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On adjacency and (signless) Laplacian spectra of centralizer and co-centralizer graphs of some finite non-abelian groups

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Abstract

Let G be a finite non abelian group. The centralizer graph of G is a simple undirected graph $\Gamma_{cent}(G)$, whose vertices are the proper centralizers of G and two vertices are adjacent if and only if their cardinalities are identical [7]. The complement of the centralizer graph is called the co-centralizer graph. In this paper, we investigate the adjacency and (signless) Laplacian spectra of centralizer and co-centralizer graphs of some classes of finite non-abelian groups and obtain some conditions on a group so that the centralizer and co-centralizer graphs are adjacency, (signless) Laplacian integral. We also demonstrate how the integrality phenomena of these graphs either align with or differ from those of the commuting and non-commuting graphs of the corresponding groups.

Keywords: Centralizer graph, Co-centralizer graph, Adjacency matrix, Laplacian matrix, signless Laplacian matrix, spectrum, integral graphs
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1. Introduction

Let G be a finite non-ableian group. In literature, there are many occasions when one associates a graph to a group G in different ways. For example, a commuting graph of a group G denoted by

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$\Gamma_{com}(G)$ is a graph with $G \setminus Z(G)$, where $Z(G)$ is the centre of G, as the vertex set and two vertices x and y are adjacent if and only if $xy = yx$. Similarly, the non-commuting graph is a graph with $G \setminus Z(G)$ as the set of vertices and two vertices x and y are adjacent if and only if $xy \neq yx$, and it is denoted by $\overline{\Gamma_{com}(G)}$. Likewise, non-nilpotent graph, cyclic, non-cyclic and conjugacy class graphs has also been associated on a group. The centralizer graph of G is a simple undirected graph $\Gamma_{cent}(G)$, whose vertices are the proper centralizers of G and two vertices are adjacent if their cardinalities are identical [7]. A brief study is done about the structure of centralizer graph in [7]. Another definition is given for centralizer graph in [6] where the centralizer graph of G is a simple undirected graph with the proper centralizers of G constituting the vertex set and two vertices are adjacent if they are same, and its complement graph is called non-centralizer graph. In this article, we consider the centralizer graph defined in [7]. Also, we define the complement of that graph as the co-centralizer graph and denote it by $\overline{\Gamma_{cent}(G)}$. A group G is called a ***CA-group*** if the centralizer $C_G(x)$ of every non-central element $x \in G$ is abelian. Examples of CA-groups include the generalized quaternion group, the dihedral group, the quasidihedral group, the metacyclic group, and the projective special linear group, among others. Further details and various properties of CA-groups can be found in [9, 10, 11].

For a simple graph H on n vertices, the adjacency matrix $A(H)$ is a matrix of order n, whose (i, j)-th entry is 1, if the i-th vertex is adjacent to the j-th vertex; otherwise it is 0. Also, the Laplacian (resp. signless Laplacian) matrix of H is defined as $L(H) = D(H) - A(H)$ (resp. $Q(H) = D(H) + A(H)$), where $D(H)$ is the diagonal matrix of order n, with degree of the i-th vertex as the i-th diagonal entry.

If M is a symmetric matrix, then the characteristic polynomial of M has only real zeroes. We will represent this family of eigenvalues (known as the *spectrum*) as

$$\sigma_M = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_p \\ m_1 & m_2 & \cdots & m_p \end{pmatrix},$$

where $\mu_1, \mu_2, \ldots, \mu_p$ are the distinct eigenvalues of M and $m_1, m_2, \ldots, m_p$ are the corresponding multiplicities. Since each of $A(H)$, $L(H)$ and $Q(H)$ is symmetric, we will refer the corresponding spectrum as *the adjacency, the Laplacian* and *the signless Laplacian spectrum*, respectively.

In [2], the adjacency spectrum of the commuting graph of some finite non-abelian groups is discussed. In [3], the Laplacian and signless Laplacian spectra of the commuting graph of some finite non-abelian groups is investigated, whereas the Laplacian spectrum of the non-commuting graph of some finite non-abelian groups is determined in [4]. In [8], the Laplacian spectrum of unitary cayley graphs are discussed. For other related results the reader can look into [2, 3, 4, 8] and the references therein.

A graph is called adjacency (respectively (signless) Laplacian) integral if the adjacency (respectively (signless) Laplacian) spectrum consists entirely of integers. In this article, we consider some finite non-abelian groups, namely the generalized quaternion group, the dihedral group, the quasidihedral group, the metacyclic group, and the projective special linear group, and investigate the adjacency, (signless) Laplacian spectra of centralizer and co-centralizer graphs of them. Moreover, we obtain some conditions so that their centralizer and co-centralizer graphs are adjacency, (signless) Laplacian integral and demonstrate the ways in which the integrality phenomena of these graphs either coincide with or deviate from those of the commuting and non-commuting graphs of the corresponding groups.

2. Preliminaries

Consider $Q_{4n} =< x, y : x^{2n} = 1, x^n = y^2, yx = x^{-1}y >$, be the generalized quaternion group of order $4n$, where $n \geq 2$, and $Z(Q_{4n})$= $\{1, x^n\}$. Then Q_{4n} can be written as $A \cup B$, where $A = \{1, x, x^2, \ldots, x^{2n-1}\}$ and $B = \{y, xy, x^2y, \ldots, x^{2n-1}y\}$, where each element of B is of order 4. It has $n + 1$ distinct centralizers with one of cardinality $2n$ and others are of cardinality 4. For illustration, we note that for any $z \in Z(Q_{4n})$, and $1 \leq i \leq 2n - 1$,

$$\begin{aligned} C_{Q_{4n}}(x) &= C_{Q_{4n}}(x^i z) \\ &= Z(Q_{4n}) \cup xZ(Q_{4n}) \cup x^2 Z(Q_{4n}) \cup \ldots \cup x^{n-1} Z(Q_{4n}) \\ &= \{1, x^n\} \cup x\{1, x^n\} \cup x^2\{1, x^n\} \cup \ldots \cup x^{n-1}\{1, x^n\} \\ &= \{1, x^n\} \cup \{x, x^{n+1}\} \cup \{x^2, x^{n+2}\} \cup \ldots \cup \{x^{n-1}, x^{2n-1}\} \\ &= \{1, x, x^2, \ldots, x^{2n-1}\}. \end{aligned}$$

