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Doubly resolving number of the corona product graphs

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Abstract

Two vertices u, v in a connected graph G are doubly resolved by vertices x, y of G if

$$d(v, x) - d(u, x) \neq d(v, y) - d(u, y).$$

A set W of vertices of the graph G is a doubly resolving set for G if every two distinct vertices of G are doubly resolved by some two vertices of W. Doubly resolving number of a graph G, denoted by $\psi(G)$, is the minimum cardinality of a doubly resolving set for G. In this paper, using adjacency resolving sets and dominating sets of graphs, we study doubly resolving sets in the corona product of graphs G and H, $G \odot H$. First, we obtain the upper and lower bounds for the doubly resolving number of the corona product $G \odot H$ in terms of the order of G and the adjacency dimension of H, then we present several conditions that make each of these bounds feasible for the doubly resolving number of $G \odot H$. Also, for some important families of graphs, we obtain the exact value of the doubly resolving number of the corona product.

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1. Introduction

Throughout this paper all graphs are simple, finite and undirected. The vertex set of a graph G is denoted by $V(G)$. We use $\overline{G}$ for the complement of the graph G. In a connected graph G,

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the distance between two vertices u and v, denoted by $d_G(u,v)$, is the length of a shortest path between u and v in G. We write it simply $d(u,v)$, when no confusion can arise. The diameter of G, denoted by $\mathrm{diam(G)}$ is $\max\{d(u,v)\colon u,v \in V\}$. If G has a cycle, then the length of a shortest cycle in G is called the girth of G and denoted by $\mathrm{girth(G)}$. The degree of a vertex v, $\deg(v)$ is the number of its neighbours. A leaf in a graph is a vertex of degree 1. A dominating set for a graph G is a subset D of $V(G)$ such that every vertex in $V(G)\setminus D$ is adjacent to at least one vertex in D. We use the notations P_n and C_n for a path of order n and a cycle of order n, respectively. The *join* of graphs G and H, denoted by $G+H$, is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)\cup\{uv \mid u\in V(G), v\in V(H)\}$. The *wheel graph* is $W_n = K_1 + C_n$ and the *fan graph* is $F_n = K_1 + P_n$.

For an ordered subset $W=\{w_1,\ldots,w_k\}$ of $V(G)$ and a vertex v of a connected graph G, the *metric representation* of v with respect to W is $r(v|W)=(d(v,w_1),\ldots,d(v,w_k))$. The set W is a *resolving set* for G if the distinct vertices of G have different metric representations, with respect to W. A resolving set W for G with minimum cardinality is a *metric basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\dim(G)$.

During the study of the metric dimension of the cartesian product of graphs Cáceres et al. [3] defined the concept of *doubly resolving sets* in graphs. Two vertices u,v in a connected graph G are doubly resolved by $x,y\in V(G)$ if

$$d(v,x)-d(u,x)\neq d(v,y)-d(u,y).$$

A set W of vertices of the graph G is a doubly resolving set for G if every two distinct vertices of G are doubly resolved by some two vertices of W. Every graph with at least two vertices has a doubly resolving set. A doubly resolving set for G with minimum cardinality is called a *doubly basis* of G and its cardinality is called the *doubly resolving number* of G and denoted by $\psi(G)$. Note that if x,y doubly resolves u,v then $d(u,x)-d(v,x)\neq 0$ or $d(u,y)-d(v,y)\neq 0$, and at least one of x and y resolves u,v. Hence a doubly resolving set is also a resolving set and $\dim(G)\le\psi(G)$.

Cáceres et al. [3] obtained doubly resolving number of trees, cycles and complete graphs. In [11] it was proved that the problem of finding doubly bases is NP-hard. Doubly resolving number of Prism graphs and Hamming graphs are computed in [4] and [12], respectively. For more results about doubly resolving sets in graphs see[3, 8, 11, 13].

Doubly resolving sets play an essential role in obtaining the metric dimension of the Cartesian product of graphs. This concept was introduced by Cáceres and his colleagues during the study of the metric dimension of the Cartesian product of graphs. They proved that the metric dimension of the Cartesian product of two graphs is at most one unit less than the sum of the metric dimension of one of them and the doubly resolving number of the other. After that, these concepts were noticed by many people.

One of the fields of work regarding each parameter in graph theory is to obtain that parameter for different products of graphs. One of the products that has been studied a lot recently is the corona product of two graphs. The *corona product*, $G\odot H$ of graphs G and H is obtained by taking one copy of G and $n(G)$ copy of H, and by joining each vertex of the i-th copy of H to the i-th vertex of G, $1\le i\le n(G)$. It is clear that if G is a connected graph then $G\odot H$ is also connected. Many results about this product have been investigated for parameters related to resolving sets, including metric dimension [16], adjacency dimension [5], edge metric dimension [14], local metric dimension and local adjacency dimension of graphs [5].

Our goal in this paper is to study doubly resolving sets for the corona product of graphs. To achieve this goal, we use the concept of dominating sets in graphs. First, we prove that for a connected graph G of order n and a non-trivial graph H

$$n\dim_2(H) \leq \psi(G \odot H) \leq n(1 + \dim_2(H)).$$

then we find some properties for the graph H that leads us to satisfying each side of this inequality. Also, this inequality implies that $n \leq \psi(G \odot H) \leq nm$, where m is the order of H. All graphs H with $\psi(G \odot H) = n$, $\psi(G \odot H) = n(m-1)$ and $\psi(G \odot H) = nm$ are determined in this paper. Also, the exact value of $\psi(G \odot H)$ for some families of graphs is computed.

