{x_i\}$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G) + 1$. If for every $i \in \{1, 2, ..., k\}$, $d_G(x_i) > 2$, then the set $S(G) \cup C(G)$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G)$ (Figure 13).

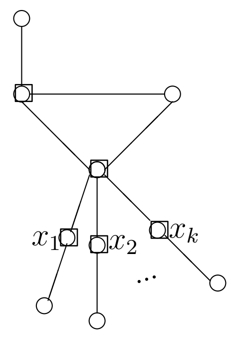


Figure **12**

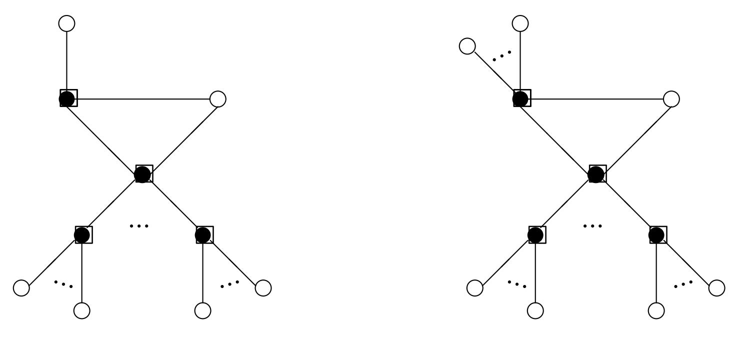


Figure 13. Graphs G with $G_p = U_{15}$.

If $G_p = U_{16}$, then the set $S(G) \cup C(G)$ is a γ_g^{oi}-set of G. If G has a vertex $u \in S(G)$ such that there is only one leaf, x, adjacent to u, then the set $(S(G) \cup \{x\}) - \{u\}$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G) + 1$. If every vertex $u \in S(G)$ is adjacent to at least two leaves (Figure 14), then the set $S(G) \cup C(G)$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G)$.

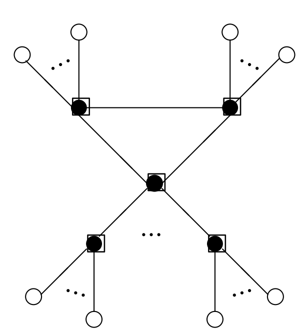


Figure 14. Graphs G with $G_p = U_{16}$ and every support vertex has at least two leaves.

If $G_p = U_{21}$, then $G = U_{21}$ or G is one of the below graphs:

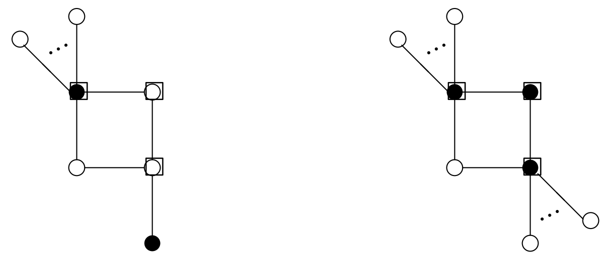


Figure 15. Graphs G with $G_p = U_{21}$.

If $G_p = U_{23}$, then the set $S(G) \cup C(G)$ is a γ_g^{oi}-set of G. If G has a vertex $u \in S(G)$ such that there is only one leaf, x, adjacent to u, then the set $(S(G) \cup \{x\}) - \{u\}$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G) + 1$. If every vertex $u \in S(G)$ is adjacent to at least two leaves (Figure 16), then the set $S(G) \cup C(G)$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G)$.

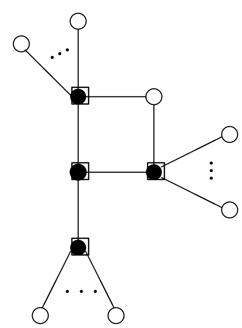
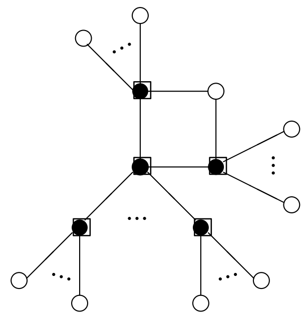


Figure 16. Graphs G with $G_p = U_{23}$ and every support vertex has at least two leaves.

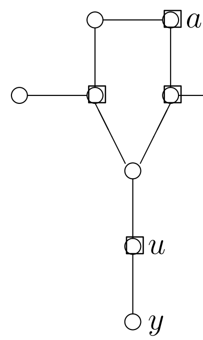
If $G_p = U_{28}$, then the set $S(G) \cup C(G)$ is a γ_g^{oi}-set of G. If G has a vertex $u \in S(G)$ such that there is only one leaf, x, adjacent to u, then the set $(S(G) \cup \{x\}) - \{u\}$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G) + 1$. If every vertex $u \in S(G)$ is adjacent to at least two leaves (Figure 17), then the set $S(G) \cup C(G)$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G)$.