Moreover, for $1 \leq j \leq n$,

$$\begin{aligned} C_{Q_{4n}}(yx^j) = C_{Q_{4n}}(yx^j z) &= Z(Q_{4n}) \cup yx^j Z(Q_{4n}) \\ &= \{1, x^n\} \cup yx^j\{1, x^n\} \\ &= \{1, x^n\} \cup \{yx^j, yx^{n+j}\}. \end{aligned}$$

Therefore, from the definition of centralizer graph, it follows that $\Gamma_{cent}(Q_{4n})$ is a graph with $n + 1$ vertices where one component is K_n and the other is an isolated vertex, i.e., $\Gamma_{cent}(Q_{4n}) \cong K_n \sqcup K_1$. As co-centralizer graph is the complement of this graph, therefore $\overline{\Gamma_{cent}(Q_{4n})} \cong K_{1,n}$.

The following result gives the adjacency characteristics polynomial for $K_{p_1,p_2,\ldots,p_n}$ and will be useful to derive some of our main results.

Lemma 2.1. [5] *The adjacency characteristics polynomial of the complete multipartite graph* $K_{p_1,p_2,\ldots,p_n}$ *, where* $p_1 + p_2 + \ldots + p_n = N$ *is:*

$$P_{A(G)}(\lambda) = \lambda^{N-n} \left[\prod_{i=1}^{n} (\lambda + p_i) - \sum_{i=1}^{n} p_i \prod_{j=1, j \neq i}^{n} (\lambda + p_j) \right]. \quad (1)$$

3. Spectra of centralizer graphs of some finite non-abelian groups

3.1. Spectra of $\Gamma_{cent}(Q_{4n})$

In this section, we consider the centralizer graph of Q_{4n} and obtain the adjacency, Laplacian and signless Laplacian spectra of it. It is well known (see [1, 3]) that when $G = K_n$, then $\sigma(Q(G)) =$

$\begin{pmatrix} 2n-2 & n-2 \\ 1 & n-1 \end{pmatrix}$. when $G = K_{m_1} \sqcup K_{m_2} \sqcup \ldots \sqcup K_{m_l}$, then

$$\sigma(A(G)) = \begin{pmatrix} -1 & m_1-1 & m_2-1 & \ldots & m_l-1 \\ \sum_{i=1}^{l} m_i - l & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Similarly, if $G = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \ldots \sqcup l_kK_{m_k}$, then

$$\sigma(L(G)) = \begin{pmatrix} 0 & m_1 & m_2 & \ldots & m_k \\ \sum_{i=1}^{k} l_i & l_1(m_1-1) & l_2(m_2-1) & \ldots & l_k(m_k-1) \end{pmatrix}.$$

Therefore, the spectra of $\Gamma_{\text{cent}}(Q_{4n})$ are

$$\sigma(A(\Gamma_{\text{cent}}(Q_{4n}))) = \begin{pmatrix} -1 & 0 & n-1 \\ n-1 & 1 & 1 \end{pmatrix}, \quad \sigma(L(\Gamma_{\text{cent}}(Q_{4n}))) = \begin{pmatrix} 0 & n \\ 2 & n-1 \end{pmatrix},$$

$$\sigma(Q(\Gamma_{\text{cent}}(Q_{4n}))) = \begin{pmatrix} 0 & n-2 & 2(n-1) \\ 1 & n-1 & 1 \end{pmatrix}.$$

Thus, $\Gamma_{cent}(Q_{4n})$ is adjacency, Laplacian and signless Laplacian integral for any n.

3.2. Comparative study of the centralizer graph $\Gamma_{\text{cent}}(Q_{4n})$ and the commuting graph $\Gamma_{\text{com}}(Q_{4n})$

As established earlier, Q_{4n} is a non-abelian CA-group that possesses $n+1$ distinct centralizers of non-central elements. Among these, one centralizer has order $2n$, while each of the remaining ones has order 4. Since $|Z(Q_{4n})| = 2$ and all centralizers of its non-central elements are abelian, it follows that

$$\Gamma_{\text{com}}(Q_{4n}) \cong K_{2n-2} \sqcup nK_2.$$

The spectra of $\Gamma_{\text{com}}(Q_{4n})$ have been explicitly determined in [2, 3], where it is shown that its adjacency, Laplacian, and signless Laplacian spectra are integral for all values of n. Thus, the integrality phenomena of the centralizer graph of Q_{4n} is consistent with that of its commuting graph.

3.3. Spectra of $\Gamma_{cent}(D_{2n})$

In this section, we consider the centralizer graph of the dihedral group $D_{2n} =< x, y : x^n = y^2 = 1, yxy^{-1} = x^{-1} >$ and obtain the adjacency, Laplacian and signless Laplacian spectra of it. For odd values of n, we have $|Z(D_{2n})| = 1$, and the group D_{2n} admits $n+1$ distinct centralizers of non-central elements. Among these, one centralizer is of order n, while each of the remaining centralizers has order 2. Consequently, the centralizer graph $\Gamma_{\text{cent}}(D_{2n})$ takes the form $K_1 \sqcup K_n$ when n is odd.

On the other hand, when n is even, we obtain $|Z(D_{2n})| = 2$, and D_{2n} possesses $\frac{n}{2}+1$ distinct centralizers of non-central elements. In this case, exactly one centralizer has order n, while all others are of order 4. Therefore, for even n, the centralizer graph $\Gamma_{\text{cent}}(D_{2n})$ is given by $K_1 \sqcup K_{\frac{n}{2}}$.