One of our important tools in this paper is the concept of adjacency resolving set, which is defined by Jannesari and Omoomi [10]. In the end of this section, we present the definition of this concept. Let G be a graph and $W = \{w_1, \ldots, w_k\} \subseteq V(G)$. For each vertex $v \in V(G)$, the *adjacency representation* of v with respect to W is the k-vector $r_2(v|W) = (a_G(v, w_1), \ldots, a_G(v, w_k))$, where $a_G(v, w_i) = min\{2, d_G(v, w_i)\}; 1 \leq i \leq k$. The set W is an *adjacency resolving set* for G if the vectors $r_2(v|W)$ for $v \in V(G)$ are distinct. The minimum cardinality of an adjacency resolving set is the *adjacency dimension* of G, denoted by $\dim_2(G)$. An adjacency resolving set of cardinality $\dim_2(G)$ is an *adjacency basis* of G.

2. Preliminaries

In this section we present some known or primary results that are necessary for our us to get the main results.

Observation 2.1. *To determine whether a given set W is a (an adjacency) resolving set for G, it is sufficient to look at the (adjacency) metric representations of vertices in $V(G)\backslash W$, because $w \in W$ is the unique vertex of G for which $d_G(w, w) = 0 (a_G(w, w) = 0)$.*

Proposition 2.2. [10] For every graph G, $\dim_2(G) = \dim_2(\overline{G})$.

Let G be a graph of order n. It is easy to see that, $1 \leq \dim_2(G) \leq n - 1$. In the following proposition, all graphs G with $\dim_2(G) = 1$ and all graphs of order n and adjacency dimension $n - 1$ are characterized.

Proposition 2.3. [10] If G is a graph of order n, then

(i) $\dim_2(G) = 1$ if and only if $G \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$.

(ii) $\dim_2(G) = n - 1$ if and only if $G = K_n$ or $G = \overline{K}_n$.

All graphs of order n and adjacency dimension $n-2$ are characterized in the following theorem.

Theorem 2.4. [9] Let G be a graph of order n. Then $\dim_2(G) = n - 2$ if and only if G or $\overline{G}$ is one of the graphs P_4, $K_{s,t}$ $(s, t \geq 1)$, $K_s + \overline{K}_t$ $(s \geq 1, t \geq 2)$, or $K_s + (K_t \cup K_1)$ $(s, t \geq 1)$.

Metric dimension of $G \odot H$ has a closed relation with order of G and adjacency dimension of H.

Theorem 2.5. [5] For any connected graph G of order n and any non-trivial graph H,

$$\dim(G \odot H) = n \dim_2(H).$$

Lemma 2.6. [8] Let v be a leaf in a connected graph G. Then v belongs to all doubly bases of G, and $\psi(G)$ is bigger than or equal to the number of leaves in G.

By the following proposition and theorem, If G is a path or a cycle then $\dim_2(G) = \dim G + K_1$.

Proposition 2.7. [10] If $n \geq 4$, then $\dim_2(C_n) = \dim_2(P_n) = \lfloor \frac{2n+2}{5} \rfloor$.

Theorem 2.8. [1, 2]

(i) If $n \notin \{3, 6\}$, then $\dim(C_n + K_1) = \lfloor \frac{2n+2}{5} \rfloor$,

(ii) If $n \notin \{1, 2, 3, 6\}$, then $\dim(P_n + K_1) = \lfloor \frac{2n+2}{5} \rfloor$.

Clearly, every doubly resolving set is also a resolving set. In the next proposition we consider resolving sets in $G \odot H$ that are not doubly resolving set.

Proposition 2.9. *Let W be a resolving set for $G \odot H$. If $x, y \in V(G \odot H)$ are not doubly resolved by any pair of vertices in W, then exactly one of them belongs to $V(G)$ and they are adjacent.*

Proof. We consider the following five cases for x and y.

Case 1: $x, y \in V(G)$. Let $x = v_i, y = v_j$ for some $i, j; 1 \leq i, j \leq n$. By Lemma 3.2, there exist vertices $u_i \in W \cap H_i$ and $u_j \in W \cap H_j$. Thus

$$d(x, u_i) - d(y, u_i) = 1 - (1 + d(x, y)) = -d(x, y) \neq d(x, y) = 1 + d(x, y) - 1 = d(x, u_j) - d(y, u_j).$$

Therefore x, y are doubly resolved by u_i and u_j, which is impossible.

Case 2: $x = v_i \in V(G)$ and $y \in V(H_j)$ for some $j \neq i$. By Lemma 3.2, there exist vertices $u_i \in W \cap H_i$ and $u_j \in W \cap H_j$. Hence

$$d(x, u_i) - d(y, u_i) = 1 - (1 + d(v_j, u_i)) = 1 - (2 + d(v_j, v_i)) = -(1 + d(v_j, v_i)) \leq -2$$

and

$$d(x, u_j) - d(y, u_j) = 1 + d(x, v_j) - d(y, u_j) \geq 2 - d(y, u_j) \geq 0.$$

Therefore x, y are doubly resolved by u_i and u_j, a contradiction.

