Electronic Journal of Graph Theory and Applications **12** (1) (2024), 17–23

Electronic Journal of
Graph Theory and Applications

A refined Turán theorem

Slobodan Filipovski

Faculty of Mathematics, Natural Sciences and Information Technologies
University of Primorska, Glagoljaška 8
6000 Koper, Slovenia

slobodan.filipovski@famnit.upr.si

Abstract

Let $G = (V, E)$ be a finite undirected graph with vertex set $V(G)$ of order $|V(G)| = n$ and edge set $E(G)$ of size $|E(G)| = m$. Let $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$ be the degree sequence of the graph G. A *clique* in a graph G is a complete subgraph of G. The *clique number* of a graph G, denoted by $\omega(G)$, is the order of a maximum clique of G. In 1907 Mantel proved that a triangle-free graph with n vertices can contain at most $\lfloor \frac{n^2}{4} \rfloor$ edges. In 1941 Turán generalized Mantel's result to graphs not containing cliques of size r by proving that graphs of order n that contain no induced K_r have at most $(1 - \frac{1}{r-1})\frac{n^2}{2}$ edges. In this paper, we give new bounds for the maximum number of edges in a K_r-free graph G of order n, minimum degree δ, and maximum degree Δ. We show that, for the families of graphs having the above properties, our bounds are slightly better than the more general bounds of Turán.

Keywords: clique, extremal graphs, Turán theorem, inequality
Mathematics Subject Classification : 05C35
DOI: 10.5614/ejgta.2024.12.1.2

1. Introduction

Let $G = (V, E)$ be a finite undirected graph with vertex set $V(G)$ of order $|V(G)| = n$ and edge set $E(G)$ of size $|E(G)| = m$. Let $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$ be the degree sequence of the graph G. A *clique* in a graph G is a complete subgraph of G. The *clique number* of a graph G, denoted by $\omega(G)$, is the order of a maximum clique of G.

Received: 21 February 2023, Revised: 24 October 2023, Accepted: 26 November 2023.

In extremal graph theory one seeks extremal values of graph parameters for graphs belonging to families with prescribed properties. One of the classical problems of extremal graph theory involves showing that a graph with sufficiently many edges must contain a specific substructure. In this sense, an *extremal graph* with respect to a family of graphs $\{H_i\}_{i\in\mathcal{I}}$ is a graph G of order n and *maximum size* m among all graphs of order n that do not contain any of the graphs $\{H_i\}_{i\in\mathcal{I}}$ as induced subgraphs. The well-known Turán graphs $T(n,r)$ constitute examples of just this type of extremal graphs. Namely, the Turán graph $T(n,r)$ has the maximum possible number of edges among all graphs on n vertices which do not contain an induced $(r+1)$-clique. If $r=2$, the Turán graph $T(n,2)$ is an undirected graph in which no three vertices form a triangle of edges; a *triangle-free graph*. One of the first results concerning extremal triangle-free graphs is due to Mantel:

Theorem 1.1 ([6]). *If G is a triangle-free graph, then $m \le \lfloor \frac{n^2}{4} \rfloor$. Furthermore, the only triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Turán's generalization of Mantel's result to cliques of size r is perhaps the best known theorem of extremal graph theory:

Theorem 1.2 ([8]). *If a graph G on n vertices contains no copy of K_r, then it contains at most $\left(1-\frac{1}{r-1}\right)\frac{n^2}{2}$ edges.*

In this paper we consider the size of the K_r-free graphs G. Since $\omega(G) \le r-1$, we focus on the lower bounds of the clique number of the graph. Myers and Liu in [7] proved that for a graph G with n vertices and m edges it holds

$$\omega(G) \ge \frac{n^2}{n^2-2m}. \tag{1}$$

Edwards and Elphick in [2] have improved this bound by proving that for a graph G with n vertices and degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$ we have

$$\omega(G) \ge \frac{n^2}{n-\sqrt{\frac{\sum_{i=1}^{n} d_i^2}{n}}}. \tag{2}$$

Independently, Caro [1] and Wei [9] showed that

$$\omega(G) \ge \frac{1}{n-d_1}+\frac{1}{n-d_2}+\cdots+\frac{1}{n-d_n}. \tag{3}$$

Based on the above lower bounds of the clique number, we derive new bounds on the maximum number of edges in K_r-free graphs in terms of the order n and maximum and minimum degrees. We show that the upper bounds on the maximum sizes of K_r-free graphs of specified maximum or minimum degrees given by our theorems are smaller than or equal to the upper bounds on the maximum sizes among all K_r-free graphs (not restricted to specific minimum and maximum degrees) derived by Turán.