Therefore, if n is odd, then

$$\sigma(A(\Gamma_{\text{cent}}(D_{2n}))) = \begin{pmatrix} -1 & 0 & n-1 \\ n-1 & 1 & 1 \end{pmatrix}, \quad \sigma(L(\Gamma_{\text{cent}}(D_{2n}))) = \begin{pmatrix} 0 & n \\ 2 & n-1 \end{pmatrix},$$

$$\sigma(Q(\Gamma_{\text{cent}}(D_{2n}))) = \begin{pmatrix} 0 & n-2 & 2(n-1) \\ 1 & n-1 & 1 \end{pmatrix}.$$

Similarly, if n is even, then

$$\sigma(A(\Gamma_{\text{cent}}(D_{2n}))) = \begin{pmatrix} -1 & 0 & \frac{n}{2}-1 \\ \frac{n}{2}-1 & 1 & 1 \end{pmatrix}, \quad \sigma(L(\Gamma_{\text{cent}}(D_{2n}))) = \begin{pmatrix} 0 & \frac{n}{2} \\ 2 & \frac{n}{2}-1 \end{pmatrix},$$

$$\sigma(Q(\Gamma_{\text{cent}}(D_{2n}))) = \begin{pmatrix} 0 & \frac{n}{2}-2 & n-2 \\ 1 & \frac{n}{2}-1 & 1 \end{pmatrix}.$$

Thus, $\Gamma_{\text{cent}}(D_{2n})$ is adjacency, Laplacian, and signless Laplacian integral for all n.

3.4. *Comparative study of the centralizer graph $\Gamma_{\text{cent}}(D_{2n})$ and the commuting graph $\Gamma_{\text{com}}(D_{2n})$*

Since D_{2n} is a CA-group, by the discussion in the previous section regarding the structure of $\Gamma_{\text{cent}}(D_{2n})$, it follows that

$$\Gamma_{\text{com}}(D_{2n}) \cong \begin{cases} K_{n-1} \sqcup nK_1, & \text{if } n \text{ is odd,} \\ K_{n-2} \sqcup \frac{n}{2}K_2, & \text{if } n \text{ is even.} \end{cases}$$

In [2, 3], the adjacency, Laplacian, and signless Laplacian spectra of $\Gamma_{\text{com}}(D_{2n})$ were determined, and from those results it is evident that the commuting graph of D_{2n} is adjacency, Laplacian, and signless Laplacian integral for all n. Thus, the property of integrality exhibited by $\Gamma_{\text{com}}(D_{2n})$ is consistent with that of $\Gamma_{\text{cent}}(D_{2n})$.

Remark 3.1. Let us consider the metacyclic group $M_{2pq} =< a, b : a^p = b^{2q} = 1, bab^{-1} = a^{-1} >$, where $p > 2$. It can be easily observed that for even p (respectively for odd p) the corresponding centralizer graph is same as that of the centralizer graph of dihedral group D_{2n} for even n (respectively for odd n), and is independent of q. Therefore the adjacency, Laplacian and signless Laplacian spectra of M_{2pq} is exactly same as the corresponding spectra of D_{2n}.

3.5. *Spectra of $\Gamma_{cent}(QD_{2^n})$*

In this section, we consider the centralizer graph of the quasidihedral group $QD_{2^n} =< a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}-1} >$, where $n \geq 4$, and obtain the adjacency, Laplacian and signless Laplacian spectra of it. It is well known that $Z(QD_{2^n}) = \{1, a^{2^{n-2}}\}$. The group QD_{2^n} admits $2^{n-2}+1$ distinct centralizers of its non-central elements. Among these, one centralizer has order 2^{n-1}, while each of the remaining centralizers has order 4. Consequently, the centralizer graph of QD_{2^n} is given by

$$\Gamma_{\text{cent}}(QD_{2^n}) \cong K_1 \sqcup K_{2^{n-2}}.$$

The corresponding spectra of $\Gamma_{\text{cent}}(QD_{2^n})$ are as follows:

$$\sigma(A(\Gamma_{\text{cent}}(QD_{2^n}))) = \begin{pmatrix} -1 & 0 & 2^{n-2}-1 \\ 2^{n-2}-1 & 1 & 1 \end{pmatrix}, \quad \sigma(L(\Gamma_{\text{cent}}(QD_{2^n}))) = \begin{pmatrix} 0 & 2^{n-2} \\ 2 & 2^{n-2}-1 \end{pmatrix},$$

$$\sigma(Q(\Gamma_{\text{cent}}(QD_{2^n}))) = \begin{pmatrix} 0 & 2^{n-2}-2 & 2^{n-1}-2 \\ 1 & 2^{n-2}-1 & 1 \end{pmatrix}.$$

Thus, $\Gamma_{cent}(QD_{2^n})$ is adjacency, Laplacian and signless Laplacian integral for any n.

3.6. Comparative study of the centralizer graph $\Gamma_{\text{cent}}(QD_{2^n})$ and the commuting graph $\Gamma_{\text{com}}(QD_{2^n})$

Since QD_{2^n} is a CA-group, it follows from the above discussion that

$$\Gamma_{\text{com}}(QD_{2^n}) \cong K_{2^{n-1}-2} \sqcup 2^{n-2}K_2.$$

From the known spectra of the commuting graph of QD_{2^n} [2, 3], it is established that $\Gamma_{\text{com}}(QD_{2^n})$ is adjacency, Laplacian, and signless Laplacian integral for every n. Thus, the integrality phenomena of the centralizer graph of QD_{2^n} coincides with that of its commuting graph.