Case 3: $x \in V(H_i), y \in V(H_j)$ for some $i \neq j$. By Lemma 3.2, there exist vertices $u_i \in W \cap H_i$ and $u_j \in W \cap H_j$. Thus

$$d(x,u_i) - d(y,u_i) = d(x,u_i) - (1 + d(v_j,u_i)) = d(x,u_i) - (2 + d(v_j,v_i)) \leq -1,$$

because $d(x,u_i) = a_H(x,u_i) \leq 2$ and $i \neq j$. On the other hand

$$d(x,u_j) - d(y,u_j) = 1 + d(v_i,u_j) - d(y,u_j) \geq 2 + d(v_i,v_j) - d(y,u_j) \geq 1.$$

because $d(y,u_j) = a_H(y,u_j) \leq 2$ and $i \neq j$. Therefore x,y are doubly resolved by u_i and u_j. That is impossible.

Case 4: $x,y \in V(H_i)$, for some i. Let $j \neq i$, by Lemma 3.2, there exist vertices $u_j \in W \cap H_j$ and $u_i \in W \cap H_i$ such that $a_H(x,u_i) \neq a_H(y,u_i)$. Hence

$$d(x,u_i)-d(y,u_i) = a_H(x,u_i)-a_H(y,u_i) \neq 0 = d(x,v_j)+1-(d(y,v_j)+1) = d(x,u_j)-d(y,u_j).$$

Therefore x,y are doubly resolved by u_i and u_j, which is impossible.

These contradictions imply that the following case is the only case that can happen.

Case 5: $x = v_i$ and $y \in V(H_i)$, for some i. Clearly $x \in V(G)$ and $y \notin V(G)$ and x,y are adjacent. □

3. Doubly resolving sets in $G \odot H$

Throughout this section G is a connected graph of order n and H is an arbitrary graph of order m. For convenience let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and H_i be the i-th copy of H in $G \odot H$, i.e. all vertices of H_i are joined to v_i in the graph $G \odot H$. When H is the trivial graph K_1, the doubly resolving number of $G \odot H$ is equal to the order of G.

Theorem 3.1. *For every connected graph G of order $n \geq 2$, $\psi(G \odot K_1) = n$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\{u_i\}$ be the vertex of the i-th copy of K_1 in $G \odot K_1$. Clearly every $u_i; 1 \leq i \leq n$, is a leaf in $G \odot K_1$ and by Lemma 2.6, each u_i belongs to every doubly basis of $G \odot K_1$. That means $\psi(G \odot K_1) \geq n$. Now let $W = \{u_1, u_2, \ldots, u_n\}$. We prove that W is a doubly resolving set for $G \odot K_1$. For every $i,j; 1 \leq i \neq j \leq n$ we have

$$d(u_i,u_i) - d(u_j,u_i) = -d(u_j,u_i) \neq d(u_i,u_j) - d(u_j,u_j),$$

$$d(v_i,u_i) - d(v_j,u_i) = -d(v_j,v_i) \neq d(v_j,v_i) = d(v_i,u_j) - d(v_j,u_j).$$

Also, for every $i,j; 1 \leq i,j \leq n$,

$$d(v_i,u_i) - d(u_j,u_i) = 1 - d(u_j,u_i) = -d(v_i,u_j) \neq d(v_i,u_j) = d(v_i,u_j) - d(u_j,u_j).$$

Therefore W is a doubly resolving set for $G \odot K_1$ and $\psi(G \odot K_1) = n$. □

Now we consider $\psi(G \odot H)$ for graphs G and H of order at least 2.

Lemma 3.2. *If W is a resolving set for $G \odot H$, then $W \cap V(H_i); 1 \leq i \leq n$, contains an adjacency resolving set for H_i.*

Proof. Let $x, y \in V(H_i)$. Clearly for every vertex $v \in V(G \odot H) \setminus V(H_i)$ we have $d(x, v) = d(y, v)$. Therefore W must contain a vertex of H_i to resolve x, y. But $d_{G\odot H}(x, y) = a_{H_i}(x, y)$. Hence W contains an adjacency resolving set for H_i. □

Through the following two theorems we will prove that $\psi(G \odot H)$ is $n \dim_2(H)$ or $n(1 + \dim_2(H))$, where n is the order of G.

Theorem 3.3. *Let H have an adjacency basis which is also a dominating set. Then for every connected graph G of order n*

$$\psi(G \odot H) = n \dim_2(H).$$

Proof. Since every doubly resolving set is a resolving set, by Theorem 2.5, we have $\psi(G \odot H) \geq n \dim_2(H)$.

For every $i; 1 \leq i \leq n$, let B_i be an adjacency basis of H_i which is also a dominating set. Let $W = \bigcup_{i=1}^{n} B_i$. By Theorem 2.5 and Lemma 3.2, W is a basis of $G \odot H$. If there exist vertices $x, y \in V(G \odot H)$ that are not doubly resolved by W, then by Proposition 2.9, exactly one of them belongs to $V(G)$ and they are adjacent. Therefor by symmetry, there exists $i; 1 \leq i \leq n$, such that $x = v_i$ and $y \in V(H_i)$. Since B_i is a dominating set for H_i, there exits a vertex $u_i \in B_i = W \cap H_i$ such that $d(y, u_i) \leq 1$. Clearly u_i is adjacent to $x = v_i$. Let $j \neq i$ and $u_j \in W \cap H_j$. Hence

$$d(x, u_i) - d(y, u_i) = 1 - d(y, u_i) \geq 0 > -1 = d(x, u_j) - (1 + d(x, u_j)) = d(x, u_j) - d(y, u_j).$$