2. New upper bounds for the size of K_r-free graphs

Let G be a K_r-free graph. Between the clique number of G and r there exist a natural relation, that is, $r-1 \geq \omega(G)$. Combining this inequality with the well-known bound $\omega(G) \geq \frac{n^2}{n^2-2m}$ we easily prove the Turán theorem. We notice that an improvement of the lower bounds of the clique number could improve the Turán theorem. We start with an improvement of the inequality between the arithmetic and the geometric mean, published in [3].

Lemma 2.1. *[3] If x and y are strictly positive real numbers, then*

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}} \geq 2+\frac{(x-y)^2}{2(x^2+y^2)}.$$

Proposition 2.1. *Let G be a graph with n vertices and m edges. If Δ is the maximum degree, δ is the minimum degree of G and $\omega(G)$ is the clique number of the graph G, then*

$$\omega(G) \geq \frac{\left(n+\frac{(\Delta-\delta)^2}{2((n-\Delta)^2+(n-\delta)^2)}\right)^2}{n^2-2m}. \tag{4}$$

Proof. Let $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$ be the vertex-degree sequence of G. From the Cauchy-Schwarz inequality we have

$$\omega(G)(n^2-2m) \geq \left(\frac{1}{n-d_1}+\frac{1}{n-d_2}+\cdots+\frac{1}{n-d_n}\right)(n^2-2m) =$$

$$= \left(\frac{1}{n-d_1}+\frac{1}{n-d_2}+\cdots+\frac{1}{n-d_n}\right)((n-d_n)+(n-d_2)+\cdots+(n-d_1)) \geq$$

$$\geq \left(\sqrt{\frac{n-d_n}{n-d_1}}+1+\cdots+1+\sqrt{\frac{n-d_1}{n-d_n}}\right)^2 = \left(n-2+\sqrt{\frac{n-d_n}{n-d_1}}+\sqrt{\frac{n-d_1}{n-d_n}}\right)^2.$$

The inequality in (4) follows from Lemma 2.1, setting $x = n-\Delta$ and $y = n-\delta$. □

Remark 2.1. Since $\frac{(\Delta-\delta)^2}{2((n-\Delta)^2+(n-\delta)^2)} > 0$ we get that the bound in (4) is better than the existing bound given by Myers and Liu [7].

Theorem 2.1. *If G is a graph of order n, with maximum degree Δ, minimum degree δ and if G contains no copy of K_r, then the size m of G is at most*

$$m \leq \frac{n^2}{2}\left(1-\frac{\left(1+\frac{(\Delta-\delta)^2}{2n((n-\Delta)^2+(n-\delta)^2)}\right)^2}{(r-1)}\right). \tag{5}$$

Proof. The proof follows directly from $r-1 \geq \omega(G)$ and the inequality in (4). □

Remark 2.2. It is clear that the bound in the above theorem is slightly better than the bound stated in the Turán theorem. Let us observe that, if the gap between Δ and δ is bigger, then the difference between our and the exiting bound is more significant. For example, if $\Delta = n-1$ and $\delta = 1$ we get $m \leq \frac{n^2}{2}\left(1 - \frac{C^2}{r-1}\right)$, where $C = \frac{2n^3-3n^2+4}{2n^3-4n^2+4n} > 1$.

Next, we use the lower bound for the clique number given by Edwards and Elphick, [2], and the following refinement of the inequality between quadratic and arithmetic means published in [3].

Proposition 2.2. *[3] If $a_1, \ldots, a_n$ are n positive real numbers such that $a_1^2 + \cdots + a_n^2 \neq 0$, then*

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n} + \frac{1}{4n} \cdot \sum_{i=1}^{n} \frac{(na_i^2 - (a_1^2 + \cdots + a_n^2))^2}{n^2 a_i^4 + (a_1^2 + \cdots + a_n^2)^2} a_i.$$

Let σ^2 be the variance of the numbers $d_1^2, d_2^2, \ldots, d_n^2$. From the above inequality we obtain the following result.