3.7. Spectra of $\Gamma_{cent}(PSL(2, 2^k))$

In this section, we consider the centralizer graph of the projective special linear group $PSL(2, 2^k)$ and obtain the adjacency, Laplacian and signless Laplacian spectra of it. The centralizer graph of $PSL(2, 2^k)$ is given by

$$\Gamma_{\text{cent}}(PSL(2, 2^k)) \cong K_{2^k+1} \sqcup K_{2^{k-1}(2^k+1)} \sqcup K_{2^{k-1}(2^k-1)}.$$

Hence, the spectra of $\Gamma_{\text{cent}}(PSL(2, 2^k))$ are as follows:

$$\sigma\big(A(\Gamma_{\text{cent}}(PSL(2, 2^k)))\big) = \begin{pmatrix} -1 & 2^k & 2^{k-1}(2^k+1)-1 & 2^{k-1}(2^k-1)-1 \\ 2^{2k}+2^k-2 & 1 & 1 & 1 \end{pmatrix},$$

$$\sigma\big(L(\Gamma_{\text{cent}}(PSL(2, 2^k)))\big) = \begin{pmatrix} 0 & 2^k+1 & 2^{k-1}(2^k+1) & 2^{k-1}(2^k-1) \\ 3 & 2^k & 2^{k-1}(2^k+1)-1 & 2^{k-1}(2^k-1)-1 \end{pmatrix}.$$

Let $\mathbb{1}_n$ (resp. $\mathbb{0}_n$) denote the $n \times 1$ vector with each entry 1 (resp. 0). Also, let J_n (resp. $\mathbf{0}_n$) denote the matrix of order n with all entries equal to 1 (resp. 0) and I_n denote the identity matrix of order n (we will write J (resp. $\mathbf{0}$) and I if the order is clear from the context). The following theorem describes the signless Laplacian spectrum of $\Gamma_{cent}(PSL(2, 2^k))$.

Theorem 3.1. *Let $\Gamma_{cent}(PSL(2, 2^k))$ be the centralizer graph of the projective special linear group. Then*

(a) $2^k - 1 \in \sigma(Q(\Gamma_{cent}(PSL(2, 2^k))))$ *with multiplicity* 2^k;
(b) $2^{k-1}(2^k+1) - 2 \in \sigma(Q(\Gamma_{cent}(PSL(2, 2^k))))$ *with multiplicity* $2^{k-1}(2^k+1) - 1$;
(c) $2^{k-1}(2^k-1) - 2 \in \sigma(Q(\Gamma_{cent}(PSL(2, 2^k))))$ *with multiplicity* $2^{k-1}(2^k-1) - 1$,

(d) $(2^k+1)(2^k-2) \in \sigma(Q(\Gamma_{cent}(PSL(2,2^k))))$ *with multiplicity* 1,
(e) $2^{2k}+2^k-2 \in \sigma(Q(\Gamma_{cent}(PSL(2,2^k))))$ *with multiplicity* 1,
(f) $2^{k+1} \in \sigma(Q(\Gamma_{cent}(PSL(2,2^k))))$ *with multiplicity* 1.

Proof. With a suitable labeling of the vertices, the signless Laplacian matrix for $\Gamma_{cent}(PSL(2,2^k))$ can be written as

$$Q(\Gamma_{cent}(PSL(2,2^k))) = \left(\begin{array}{c|c|c} J+(2^k-1)I & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & J+(2^{k-1}(2^k+1)-2)I & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & J+(2^{k-1}(2^k-1)-2)I \end{array}\right).$$

Now, $Q(\Gamma_{cent}(PSL(2,2^k)))\begin{pmatrix} -1 \\ 1 \\ \mathbb{O}_{2^k-1} \\ \hline \mathbb{O}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix} = (2^k-1)\begin{pmatrix} -1 \\ 1 \\ \mathbb{O}_{2^k-1} \\ \hline \mathbb{O}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix}$.

Therefore, (2^k-1) is an eigenvalue of $Q(\Gamma_{cent}(PSL(2,2^k)))$, and the following set V_1 lists the set of 2^k linearly independent eigenvectors corresponding to the eigenvalue 2^k-1;

$$V_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ \mathbb{O}_{2^k-1} \\ \hline \mathbb{O}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ \mathbb{O}_{2^k-2} \\ \hline \mathbb{O}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix}, \dots, \begin{pmatrix} -1 \\ \mathbb{O}_{2^k-1} \\ 1 \\ \hline \mathbb{O}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix} \right\}.$$

Again, $Q(\Gamma_{cent}(PSL(2,2^k)))\begin{pmatrix} \mathbb{O}_{2^k+1} \\ \hline -1 \\ 1 \\ \mathbb{O}_{2^{k-1}(2^k+1)-2} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix} = (2^{k-1}(2^k+1)-2)\begin{pmatrix} \mathbb{O}_{2^k+1} \\ \hline -1 \\ 1 \\ \mathbb{O}_{2^{k-1}(2^k+1)-2} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix}$. Therefore, $(2^{k-1}(2^k+1)-2)$ is an eigenvalue of $Q(\Gamma_{cent}(PSL(2,2^k)))$, and the set V_2 gives $2^{k-1}(2^k+1)-1$ linearly independent eigenvectors corresponding to it;

$$V_2 = \left\{ \begin{pmatrix} \mathbb{O}_{2^k+1} \\ \hline -1 \\ 1 \\ \mathbb{O}_{2^{k-1}(2^k+1)-2} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix}, \begin{pmatrix} \mathbb{O}_{2^k+1} \\ \hline -1 \\ 0 \\ 1 \\ \mathbb{O}_{2^{k-1}(2^k+1)-3} \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix}, \dots, \begin{pmatrix} \mathbb{O}_{2^k+1} \\ \hline -1 \\ \mathbb{O}_{2^{k-1}(2^k+1)-2} \\ 1 \\ \hline \mathbb{O}_{2^{k-1}(2^k-1)} \end{pmatrix} \right\}.$$

Similarly,

$$Q(\Gamma_{cent}(PSL(2,2^k)))\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ 1\\ \mathbb{0}_{2^{k-1}(2^k-1)-2}\end{pmatrix} = (2^{k-1}(2^k-1)-2)\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ 1\\ \mathbb{0}_{2^{k-1}(2^k-1)-2}\end{pmatrix}.$$