Therefore W is a doubly resolving set for $G \odot H$ with cardinality $n \dim_2(H)$. □

Theorem 3.4. *If no adjacency basis of H is a dominating set, then for every connected graph G of order n*

$$\psi(G \odot H) = n(1 + \dim_2(H)).$$

Proof. Since every doubly resolving set is a resolving set, Lemma 3.2 implies that every doubly resolving set for $G \odot H$ contains an adjacency resolving set of each $H_i; 1 \leq i \leq n$. But by the assumption every adjacency basis B_i of H_i is not a dominating set and so there is a vertex $t_i \in V(H_i) \setminus B_i$ that is not adjacent to any vertex of B_i. Hence for each $u_i \in B_i$ we have $d(t_i, u_i) - d(v_i, u_i) = 1$. Moreover, for all vertices $v \in V(G \odot H) \setminus V(H_i)$, we have $d(t_i, v) - d(v_i, v) = 1$. Therefore every doubly resolving set for $G \odot H$ contains at least $\dim_2(H) + 1$ vertices from each H_i. This means $\psi(G \odot H) \geq n(1 + \dim_2(H))$.

Now we obtain a doubly resolving set for $G \odot H$ with cardinality $n(1 + \dim_2(H))$. For each i, let B_i be an adjacency basis of H_i and $t_i \in V(H_i) \setminus B_i$ be the vertex that does not have any adjacent in B_i. Since B_i is an adjacency basis, t_i is unique. Thus $B_i \cup \{t_i\}$ is a dominating set for H_i. Set $W = \bigcup_{i=1}^{n}(B_i \cup \{t_i\})$. By Theorem 2.5 and Lemma 3.2, W is a resolving set for $G \odot H$. If there exist vertices $x, y \in V(G \odot H)$ that are not doubly resolved by W, then by Proposition 2.9, exactly one of them belongs to $V(G)$ and they are adjacent. Therefore by symmetry, there exists

i, such that $x = v_i$ and $y \in V(H_i)$. Since $B_i \cup \{t_i\}$ is a dominating set for H_i, there exits a vertex $u_i \in B_i \cup \{t_i\} = W \cap H_i$ such that $d(y, u_i) \leq 1$. Clearly u_i is adjacent to $x = v_i$. Let $j \neq i$ and $u_j \in W \cap H_j$. Hence

$$d(x,u_i) - d(y,u_i) = 1 - d(y,u_i) \geq 0 > -1 = d(x,u_j) - (1 + d(x,u_j)) = d(x,u_j) - d(y,u_j).$$

Therefore W is a doubly resolving set for $G \odot H$ with cardinality $n(1 + \dim_2(H))$. □

Theorems 3.3 and 3.4 imply the following corollary.

Corollary 3.5. *Let G be a connected graph of order $n \geq 2$ and H be a non-trivial graph. Then*

$$n \dim_2(H) \leq \psi(G \odot H) \leq n(1 + \dim_2(H)).$$

To find that $\psi(G \odot H)$ is which one of $n \dim_2(H)$ or $n(1 + \dim_2(H)$, we need to know that is there an adjacency basis for H that is also a dominating set. In the following, we investigate the conditions that provide this property.

Definition 3.6. *Let W be a subset of $V(G)$, a vertex $v \in V(G) \setminus W$ is called a dominant vertex for W if v is adjacent to all vertices of W.*

Clearly an adjacency basis B for H is a dominating set for H if and only if there is no dominant vertex for B in $\overline{H}$. Therefore we have the following corollary.

Corollary 3.7. *Let G be a connected graph of order $n \geq 2$ and H be a non-trivial graph.*

(i) If H has an adjacency basis B such that there is not any dominant vertex for B in $V(H) \setminus B$, then $\psi(G \odot \overline{H}) = n \dim_2(H)$.

(ii) If for every adjacency basis of H there exists a dominant vertex in $V(H)$, then $\psi(G \odot \overline{H}) = n(\dim_2(H) + 1)$.

Clearly every adjacency basis of $K_{r,s}; r, s \geq 2$, is a dominating set and there is no dominant vertex for any adjacency basis of $K_{r,s}$. Also note that every adjacency basis of $K_m; m \geq 2$, is a dominating set and there is a dominant vertex for every adjacency basis of K_m. Thus we have the following observation.

Observation 3.8. *Let G be a connected graph of order $n \geq 2$.*

(i) For every $r, s \geq 2$ we have $\psi(G \odot K_{r,s}) = \psi(G \odot \overline{K_{r,s}}) = n(r + s - 2)$.

(ii) For every $m \geq 2$ we have $\psi(G \odot K_m) = n(m - 1)$ and $\psi(G \odot \overline{K_m}) = nm$.

The next two propositions specify the conditions under which the path, P_m, and cycle, C_m, have an adjacency basis that is also a dominating set.

Proposition 3.9. *Let $m = 5k + r, k \geq 1$ and $0 \leq r \leq 4$. Then P_m has an adjacency basis that is a dominating set, if and only if r is even.*

Proof. We use induction on k.

Basis step: If $k = 1$, then $5 \leq n \leq 9$ and by checking each case, the claim is true.

