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About the second neighborhood problem in tournaments missing disjoint stars

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Abstract

Let D be a digraph without digons. Seymour's second neighborhood conjecture states that D has a vertex v such that $d^+(v) \leq d^{++}(v)$. Under some conditions, we prove this conjecture for digraphs missing n disjoint stars. Weaker conditions are required when $n = 2$ or 3. In some cases we exhibit two such vertices.

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1. Introduction

In this paper, a digraph D is a pair of two disjoint finite sets (V, E) such that $E \subseteq V \times V$. E is the arc set and V is the vertex set and they are denoted by $E(D)$ and $V(D)$ respectively. An oriented graph is a digraph without loop and digon (directed cycles of length two). If $K \subseteq V(D)$ then the induced restriction of D to K is denoted by $D[K]$. As usual, $N^+_D(v)$ (resp. $N^-_D(v)$) denotes the (first) out-neighborhood (resp. in-neighborhood) of a vertex $v \in V$. $N^{++}_D(v)$ (resp.

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$N_D^{--}(v)$) denotes the second *out-neighborhood* (*in-neighborhood*) of v, which is the set of vertices that are at distance 2 from v (resp. to v). We also denote $d_D^+(v) = |N_D^+(v)|$, $d_D^{++}(v) = |N_D^{++}(v)|$, $d_D^-(v) = |N_D^-(v)|$ and $d_D^{--}(v) = |N_D^{--}(v)|$. We omit the subscript if the digraph is clear from the context. For short, we write $x \rightarrow y$ if the arc $(x, y) \in E$. A vertex $v \in V(D)$ is called *whole* if it is adjacent to every vertex in $V(D) - \{v\}$. A *sink* v is a vertex with $d^+(v) = 0$, while a source v is a vertex with $d^-(v) = 0$. For $x, y \in V(D)$, we say xy is a *missing edge* of D if neither (x, y) nor (y, x) are in $E(D)$. The *missing graph* G of D is the graph whose edges are the missing edges of D and whose vertices are the non whole vertices of D. In this case, we say that D is *missing* G. So, a tournament does not have any missing edge. A star of center x is a graph whose edge set has the form $\{a_i x; i = 1, ..., k\}$. In this paper, n stars are said to be disjoint if any two of them do not share a common vertex.

A vertex v of D is said to have the *second neighborhood property* (SNP) if $d_D^+(v) \leq d_D^{++}(v)$. In 1990, Seymour conjectured the following:

Conjecture 1. ***(Seymour's Second Neighborhood Conjecture (SNC))[1]*** *Every oriented graph has a vertex with the SNP.*

In 1996, Fisher [3] solved the SNC for tournaments by using a certain probability distribution on the vertices. Another proof of Dean's conjecture was established in 2000 by Havet and Thomassé [7]. Their short proof uses a tool called median orders. Furthermore, they have proved that if a tournament has no sink vertex then there are at least two vertices with the SNP. In 2007 Fidler and Yuster [2] proved, using median orders and dependency digraphs, that SNC holds for digraphs missing a matching, a star or a complete graph. Ghazal proved more general statements in [4, 6] and proved that the SNC holds for some other classes of digraphs [5].

2. Definitions and Preliminary Results

Let $L = v_1 v_2 ... v_n$ be an ordering of the vertices of a digraph D. An arc $e = (v_i, v_j)$ is *forward* with respect to L if $i < j$. Otherwise e is a *backward* arc. The weight of L is $\omega(L) = |\{(v_i, v_j) \in E(D); i < j\}|$. L is called a *median order* of D if $\omega(L) = max\{\omega(L'); L'$ is an ordering of the vertices of $D\}$; that is L maximizes the number of forward arcs. In fact, the median order L satisfies the *feedback property*: For all $1 \leq i \leq j \leq n$:

$$d^+_{D[i,j]}(v_i) \geq d^-_{D[i,j]}(v_i)$$

and

$$d^-_{D[i,j]}(v_j) \geq d^+_{D[i,j]}(v_j)$$

where $[i, j] := \{v_i, v_{i+1}, ..., v_j\}$ (See [7]).

It is also known that if we reverse the orientation of a backward arc $e = (v_i, v_j)$ of D with respect to L, then L is again a weighted median order of the new digraph $D' = D - (v_i, v_j) + (v_j, v_i)$

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(See [5]).

Let $L = v_1v_2...v_n$ be a median order. Among the vertices not in $N^+(v_n)$ two types are distinguished: A vertex v_j is *good* if there is $i \leq j$ such that $v_n \rightarrow v_i \rightarrow v_j$, otherwise v_j is a *bad vertex*. The set of good vertices of L is denoted by G_L^D [7] (or G_L if there is no confusion). Clearly, $G_L \subseteq N^{++}(v_n)$. The last vertex v_n is called a feed vertex of D.

We say that a missing edge x_1y_1 *loses to* a missing edge x_2y_2 if: $x_1 \rightarrow x_2$, $y_2 \notin N^+(x_1) \cup N^{++}(x_1)$, $y_1 \rightarrow y_2$ and $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$. The *dependency digraph* Δ of D is defined as follows: Its vertex set consists of all the missing edges and $(ab, cd) \in E(\Delta)$ if ab loses to cd [2, 5]. Note that Δ may contain digons.