Therefore, $(2^{k-1}(2^k-1)-2)$ is an eigenvalue of $Q(\Gamma_{cent}(PSL(2,2^k)))$, and the set set V_3 gives $(2^{k-1}(2^k-1)-1)$ linearly independent eigenvectors corresponding to it;

$$V_3 = \left\{\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ 1\\ \mathbb{0}_{2^{k-1}(2^k-1)-2}\end{pmatrix}, \begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ 0\\ 1\\ \mathbb{0}_{2^{k-1}(2^k-1)-3}\end{pmatrix}, \dots, \begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ \mathbb{0}_{2^{k-1}(2^k-1)-2}\\ 1\end{pmatrix}\right\}.$$

Moreover,

$$\begin{aligned}
Q(\Gamma_{cent}(PSL(2,2^k)))\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{1}_{2^{k-1}(2^k-1)}\end{pmatrix} &= (2^k+1)(2^k-2)\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{1}_{2^{k-1}(2^k-1)}\end{pmatrix},\\
Q(\Gamma_{cent}(PSL(2,2^k)))\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{1}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{pmatrix} &= (2^{2k}+2^k-2)\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{1}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{pmatrix},\\
Q(\Gamma_{cent}(PSL(2,2^k)))\begin{pmatrix}\mathbb{1}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{pmatrix} &= (2^{k+1})\begin{pmatrix}\mathbb{1}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{pmatrix}.
\end{aligned}$$

Therefore, $(2^k+1)(2^k-2)$, $(2^{2k}+2^k-2)$, and (2^{k+1}) are the eigenvalues of $Q(\Gamma_{cent}(PSL(2,2^k)))$ with $\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{1}_{2^{k-1}(2^k-1)}\end{pmatrix}$, $\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{1}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{pmatrix}$, and $\begin{pmatrix}\mathbb{1}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{pmatrix}$ as the corresponding eigenvectors, respectively.

We note that $V_1\cup V_2\cup V_3\cup\left\{\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{1}_{2^{k-1}(2^k-1)}\end{pmatrix}\right\}\cup\left\{\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{1}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{pmatrix}\right\}\cup\left\{\begin{pmatrix}\mathbb{1}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{pmatrix}\right\}$ is a set of mutually orthogonal eigenvectors for $\Gamma_{cent}(PSL(2,2^k))$. Since the order of $\Gamma_{cent}(PSL(2,2^k))$ is $2^{2k}+2^k+1$, the result follows. □

Thus, $\Gamma_{cent}(PSL(2,2^k))$ is adjacency, Laplacian and signless Laplacian integral for any k.

3.8. Comparative study of the centralizer graph $\Gamma_{\text{cent}}(PSL(2,2^k))$ and the commuting graph $\Gamma_{\text{com}}(PSL(2,2^k))$

It follows from [2, 3] that,

$$\Gamma_{\text{com}}(PSL(2,2^k)) \cong (2^k+1)K_{2^k-1} \sqcup 2^{k-1}(2^k+1)K_{2^k-2} \sqcup 2^{k-1}(2^k-1)K_{2^k},$$

and this graph is adjacency, Laplacian, and signless Laplacian integral for all values of k. Hence, the integrality phenomena of the commuting graph of $PSL(2,2^k)$ is consistent with that of its centralizer graph $\Gamma_{\text{cent}}(PSL(2,2^k))$.

4. Spectra of co-centralizer graphs of some finite non-abelian groups

4.1. Spectra of $\overline{\Gamma_{cent}(Q_{4n})}$

In this section, we consider the co-centralizer graph of Q_{4n} and obtain the adjacency, Laplacian and signless Laplacian spectra of it. It is well known (see [1]) that the adjacency spectra of a complete bipartite graph $K_{m,n}$ is $\begin{pmatrix} \sqrt{mn} & -\sqrt{mn} & 0 \\ 1 & 1 & m+n-2 \end{pmatrix}$. As it is already observed in Section 2, $\overline{\Gamma_{cent}(Q_{4n})} = K_{1,n}$. Therefore, $\sigma(A(\overline{\Gamma_{cent}(Q_{4n})})) = \begin{pmatrix} \sqrt{n} & -\sqrt{n} & 0 \\ 1 & 1 & n-1 \end{pmatrix}$. Therefore, $\overline{\Gamma_{cent}(Q_{4n})}$ is adjacency integral if n is a perfect square. Also, by Lemma 5 of [8], $\sigma(L(\overline{\Gamma_{cent}(Q_{4n})})) = \begin{pmatrix} 0 & 1 & 1+n \\ 1 & n-1 & 1 \end{pmatrix}$. Since, for a bipartite graph the Laplacian spectrum coincides with the signless Laplacian spectrum, we have $\sigma(L(\overline{\Gamma_{cent}(Q_{4n})})) = \sigma(Q(\overline{\Gamma_{cent}(Q_{4n})}))$. Thus, $\overline{\Gamma_{cent}(Q_{4n})}$ is Laplacian and signless Laplacian integral for any value of n.

4.2. Comparative study of the co-centralizer graph $\overline{\Gamma_{\text{cent}}(Q_{4n})}$ and the non-commuting graph $\overline{\Gamma_{\text{com}}(Q_{4n})}$

In [4, 12, 13], the spectra of $\overline{\Gamma_{\text{com}}(Q_{4n})}$ have been thoroughly investigated, and it has been shown that this graph is integral precisely when $(n-1)(5n-1)$ is a perfect square. On the other hand, $\overline{\Gamma_{\text{cent}}(Q_{4n})}$ is integral whenever n is a perfect square. Hence, the integrality phenomenon differs between the two graphs.