Induction step: $k > 1$. Assume the claim for $k - 1$. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$, where v_i is adjacent to v_{i+1} for all $i; 1 \leq i \leq m-1$, and H be the induced subgraph $\langle v_6, v_7, \ldots, v_m \rangle$ of P_m. If r is even, then by the induction hypothesis, there exists an adjacency basis B for H which is a dominating set for H. By Proposition 2.7, $\dim_2(H) = \dim_2(P_m) - 2$. Now let $W = B \cup \{v_2, v_4\}$. Clearly W is a dominating set for P_m and $|W| = \dim_2(P_m)$. Note that $r_2(v_1|\{v_2, v_4\}) = (1, 2)$, $r_2(v_3|\{v_2, v_4\}) = (1, 1)$, $r_2(v_5|\{v_2, v_4\}) = (2, 1)$ and for every vertex v of H we have $r_2(v|\{v_2, v_4\}) = (2, 2)$. Therefore W is an adjacency basis of G that is also a dominating set.

If r is odd, suppose on the contrary that P_m has an adjacency basis B that is also a dominating set. Clearly $|B \cap \{v_1, v_2, \ldots, v_5\}| \geq 2$, otherwise the adjacency representations of some vertices in $\{v_1, v_2, \ldots, v_5\}$ with respect to B are the same. Since B is a dominating set, $B' = B \cap V(H)$ is an adjacency resolving set for H. By Proposition 2.7, $\dim_2(H) = \dim_2(P_m) - 2$, hence B' is an adjacency basis for H and $|B \cap \{v_1, v_2, \ldots, v_5\}| = 2$. By induction hypothesis B' is not a dominating set for H. Since B is a dominating set and B' is not a dominating set, v_6 has no neighbour in B'. That means $v_6, v_7 \notin B$ and $v_5 \in B$. Thus $|B \cap \{v_1, v_2, \ldots, v_4\}| = 1$. Since B is a dominating set we have $v_2 \in B$. Therefore $r_2(v_4|B) = r_2(v_6|B)$. This contradiction implies that B is not a dominating set for P_m. □

Proposition 3.10. *Let $m = 5k + r, k \geq 1$ and $0 \leq r \leq 4$. Then C_m has an adjacency basis that is a dominating set, if and only if r is even.*

Proof. Let $V(C_m) = \{v_1, v_2, \ldots, v_m\}$, where v_i is adjacent to v_{i+1} for all $i; 1 \leq i \leq m-1$, and v_1 is adjacent to v_m. First, let r be an even number. Suppose that H is the resulting graph from C_m by removing the edge between v_1 and v_m. Clearly H is a path on m vertices. Proposition 3.9 implies that H has an adjacency basis B that is also a dominating set for H. If $|B \cap \{v_1, v_n\}| \in \{0, 2\}$, then for every $v \in \{v_1, v_2, \ldots, v_m\} \setminus B$ the adjacency representations of v with respect to B in graphs C_m and H are the same. Hence B is an adjacency basis for C_m that is also a dominating set. If $|B \cap \{v_1, v_n\}| = 1$, say $v_n \in B$. If $r_2(v|B) = r_2(u|B)$ for some $u, v \in V(C_m) \setminus B$, then $\{u, v\} = \{v_1, v_{n-1}\}$. Since B is a dominating set for H, v_2 must be in B. Since $m \geq 5$, we have $a_{C_m}(v_1, v_2) \neq a_{C_m}(v_{n-1}, v_2)$. Hence $r_2(v_1|B) \neq r_2(v_{n-1}|B)$. Therefore B is an adjacency basis for C_m that is also a dominating set.

If r is odd, suppose on the contrary that C_m has an adjacency basis B that is also a dominating set. By Proposition 2.7, there exist two adjacent vertices $v_i, v_{i+1} \in V(C_m) \setminus B$, otherwise $\lfloor \frac{2m+2}{5} \rfloor = |B| \geq \lfloor \frac{m}{2} \rfloor > \lfloor \frac{2m+2}{5} \rfloor$. By removing the edge $v_i v_{i+1}$ we get a graph H that is a path on m vertices. Clearly B is an adjacency basis for H that is also a dominating set. This contradicts Proposition 3.9. □

Now we can obtain $\psi(G \odot H)$ when H or $\overline{H}$ is a path or a cycle.

Proposition 3.11. *Let G be a connected graph of order $n \geq 2$ and $H \in \{P_m, C_m\}$, $m = 5k + r$, where $k \geq 1$ and $0 \leq r \leq 4$.*

(i) If r is even, then $\psi(G \odot H) = \psi(G \odot \overline{H}) = n\lfloor \frac{2m+2}{5} \rfloor$.

(ii) If $m = 6$, then $\psi(G \odot H) = \psi(G \odot \overline{H}) = n(\lfloor \frac{2m+2}{5} \rfloor + 1)$.

(iii) If r is odd and $m \neq 6$, then $\psi(G \odot H) = n(\lfloor \frac{2m+2}{5} \rfloor + 1)$ and $\psi(G \odot \overline{H}) = n\lfloor \frac{2m+2}{5} \rfloor$.