H_{17}

Figure 17. Graphs G with $G_p = U_{28}$ and every support vertex has at least two leaves.

If $G_p = U_{35}$ (Figure 18), then the set $S(G) \cup \{a\}$ is a γ_g^{oi}-set of G.



U_{35}

Figure 18.

If $d_G(u) = 2$ and y is the leaf adjacent to u, then the set $(S(G) \cup \{y\}) - \{u\}$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G) + 1$. If $d_G(u) > 2$, then the set $S(G) \cup C(G)$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G)$ (Figure 19).

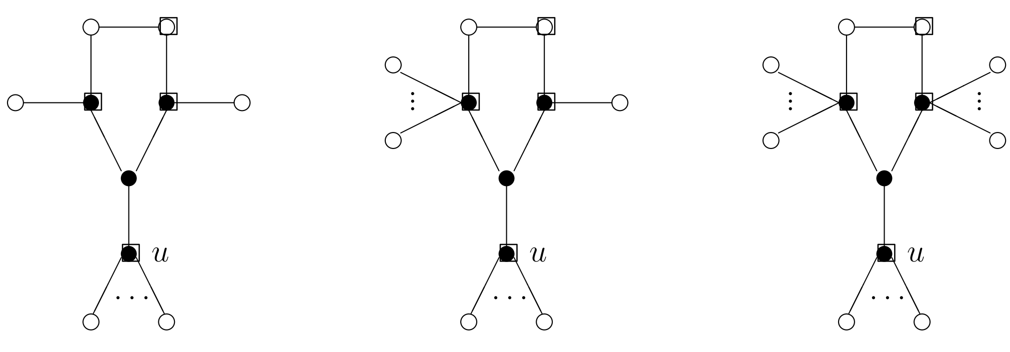


Figure 19. Graphs G with $G_p = U_{35}$ and $d(u) \geq 3$.

If $G_p = U_{41}$ (Figure 20), then the set $S(G) \cup \{a\}$ is a γ_g^{oi}-set of G.

If there exists $i \in \{1, 2, ..., k\}$ such that $d_G(x_i) = 2$ and y_i is the leaf adjacent to x_i, then the set $(S(G) \cup \{y_i\}) - \{x_i\}$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G) + 1$. If for every $i \in \{1, 2, ..., k\}$, $d_G(x_i) > 2$, then the set $S(G) \cup C(G)$ is a γ_g-set of G, so $\gamma_g^{oi}(G) = \gamma_g(G)$ (Figure 21).

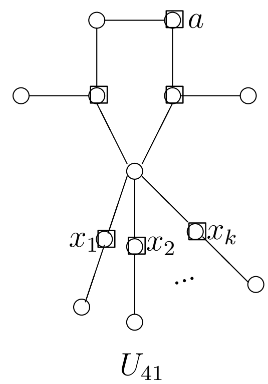


Figure 20.

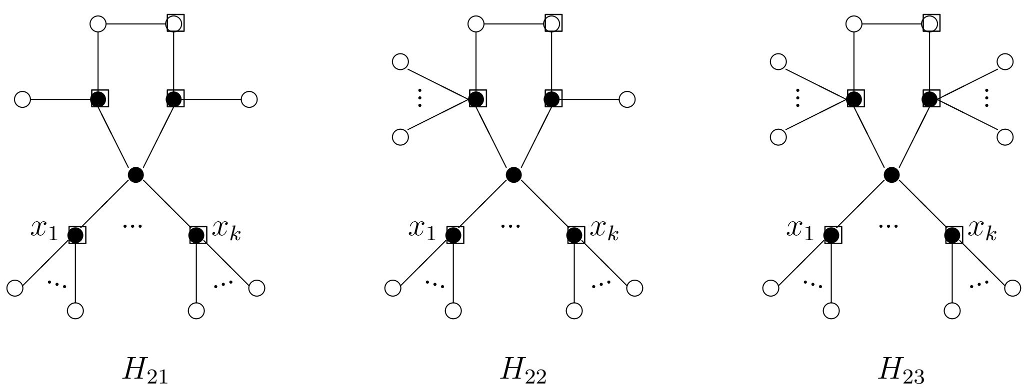


Figure 21. Graphs G with $G_p = U_{41}$ and $d(x_i) \geq 3$, $(1 \leq i \leq k)$.

According to the presented γ_g-sets and γ_g^{oi}-sets for graphs $G = H_i$, $i = 1, 2, ..., 23$, we have $\gamma_g^{oi}(H_i) = \gamma_g(H_i)$ only for $i = 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, ..., 23$ ($G \in C$). □

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