Theorem 2.2. *Let G be a connected graph on n vertices, with maximum degree Δ, minimum degree δ and let G contain no copy of K_r. If σ^2 is the variance of the squares of the degrees of G, then*

$$m \leq \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) - \frac{\delta n}{16\Delta^4} \cdot \sigma^2. \tag{6}$$

Proof. Setting $a_i = d_i$ in Proposition 2.2 we get

$$\sqrt{\frac{M_1}{n}} = \sqrt{\frac{d_1^2 + \cdots + d_n^2}{n}} \geq \frac{2m}{n} + \frac{\delta}{4n} \cdot \sum_{i=1}^{n} \frac{(d_i^2 - \frac{M_1}{n})^2}{a_i^4 + \left(\frac{M_1}{n}\right)^2}. \tag{7}$$

From $d_i^4 + \left(\frac{M_1}{n}\right)^2 \leq 2\Delta^4$ for each $i = 1, 2, \ldots, n$ and from $\sigma^2 = \frac{\sum_{i=1}^{n}\left(d_i^2 - \frac{M_1}{n}\right)^2}{n}$ we get the inequality $\sqrt{\frac{M_1}{n}} \geq \frac{2m}{n} + \frac{\delta}{8\Delta^4}\sigma^2$. Now, the desired inequality follows directly from (2).

□

Remark 2.3. From the bound in (6) we can notice that among all graphs on a fixed number of vertices, fixed r, with fixed maximum and minimum degree, the graphs whose degrees are spread further away from the mean, have larger variance σ^2, thus have smaller size m. In other words, K_r-free graphs with larger irregularity have smaller size.

Li et al. [4, 5] introduced the generalized version of the first Zagreb index, defined as

$$Z_p(G) = M_1^p(G) = d_1^p + d_2^p + \cdots + d_n^p$$

where p is a real number. This graph invariant is nowadays known under the name *general first Zagreb index*, and has also been much investigated. The lower bound for the clique number of G could be expressed in terms of the general first Zagreb indices.

Proposition 2.3. *Let G be a graph on n vertices and let Z_i be the general first Zagreb index. Then*

$$\omega(G) \geq 1 + \sum_{i=1}^{\infty} \frac{Z_i}{n^{i+1}}.$$

Proof. We have

$$\omega(G) \geq \frac{1}{n-d_1} + \cdots + \frac{1}{n-d_n} = \frac{1}{n(1-\frac{d_1}{n})} + \cdots + \frac{1}{n(1-\frac{d_n}{n})} =$$

$$= \frac{1}{n}\left(\sum_{i=0}^{\infty}\left(\frac{d_1}{n}\right)^i + \cdots + \sum_{i=0}^{\infty}\left(\frac{d_n}{n}\right)^i\right) =$$

$$= \frac{1}{n}\left(n + \sum_{i=1}^{\infty}\frac{Z_i}{n^i}\right) = 1 + \sum_{i=1}^{\infty}\frac{Z_i}{n^{i+1}}.$$

□

Remark 2.4. From the power mean inequality we obtain $\sqrt[k]{\frac{d_1^k+\cdots+d_n^k}{n}} \geq \frac{d_1+\cdots+d_n}{n} = \frac{2m}{n}$, that is, $Z_i \geq \frac{(2m)^i}{n^{i-1}}$. Therefore $\omega(G) \geq 1 + \sum_{i=1}^{\infty}\left(\frac{2m}{n^2}\right)^i = \frac{n^2}{n^2-2m}$, which is a well-known lower bound for $\omega(G)$.