Proof. (i) If r is even, then Propositions 3.9 and 3.10 imply that H has an adjacency basis that is also a dominating set. Hence, by Theorem 3.3 and Proposition 2.7, $\psi(G \odot H) = n\dim_2(H) = n\lfloor \frac{2m+2}{5} \rfloor$. If $m > 5$, then Proposition 2.7 yields $\dim_2(H) \geq 3$. Let B be an adjacency basis for H. Since the degree of each vertex of H is at most 2 there is no dominant vertex for B in H. Thus by Corollary 3.7 and Proposition 2.7 we have $\psi(G \odot \overline{H}) = n\dim_2(H) = n\lfloor \frac{2m+2}{5} \rfloor$. for $m = 5$ it is easy to see that there exists an adjacency basis for H such that there is no dominant vertex for it in H. Therefore by a same argument as in the case $m > 5$, we have $\psi(G \odot \overline{H}) = n\dim_2(H) = n\lfloor \frac{2m+2}{5} \rfloor$.

(ii) if $m = 6$, it is easy to see that every adjacency basis B of H is not a dominating set and there is a dominant vertex for B in H. Therefore Theorem 3.4 and Corollary 3.7 imply that $\psi(G \odot H) = \psi(G \odot \overline{H}) = n(\dim_2(H) + 1) = n(\lfloor \frac{2m+2}{5} \rfloor + 1)$.

(iii) If r is odd and $m \neq 6$, then Propositions 3.9 and 3.10 imply that no adjacency basis of H is a dominating set, and by Theorem 3.4 and Proposition 2.7, $\psi(G \odot H) = n(\dim_2(H) + 1) = n(\lfloor \frac{2m+2}{5} \rfloor + 1)$. On the other hand, by Proposition 2.7 we have $\dim_2(H) \geq 3$ and so there is no dominant vertex for B in H, because the degree of each vertex of H is at most 2. Thus Corollary 3.7 and Proposition 2.7 imply $\psi(G \odot \overline{H}) = n\dim_2(H) = n\lfloor \frac{2m+2}{5} \rfloor$. □

In the following, we investigate the relations between $\psi(G \odot H)$ and some parameters of the graph H or $\overline{H}$ such as maximum degree, minimum degree, diameter and girth. The first result is about maximum and minimum degree.

Lemma 3.12. *Let H be a non-trivial graph of order m, maximum degree $\Delta(H)$ and minimum degree $\delta(H)$.*

(i) If $\dim_2(H) > \Delta(H)$, then there is no dominant vertex for any adjacency basis of H.

(ii) If $\delta(H) \geq m - \dim_2(H)$, then every adjacency basis of H is a dominating set.

Proof. (i) Let $\dim_2(H) > \Delta(H)$ and B be an adjacency basis of H. If there is a dominant vertex x for B, then x is adjacent to all vertices in B. Therefore $\deg(x) \geq \dim_2(H) > \Delta(H)$, which is a contradiction.

(ii) Let $\delta(H) > m - \dim_2(H)$ and B be an adjacency basis of H. If B is not a dominating set for H, then there exists a vertex $x \in V(H) \setminus B$ such that x is not adjacent to any member of B. Therefore $\deg(x) \leq m - 1 - \dim_2(H) < \delta(H)$, which is impossible. □

Corollary 3.13. *Let G be a connected graph of order $m \geq 2$ and H be a non-trivial graph of order m, maximum degree $\Delta(H)$ and minimum degree $\delta(H)$.*

(i) If $\dim_2(H) > \Delta(H)$*, then* $\psi(G \odot \overline{H}) = n \dim_2(H)$*.*

(ii) If $\delta(H) \geq m - \dim_2(H)$*, then* $\psi(G \odot H) = n \dim_2(H)$*.*

The next proposition express some conditions on diameter of H those are enough for existence of an adjacency basis with no dominant vertex for it.

Proposition 3.14. *Let* H *be a graph such that for each adjacency basis of* H *there exists a dominant vertex.*

(i) If there exists an adjacency basis for H *that is also a dominating set, then* $\mathrm{diam(H)} \leq 4$*.*

(ii) If any adjacency basis of H *is not a dominating set, then* $\mathrm{diam(H)} \leq 5$*.*

Proof. (i) Suppose that B is an adjacency basis for H that is also a dominating set. Let b be a dominant vertex for B. Hence every two vertices of B are at distance at most 2. Since B is a dominating set, every vertex of $V(H) \setminus B$ has a neighbour in B. Therefor the distance between two of them is at most 4. Hence $\mathrm{diam(H)} \leq 4$.

(ii) Let B be an adjacency basis of H and b be a dominant vertex for B. Since B is not a dominating set, there exists a vertex $v \in V(H) \setminus B$ such that v is not adjacent to any vertex in B. But each neighbour of v has a neighbour in B, because B is an adjacency basis. Therefore $\mathrm{diam(H)} \leq 5$.

□

The following proposition specify some conditions on girth of H those are enough for existence of an adjacency basis with no dominant vertex for it.

Proposition 3.15. *Let* H *be a graph such that has a cycle and for each adjacency basis of* H *there exists a dominant vertex.*

(i) If there exists an adjacency basis for H *that is also a dominating set, then* $\mathrm{girth(H)} \leq 5$*.*

(ii) If any adjacency basis of H *is not a dominating set, then* $\mathrm{girth(H)} \leq 6$*.*

Proof. (i) Suppose that B is an adjacency basis for H that is also a dominating set. Let b be a dominant vertex for B. Hence every two vertices of B are at distance at most 2. If there exists an edge between two vertices in B, then the ends of this edge along with b makes a cycle. Now let there is no edge between any two vertices in B. Let C be a cycle in H with $\mathrm{girth(H)}$ vertices. Therefore C has a vertex u in $V(H) \setminus (B \cup \{b\})$. Since B is a dominating set, every vertex of $V(H) \setminus B$ has a neighbour in B. If u is adjacent to at least two vertices of $B \cup \{b\}$ then $\mathrm{girth(H)} \leq 4$. Otherwise u has a neighbour u' out of $B \cup \{b\}$. Since u' has a neighbour in B we have $\mathrm{girth(H)} \leq 5$.