It is further observed that both $\overline{\Gamma_{\text{com}}(Q_{4n})}$ and $\overline{\Gamma_{\text{cent}}(Q_{4n})}$ are Laplacian integral for all values of n. Moreover, $\overline{\Gamma_{\text{com}}(Q_{4n})}$ is signless Laplacian integral exactly when $(8n^2-16n+9)$ is a perfect square, whereas $\overline{\Gamma_{\text{cent}}(Q_{4n})}$ is signless Laplacian integral for every n. Thus, once again, the integrality behavior of these two graphs exhibits a clear distinction.

4.3. Spectra of $\overline{\Gamma_{cent}(D_{2n})}$

In this section, we consider the co-centralizer graph of dihedral group D_{2n} and obtain the adjacency, Laplacian and signless Laplacian spectra of it. The co centralizer graph of D_{2n} is $\overline{\Gamma_{cent}(D_{2n})} = \begin{cases} K_{1,n}, & \text{if } n \text{ is odd,} \\ K_{1,\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$

Therefore, $\sigma(A(\overline{\Gamma_{cent}(D_{2n})})) = \begin{cases} \begin{pmatrix} \sqrt{n} & -\sqrt{n} & 0 \\ 1 & 1 & n-1 \end{pmatrix}, & \text{if } n \text{ is odd,} \\ \begin{pmatrix} \sqrt{\frac{n}{2}} & -\sqrt{\frac{n}{2}} & 0 \\ 1 & 1 & \frac{n}{2}-1 \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases}$ Thus, $\overline{\Gamma_{cent}(D_{2n})}$ is adjacency integral if n is a perfect square for odd n, and $\frac{n}{2}$ is a perfect square for even n.

As discussed in Subsection 4.1, it can be seen that

$$\sigma(L(\overline{\Gamma_{cent}(D_{2n})})) = \sigma(Q(\overline{\Gamma_{cent}(D_{2n})})) = \begin{cases} \begin{pmatrix} 0 & 1+n & 1 \\ 1 & 1 & n-1 \end{pmatrix}, & \text{if } n \text{ is odd,} \\ \begin{pmatrix} 0 & 1+\frac{n}{2} & 1 \\ 1 & 1 & \frac{n}{2}-1 \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases}$$

Hence for any value of n, $\overline{\Gamma_{cent}(D_{2n})})$ is both Laplacian and signless Laplacian integral.

Remark 4.1. By virtue of Remark 3.1, we can conclude that the adjacency, Laplacian and signless Laplacian spectrum of $\overline{\Gamma_{cent}(M_{2pq})}$ is exactly same as the corresponding spectrum of $\overline{\Gamma_{cent}(D_{2n})}$.

4.4. Comparative study of the co-centralizer graph $\overline{\Gamma_{\text{cent}}(D_{2n})}$ and the non-commuting graph $\overline{\Gamma_{\text{com}}(D_{2n})}$

In [4, 12, 13], the spectra of $\overline{\Gamma_{\text{com}}(D_{2n})}$ have been studied in detail. The comparison with $\overline{\Gamma_{\text{cent}}(D_{2n})}$ reveals distinct integrality phenomena, which we summarize below.

(i) **When n is odd:** The graph $\overline{\Gamma_{\text{com}}(D_{2n})}$ is integral if and only if $(n-1)(5n-1)$ is a perfect square, whereas $\overline{\Gamma_{\text{cent}}(D_{2n})}$ is integral precisely when n itself is a perfect square. Thus, for odd n, the integrality phenomenon differs between the two graphs. Both $\overline{\Gamma_{\text{com}}(D_{2n})}$ and $\overline{\Gamma_{\text{cent}}(D_{2n})}$ are Laplacian integral for all values of n. Moreover, $\overline{\Gamma_{\text{com}}(D_{2n})}$ is signless Laplacian integral when $(8n^2-16n+9)$ is a perfect square, while $\overline{\Gamma_{\text{cent}}(D_{2n})}$ is signless Laplacian integral for every n. Hence, the distinction in integrality persists in the signless Laplacian case.

(ii) **When n is even:** The graph $\overline{\Gamma_{\text{com}}(D_{2n})}$ is integral if and only if $\left(\frac{n}{2}-1\right)\left(\frac{5n}{2}-1\right)$ is a perfect square, while $\overline{\Gamma_{\text{cent}}(D_{2n})}$ is integral precisely when $\frac{n}{2}$ is a perfect square. Thus, for even n, the integrality phenomenon again differs between the two graphs. Both the graphs are Laplacian integral for all values of n. Furthermore, $\overline{\Gamma_{\text{com}}(D_{2n})}$ is signless Laplacian integral exactly when $(2n^2-8n+9)$ is a perfect square, whereas $\overline{\Gamma_{\text{cent}}(D_{2n})}$ remains signless Laplacian integral for all n. Once again, the two graphs exhibit distinct integrality behavior.

4.5. Spectra of Quasidihedral group $\overline{\Gamma_{cent}(QD_{2^n})}$

In this section, we consider the co-centralizer graph of the Quasidihedral group QD_{2^n}, where $n \geq 4$, and obtain its adjacency, Laplacian and signless Laplacian spectra. Since $\overline{\Gamma_{cent}(QD_{2^n})} = K_{1,2^{n-2}}$, it follows that

$$\sigma(A(\overline{\Gamma_{cent}(QD_{2^n})})) = \begin{pmatrix} \sqrt{2^{n-2}} & -\sqrt{2^{n-2}} & 0 \\ 1 & 1 & 2^{n-2}-1 \end{pmatrix}.$$

Thus, $\overline{\Gamma_{cent}(QD_{2^n})}$ is adjacency integral, if 2^{n-2} is a perfect square. Also, $\sigma(L(\overline{\Gamma_{cent}(QD_{2^n})})) = \sigma(Q(\overline{\Gamma_{cent}(QD_{2^n})})) = \begin{pmatrix} 2^{n-2}+1 & 0 & 1 \\ 1 & 1 & 2^{n-2}-1 \end{pmatrix}$, showing that for any value of n, $\overline{\Gamma_{cent}(QD_{2^n})}$ is both Laplacian and signless Laplacian integral.