Theorem 2.3. *If G is a graph of order n and maximum degree Δ, and G contains no copy of K_r, the size m of G is at most*

$$\left\lfloor \frac{n^2-n+\Delta}{2} - \frac{(n-1)^2(n-\Delta)}{2((r-1)(n-\Delta)-1)} \right\rfloor. \tag{8}$$

Proof. Since G does not contain a copy of K_r, then $\omega(G) \leq r-1$. Thus, applying (3) and the inequality between arithmetic and harmonic means, we get

$$r-1 \geq \frac{1}{n-\Delta} + \frac{(n-1)^2}{n^2-n-2m+\Delta}$$

which is equivalent to

$$n^2-n-2m+\Delta \geq \frac{(n-1)^2(n-\Delta)}{(r-1)(n-\Delta)-1}.$$

This last inequality implies the desired bound (8). □

We prove next that the bound (8) from Theorem 2.3 applied to a graph with a specific Δ is better than the more general bound of Turán, that is, we show that

$$\frac{n^2}{2}\left(1-\frac{1}{r-1}\right) \geq \frac{n^2-n+\Delta}{2} - \frac{(n-1)^2(n-\Delta)}{2((r-1)(n-\Delta)-1)}, \tag{9}$$

for all $1 \leq \Delta \leq n-1$.

The inequality (9) is equivalent to the inequality

$$\frac{(n-\Delta)((n-1)^2+(r-1)(n-\Delta)-1)}{(r-1)(n-\Delta)-1} \geq \frac{n^2}{r-1}. \tag{10}$$

Taking $S=(r-1)(n-\Delta)-1$ and $r-1=\frac{S+1}{n-\Delta}$, the inequality (10) becomes

$$\frac{(n-1)^2+S}{S} \geq \frac{n^2}{S+1}.$$

The last inequality is equivalent to $(n-1)^2-2S(n-1)+S^2 \geq 0$, that is, $(n-1-S)^2 \geq 0$, which is true. Note that our bound matches Turán's bound if $n-1=S$, that is, if $r-1=\frac{n}{n-\Delta}$.

The proof of our next theorem is analogous to the proof of Theorem 2.3.

Theorem 2.4. *If G is a graph of order n and minimum degree δ, and G contains no copy of K_r, the size m of G is at most*

$$\left\lfloor \frac{n^2-n+\delta}{2} - \frac{(n-1)^2(n-\delta)}{2((r-1)(n-\delta)-1)} \right\rfloor. \tag{11}$$

Theorems 2.3 and 2.4 yield the following obvious corollary.

Corollary 2.1. *The maximum number of edges in a graph G that contains no copy of K_r, having n vertices, maximum degree Δ, and minimum degree δ, is bounded from above by*

$$\min\left\{ \left\lfloor \frac{n^2-n+\Delta}{2} - \frac{(n-1)^2(n-\Delta)}{2((r-1)(n-\Delta)-1)} \right\rfloor, \left\lfloor \frac{n^2-n+\delta}{2} - \frac{(n-1)^2(n-\delta)}{2((r-1)(n-\delta)-1)} \right\rfloor \right\}.$$

Acknowledgement

This work is supported in part by the Slovenian Research Agency (research program P1-0285 and research projects N1-0210, J1-3001, J1-3003, J1-4414 and BI-VB/23-25-002).

References

[1] Y. Caro, *New Results on the Independence Number*, Technical report, Tel Aviv University, (1979).

[2] C.S. Edwards and C.H. Elphick, Lower bounds for the clique and the chromatic numbers of a graph, *Discret. Appl. Math.,* **5** (1983), 51–64.

[3] S. Filipovski, Improved Cauchy-Schwarz inequality and its applications, *Turkish J. Ineq*. **3(2)** (2019), 8–12.

[4] X. Li and H. Zhao, Trees with the first three smallest and largest generalized topological indices, *MATCH Commun. Math. Comput. Chem.* **50** (2004), 57–62.

[5] X. Li and J. Zheng, A unified approach to the extremal trees for different indices, *MATCH Commun. Math. Comput. Chem.* **54** (2005), 195–208.

[6] W. Mantel, Solution to Problem 28, by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh, and W.A. Wythoff, *Wiskundige Opgaven* **10** (1907), 60–61.

[7] B.R. Myers and R. Liu, Alower bound on the chromatic number of a graph, *Networks* **1** (1972), 273–277.

[8] P. Turán, On an extremal problem in graph theory, *Mat. Lapok* **48** (1941), 436–452 (in Hungarian).

[9] V.K. Wei, *A Lower Bound on the Stability Number of a Simple Graph*, Technical Report 81–11217–9, Bell Laboratories, (1981).