(ii) Let B be an adjacency basis of H and b be a dominant vertex for B. Since B is not a dominating set, there exists a vertex $v \in V(H) \setminus B$ such that v is not adjacent to any vertex in B. If there exists an edge between two vertices in B, then the ends of this edge and b

makes a cycle. Now let there is no edge between any two vertices in B. Let C be a cycle in H with $\mathrm{girth(H)}$ vertices. If $v \notin V(C)$ then similar to previous case we have $\mathrm{girth(H)} \leq 5$. Now let $v \in V(C)$ and u, u' be neighbours of v on C. Clearly u, u' are not in B. Since B is an adjacency basis, every vertex of $V(H) \setminus B$ has a neighbour in B. Therefore u, u' has neighbours in B and so $\mathrm{girth(H)} \leq 6$.

□

Propositions 3.14 and 3.15 conclude the following corollary.

Corollary 3.16. *If H is a graph such that $\mathrm{diam(H)} > 5$ or $\mathrm{girth(H)} > 6$, then for every connected graph G of order n, we have*

$$\psi(G \odot \overline{H}) = n \dim_2(H).$$

The join of the graph K_1 with another graph is an interesting graph. In the following we investigate adjacency bases of $K_1 + H$.

Proposition 3.17. *Let H be a graph. Then for every adjacency basis of $K_1 + H$ there exists a dominant vertex.*

Proof. Suppose on the contrary that there exists an adjacency basis B of $K_1 + H$ with no dominant vertex. Let v be the vertex of K_1 in $K_1 + H$. Clearly $v \in B$, otherwise v is a dominant vertex for B. Hence $B' = B \setminus \{v\}$ is not an adjacency resolving set for $K_1 + H$. Note that for every vertices $x, y \in V(K_1 + H) \setminus \{v\}$ we have, $r_2(x|B) \neq r_2(y|B)$ if and only if $r_2(x|B') \neq r_2(y|B')$. Therefore, there exists a vertex $u \in V(K_1 + H) \setminus B$ such that $r_2(u|B') = r_2(v|B') = (1, 1, \ldots, 1)$. Hence $r_2(u|B) = (1, 1, \ldots, 1)$ and so u is a dominant vertex for B, which is a contradiction. □

Corollary 3.18. *For every connected graph G of order n and arbitrary graph H,*

$$\psi(G \odot \overline{K_1 + H}) = n(\dim_2(K_1 + H) + 1).$$

In the following Proposition we compute $\psi(G \odot H)$ and $\psi(G \odot \overline{H})$, where H is a wheel or a fan graph of order at least 7.

Proposition 3.19. *Let G be a connected graph of order $n \geq 2$ and $H \in \{W_m, F_m\}$, $m = 5k + r$, where $m \geq 7$ and $0 \leq r \leq 4$.*

(i) If r is even, then $\psi(G \odot H) = n\lfloor \frac{2m+2}{5} \rfloor$.

(ii) If r is odd, then $\psi(G \odot H) = n(\lfloor \frac{2m+2}{5} \rfloor + 1)$.

(iii) $\psi(G \odot \overline{H}) = n(\lfloor \frac{2m+2}{5} \rfloor + 1)$.

Proof. By Theorem 2.8, $\dim(H) = \lfloor \frac{2m+2}{5} \rfloor$. Since the diameter of H is two we have $\dim(H) = \dim_2(H)$. Proposition 2.7 implies that $\dim_2(H) = \dim_2(P_m) = \dim_2(C_m) = \lfloor \frac{2m+2}{5} \rfloor$.

(i) Since r is even, Proposition 3.9 implies that there exists an adjacency basis B for P_m that is also a dominating set. By Proposition 2.7, $|B| \geq 3$. Hence, there is no dominant vertex for B in P_m, because the degree of every vertex of P_m is at most 2. Now consider B as a subset of $V(F_m)$, thus for every $x, y \in V(P_m)$ we have $a_{P_m}(x, y) = a_{F_m}(x, y)$. Therefore for a vertex $v \in V(P_m)$, the adjacency representation of v, as a vertex of P_m, with respect to B is the same as its adjacency representation of v, as a vertex of F_m, with respect to B. Also, the adjacency representation of $u \in V(F_m) \setminus V(P_m)$ is $(1, 1, \ldots, 1)$. Since there is no dominant vertex for B in P_m, the adjacency representation of u is different from all other vertices of F_m. Hence, B is an adjacency resolving set for F_m. Since $\dim_2(F_m) = \dim_2(P_m)$, B is an adjacency basis for F_m. Note that B is also a dominating set for F_m. Therefore, by Theorem 3.3 we have $\psi(G \odot F_m) = n \dim_2(F_m) = n\lfloor \frac{2m+2}{5} \rfloor$. For $H = W_m$, the proof is the same.