4.6. *Comparative study of the co-centralizer graph $\overline{\Gamma_{\text{cent}}(QD_{2^n})}$ and the non-commuting graph $\overline{\Gamma_{\text{com}}(QD_{2^n})}$*

In [4, 12], the Laplacian and signless Laplacian spectra of $\overline{\Gamma_{\text{com}}(QD_{2^n})}$ have been discussed in detail. It is observed that both $\overline{\Gamma_{\text{cent}}(QD_{2^n})}$ and $\overline{\Gamma_{\text{com}}(QD_{2^n})}$ are Laplacian integral for all values of n. However, $\overline{\Gamma_{\text{com}}(QD_{2^n})}$ is signless Laplacian integral only when $(2^{2n-1} - 2^{n+2} + 9)$ is a perfect square, whereas $\overline{\Gamma_{\text{cent}}(QD_{2^n})}$ is signless Laplacian integral for every n. Thus, once again, the integrality behavior of the two graphs differs from each other.

4.7. *Spectra of $\overline{\Gamma_{cent}(PSL(2,2^k))}$*

As observed in Subsection 3.7, $\overline{\Gamma_{cent}(PSL(2,2^k))}$ is the complete tripartite graph $K_{2^k+1,2^{k-1}(2^k-1),2^{k-1}(2^k+1)}$. Therefore, by equation (1) we get,

$$\begin{aligned} P_{A(\overline{\Gamma_{cent}(PSL(2,2^k))})}(\lambda) &= \lambda^{2^k+2^{2k}-2}[\lambda^3 - \{2^{4k-2} + 2^{3k} + 3 \times 2^{2k-2}\}\lambda + \\ &\quad (-2^{5k-1} - 2^{4k-1} + 2^{3k-1} + 2^{2k-1})]. \end{aligned}$$

Hence, we have the following theorem which describes the adjacency spectrum of $\overline{\Gamma_{cent}(PSL(2,2^k))}$.

Theorem 4.1. *Let $\overline{\Gamma_{cent}(PSL(2,2^k))}$ be the co-centralizer graph of the projective special linear group. Then $\sigma(A(\overline{\Gamma_{cent}(PSL(2,2^k))}))$ consists of*

(a) 0 *with multiplicity* $2^k + 2^{2k} - 2$;
(b) *three roots of the equation* $x^3 - (2^{4k-2} + 3(2^{2k-2}) + 2^{3k})x + (-2^{5k-1} - 2^{4k-1} + 2^{3k-1} + 2^{2k-1}) = 0$.

Also,

$$\begin{aligned} &\sigma(L(\overline{\Gamma_{cent}(PSL(2,2^k))})) \\ = &\begin{pmatrix} 0 & 2^{2k} & 2^{k-1}+2^{2k-1}+1 & 2^{2k-1}+3(2^{k-1})+1 & 2^{2k}+2^k+1 \\ 1 & 2^k & 2^{k-1}(2^k+1)-1 & 2^{k-1}(2^k-1)-1 & 2 \end{pmatrix} \end{aligned}$$

Thus, $\overline{\Gamma_{cent}(PSL(2,2^k))}$ is Laplacian integral for all values of k. The following theorem describes the signless Laplacian spectrum of $\overline{\Gamma_{cent}(PSL(2,2^k))}$.

Theorem 4.2. *Let $\overline{\Gamma_{cent}(PSL(2,2^k))}$ be the co-centralizer graph of the projective special linear group $PSL(2,2^k)$. Then its signless Laplacian spectrum consists of:*

(a) 2^{2k} *with multiplicity* 2^k,
(b) $(2^{k-1} + 2^{2k-1} + 1)$ *with multiplicity* $2^{k-1}(2^k+1) - 1$,
(c) $3 \times 2^{k-1} + 2^{2k-1} + 1$ *with multiplicity* $2^{k-1}(2^k-1) - 1$, *and*

(d) *the three eigenvalues of the matrix*

$$\mathfrak{L}_P = \left[\begin{array}{c|c|c} 2^{2k} & 2^{k-1}(2^k+1) & 2^{k-1}(2^k-1) \\ \hline 2^k+1 & 2^{k-1}+2^{2k-1}+1 & 2^{k-1}(2^k-1) \\ \hline 2^k+1 & 2^{k-1}(2^k+1) & 3\times 2^{k-1}+2^{2k-1}+1 \end{array}\right].$$

Proof. With a suitable labeling of the vertices, the signless Laplacian matrix for $\overline{\Gamma_{cent}(PSL(2,2^k))}$ can be written as

$$Q(\overline{\Gamma_{cent}(PSL(2,2^k))}) = \left[\begin{array}{c|c|c} 2^{2k}I & J & J \\ \hline J & (2^{k-1}+2^{2k-1}+1)I & J \\ \hline J & J & (3(2^{k-1})+2^{2k-1}+1)I \end{array}\right].$$

Now, $Q(\overline{\Gamma_{cent}(PSL(2,2^k))}) \left(\begin{array}{c} -1 \\ 1 \\ \mathbb{0}_{2^k-1} \\ \hline \mathbb{0}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right) = 2^{2k} \left(\begin{array}{c} -1 \\ 1 \\ \mathbb{0}_{2^k-1} \\ \hline \mathbb{0}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right).$

Therefore, 2^{2k} is an eigenvalue of $Q(\overline{\Gamma_{cent}(PSL(2,2^k))})$ with the following set S_1 of 2^k linearly independent eigenvectors;

$$S_1 = \left\{ \left(\begin{array}{c} -1 \\ 1 \\ 0_{2^k-1} \\ \hline \mathbb{0}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right), \left(\begin{array}{c} -1 \\ 0 \\ 1 \\ \mathbb{0}_{2^k-2} \\ \hline \mathbb{0}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right), \dots, \left(\begin{array}{c} -1 \\ \mathbb{0}_{2^k-1} \\ 1 \\ \hline \mathbb{0}_{2^{k-1}(2^k+1)} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right) \right\}.$$

Similarly,

$$Q(\overline{\Gamma_{cent}(PSL(2,2^k))}) \left(\begin{array}{c} \mathbb{0}_{2^k+1} \\ \hline -1 \\ 1 \\ \mathbb{0}_{2^{k-1}(2^k+1)-2} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right) = (2^{k-1}+2^{2k-1}+1) \left(\begin{array}{c} \mathbb{0}_{2^k+1} \\ \hline -1 \\ 1 \\ \mathbb{0}_{2^{k-1}(2^k+1)-2} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right),$$

shows that $(2^{k-1}+2^{2k-1}+1)$ is an eigenvalue of $Q(\overline{\Gamma_{cent}(PSL(2,2^k))})$ and in this way we can construct the following set S_2 of $2^{k-1}(2^k+1)-1$ linearly independent eigenvectors corresponding to $(2^{k-1}+2^{2k-1}+1)$;

$$S_2 = \left\{ \left(\begin{array}{c} \mathbb{0}_{2^k+1} \\ \hline -1 \\ 1 \\ \mathbb{0}_{2^{k-1}(2^k+1)-2} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right), \left(\begin{array}{c} \mathbb{0}_{2^k+1} \\ \hline -1 \\ 0 \\ 1 \\ \mathbb{0}_{2^{k-1}(2^k+1)-3} \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right), \dots, \left(\begin{array}{c} \mathbb{0}_{2^k+1} \\ \hline -1 \\ \mathbb{0}_{2^{k-1}(2^k+1)-2} \\ 1 \\ \hline \mathbb{0}_{2^{k-1}(2^k-1)} \end{array}\right) \right\}.$$

Moreover,

$$Q(\overline{\Gamma_{cent}(PSL(2,2^k))})\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ 1\\ \mathbb{0}_{2^{k-1}(2^k-1)-2}\end{pmatrix} = (3(2^{k-1})+2^{2k-1}+1)\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ 1\\ \mathbb{0}_{2^{k-1}(2^k-1)-2}\end{pmatrix}.$$

So, $(3(2^{k-1})+2^{2k-1}+1)$ is an eigenvalue of $Q(\overline{\Gamma_{cent}(PSL(2,2^k))})$, and the following set S_3 lists $2^{k-1}(2^k-1)-1$ independent eigenvectors corresponding to this eigenvalue;

$$S_3=\left\{\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ 1\\ \mathbb{0}_{2^{k-1}(2^k-1)-2}\end{pmatrix},\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ 0\\ 1\\ \mathbb{0}_{2^{k-1}(2^k-1)-3}\end{pmatrix},\dots,\begin{pmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline -1\\ \mathbb{0}_{2^{k-1}(2^k-1)-2}\\ 1\end{pmatrix}\right\}.$$

Thus, we have obtained $2^k+2^{k-1}(2^k+1)-1+2^{k-1}(2^k-1)-1=2^k+2^{2k}-2$ eigenvalues of $Q(\overline{\Gamma_{cent}(PSL(2,2^k))})$. Moreover, we note that all the eigenvectors constructed so far, are orthogonal to $\begin{bmatrix}\mathbb{1}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{bmatrix}$, $\begin{bmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{1}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{0}_{2^{k-1}(2^k-1)}\end{bmatrix}$ and $\begin{bmatrix}\mathbb{0}_{2^k+1}\\ \hline \mathbb{0}_{2^{k-1}(2^k+1)}\\ \hline \mathbb{1}_{2^{k-1}(2^k-1)}\end{bmatrix}$. Therefore, these three vectors span the remaining three eigenvectors of $Q(\overline{\Gamma_{cent}(PSL(2,2^k))})$. Thus, the remaining eigenvectors of $Q(\overline{\Gamma_{cent}(PSL(2,2^k))})$ are of the form $\begin{bmatrix}a\mathbb{1}_{2^k+1}\\ \hline b\mathbb{1}_{2^{k-1}(2^k+1)}\\ \hline c\mathbb{1}_{2^{k-1}(2^k-1)}\end{bmatrix}$, for some $(a,b,c)\neq(0,0,0)$. Therefore if μ is an eigenvalue of $Q(\overline{\Gamma_{cent}(PSL(2,2^k))})$ with eigenvector $\begin{bmatrix}a\mathbb{1}_{2^k+1}\\ \hline b\mathbb{1}_{2^{k-1}(2^k+1)}\\ \hline c\mathbb{1}_{2^{k-1}(2^k-1)}\end{bmatrix}$, then a,b,c are the solution of the following system of equation

$$\begin{aligned}\left(2^{2k}\right)a+\left(2^{k-1}\times(2^k+1)\right)b+\left(2^{k-1}\times(2^k-1)\right)c &= 0\\ (2^k+1)a+\left(2^{k-1}+2^{2k-1}+1\right)b+\left(2^{k-1}\times(2^k-1)\right)c &= 0\\ (2^k+1)a+\left(2^{k-1}\times(2^k+1)\right)b+\left(3\times 2^{k-1}+2^{2k-1}+1\right)c &= 0.\end{aligned}$$

Therefore, the remaining three eigenvalues of $Q(\overline{\Gamma_{cent}(PSL(2,2^k))})$ are the eigenvalues of the matrix $\mathfrak{L}_P$. □

Hence, by Theorem 4.2, $\overline{\Gamma_{cent}(PSL(2,2^k))}$ is signless Laplacian integral if $\mathfrak{L}_P$ have integral spectrum.

5. Conclusion

In this article, we have investigated the adjacency, (signless) Laplacian spectra of centralizer and co-centralizer graphs of the generalized quaternion group, the dihedral group, the quasidihedral group, the metacyclic group, and the projective special linear group. We also obtained conditions under which these graphs will be adjacency, (signless) Laplacian integral and we have demonstrated how the integrality phenomena of these graphs either align with or differ from those of the commuting and non-commuting graphs of the corresponding groups.

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