(ii) Let B be an adjacency basis of F_m. Note that for every $x, y \in V(P_m)$ we have $a_{P_m}(x, y) = a_{F_m}(x, y)$. Therefore B is an adjacency resolving set for P_m. $\dim_2(F_m) = \dim_2(P_m)$ concludes that B is an adjacency basis for P_m. Proposition 3.9 implies that B is not a dominating set for P_m, because r is odd. Since for every $x, y \in V(P_m)$ we have $a_{P_m}(x, y) = a_{F_m}(x, y)$, B is not a dominating set for F_m. Therefor, by Theorem 3.4 we have $\psi(G \odot F_m) = n(\dim_2 F_m + 1) = n(\lfloor \frac{2m+2}{5} \rfloor + 1)$. If $H = W_m$, the proof is the same.

(iii) It immediately comes from Corollary 3.18.

□

Let G be a connected graph of order $n \geq 2$ and H be a non-trivial graph of order m. Corollary 3.5 concludes $n \dim_2(H) \leq \psi(G \odot H) \leq n(\dim_2(H) + 1)$. Since $1 \leq \dim_2(H) \leq m - 1$, we have $1 \leq \psi(G \odot H) \leq nm$. The following theorem determines all graph H that $\psi(G \odot H)$ gets one of the numbers n, $n(m-1)$ or nm.

Theorem 3.20. *Let G be a connected graph of order n and H be an arbitrary graph of order m. Then*

(i) $\psi(G \odot H) = n$ if and only if H is P_1 or P_2.

(ii) $\psi(G \odot H) = nm$ if and only if $H = \overline{K_m}$.

(iii) $\psi(G \odot H) = n(m-1)$ if and only if H is one of the following graphs

$$K_m\ (m \geq 2),\ K_{1,t}\ (t \geq 2),\ \overline{K_{1,t}}\ (t \geq 2),\ \overline{K_s + \overline{K_t}}\ (s, t \geq 2),\ \overline{K_s + (K_t \cup K_1)}\ (s, t \geq 1).$$

Proof. By Corollary 3.5 we have

$$n \dim_2(H) \leq \psi(G \odot H) \leq n(\dim_2(H) + 1). \tag{1}$$

(i) If $\psi(G \odot H) = n$, then Inequality 1 implies that $\dim_2(H) = 1$. Hence, Theorem 2.3 concludes that $H \in \{P_1, P_2, P_3, \overline{P}_2, \overline{P}_3\}$. If $H \in \{P_3, \overline{P}_2, \overline{P}_3\}$, then there is no adjacency basis for H that is also a dominating set and by Theorem 3.4, $\psi(G \odot H) = 2n$. On the other hand by Theorem 3.3, $\psi(G \odot P_2) = n$ and by Theorem 3.1, $\psi(G \odot P_1) = n$.

(ii) If $\psi(G \odot H) = nm$, then Inequality 1 implies that $\dim_2(H) = m - 1$, because $\dim_2(H) \leq m - 1$. Hence, Theorem 2.3 concludes that $H \in \{K_m, \overline{K_m}\}$. Since every adjacency basis of K_m is a dominating set, Theorem 3.3 yields $\psi(G \odot K_m) = n(m-1)$. On the other hand for every adjacency basis of K_m there exists a dominant vertex, thus by Corollary 3.7 we have $\psi(G \odot \overline{K_m}) = nm$.

(iii) Let $\psi(G \odot H) = n(m-1)$. Since $\dim_2(H) \leq m - 1$, Inequality 1 implies that $\dim_2(H) \in \{m-2, m-1\}$. If $\dim_2(H) = m - 1$, then part (ii) of this proposition yields $H = K_m$. On the other hand $\psi(G \odot K_m) = n(m-1)$, because every adjacency basis of K_m is a dominating set. If $\dim_2(H) = m - 2$, then Theorem 2.3 implies that H or $\overline{H}$ is one of the graphs P_4, $K_{s,t}$ $(s, t \geq 1), K_s + \overline{K}_t$ $(s \geq 1, t \geq 2)$, or $K_s + (K_t \cup K_1)$ $(s, t \geq 1)$. It is easy to see that if H is one of the graphs

$$P_4,\ K_{s,t}\ (s,t \geq 2),\ K_s + \overline{K}_t\ (s,t \geq 2),\ K_s + (K_t \cup K_1)\ (s,t \geq 1),$$

then there exists an adjacency basis for H that is a dominating set. Therefore, Theorem 3.3 concludes that $\psi(G \odot H) = n(m-2)$. Also if H is one of the graphs P_4 or $K_{s,t}$ $(s, t \geq 2)$, then there is an adjacency basis for H that there is no dominant vertex for it. Hence, Corollary 3.7 implies that $\psi(G \odot \overline{H}) = n(m-2)$. If H is one of the graphs

$$K_m\ (m \geq 2),\ K_{1,t}\ (t \geq 2),\ \overline{K_{1,t}}\ (t \geq 2),\ \overline{K_s + \overline{K_t}}\ (s,t \geq 2),\ \overline{K_s + (K_t \cup K_1)}\ (s,t \geq 1),$$

then no adjacency basis of H is a dominating set and by Theorem 3.3 we have $\psi(G \odot H) = n(m-1)$.

□

Data Availability

All data generated or analysed during this study are included in this article.

Conflicts of Interest

The author declare no conflict of interest.

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