Definition 1. [4] In a digraph D, a missing edge ab is called a *good missing edge* if:
(i) $(\forall v \in V\backslash\{a, b\})[(v \rightarrow a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ or
(ii) $(\forall v \in V\backslash\{a, b\})[(v \rightarrow b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$.
If ab satisfies (i) we say that (a, b) is a convenient orientation of ab.
If ab satisfies (ii) we say that (b, a) is a convenient orientation of ab.

We will need the following observation:

Lemma 2.1. *([2], [5]) Let D be an oriented graph and let Δ denote its dependency digraph. A missing edge ab is good if and only if its in-degree in Δ is zero.*

Let D be a digraph and let Δ denote its dependency digraph. Let C be a connected component of Δ. Set $K(C) = \{u \in V(D)$; there is a vertex v of D such that uv is a missing edge and belongs to C $\}$. The *interval graph of* D, denoted by $\mathcal{I}_D$ is defined as follows. Its vertex set consists of the connected components of Δ and two vertices C_1 and C_2 are adjacent if $K(C_1) \cap K(C_2) \neq \phi$. So $\mathcal{I}_D$ is the intersection graph of the family $\{K(C); C$ is a connected component of Δ $\}$. Let ξ be a connected component of $\mathcal{I}_D$. We set $K(\xi) = \cup_{C \in \xi} K(C)$. Clearly, if uv is a missing edge in D then there is a unique connected component ξ of $\mathcal{I}_D$ such that u and v belong to $K(\xi)$. For $f \in V(D)$, we set $J(f) = \{f\}$ if f is a whole vertex, otherwise $J(f) = K(\xi)$, where ξ is the unique connected component of $\mathcal{I}_D$ such that $f \in K(\xi)$. Clearly, if $x \in J(f)$ then $J(f) = J(x)$ and if $x \notin J(f)$ then x is adjacent to every vertex in $J(f)$.

Let $L = x_1 \cdots x_n$ be a median order of a digraph D. For $i < j$, the sets $[i, j] := [x_i, x_j] := \{x_i, x_{i+1}, ..., x_j\}$ and $]i, j[= [i, j]\backslash\{x_i, x_j\}$ are called *intervals* of L. We recall that $K \subseteq V(D)$ is an *interval of* D if for every $u, v \in K$ we have: $N^+(u)\backslash K = N^+(v)\backslash K$ and $N^-(u)\backslash K = N^-(v)\backslash K$. The following shows a relation between the intervals of D and the intervals of L.

Proposition 2.1. *[6] Let $\mathcal{I} = \{I_1, ..., I_r\}$ be a set of pairwise disjoint intervals of D. Then for every median order L of D, there is a weighted median order L' of D such that: L and L' have the same feed vertex and every interval in $\mathcal{I}$ is an interval of L'.*

We say that D is *good digraph* if the sets $K(\xi)$'s are intervals of D. By the previous proposition, every good digraph has a median order L such that the $K(\xi)$'s form intervals of L. Such an enumeration is called a *good median order* of the good digraph D [6].

Theorem 2.1. *[6] Let D be a good oriented graph and let L be a good median order of D, with feed vertex f. Then for every $x \in J(f)$, we have $|N^+(x)\backslash J(f)| \leq |G_L\backslash J(f)|$. So if x has the SNP in $D[J(f)]$, then it has the SNP in D.*

Corollary 2.1. *([7]) Let L be a median order of a tournament with feed vertex f. Then $|N^+(f))| \leq |G_L|$.*

Let L be a good median order of a good oriented graph D and let f denote its feed vertex. By theorem 2.1, for every $x \in J(f)$, $|N^+(x)\backslash J(f)| \leq |G_L\backslash J(f)|$. Let $b_1, \cdots, b_r$ denote the bad vertices of L not in $J(f)$ and $v_1, \cdots, v_s$ denote the non bad vertices of L not in $J(f)$, both enumerated in increasing order with respect to their index in L.
If $|N^+(x)\backslash J(f)| < |G_L\backslash J(f)|$, we set $Sed(L) = L$. If $|N^+(x)\backslash J(f)| = |G_L\backslash J(f)|$, we set $sed(L) = b_1 \cdots b_r J(f) v_1 \cdots v_s$. This new order is called the *sedimentation of L*.

Lemma 2.2. *[6] Let L be a good median order of a good oriented graph D. Then $Sed(L)$ is a good median order of D.*

In the rest of this section, D is an oriented graph missing a matching and Δ denotes its dependency digraph. We begin by the following lemma:

Lemma 2.3. *[2] The maximum out-degree of Δ is one and the maximum in-degree of Δ is one. Thus Δ is composed of vertex disjoint directed paths and directed cycles.*

Proof. Assume that a_1b_1 loses to a_2b_2 and a_1b_1 loses to $a'_2b'_2$, with $a_1 \rightarrow a_2$ and $a_1 \rightarrow a'_2$. The edge a'_2b_2 is not a missing edge of D. If $a'_2 \rightarrow b_2$ then $b_1 \rightarrow a'_2 \rightarrow b_2$, a contradiction. If $b_2 \rightarrow a'_2$ then $b_1 \rightarrow b_2 \rightarrow a'_2$, a contradiction. Thus, the maximum out-degree of Δ is one. Similarly, the maximum in-degree is one. □

In the following, $C = a_1b_1, ..., a_kb_k$ denotes a directed cycle of Δ, namely $a_i \rightarrow a_{i+1}$, $b_{i+1} \notin N^{++}(a_i) \cup N^+(a_i)$, $b_i \rightarrow b_{i+1}$ and $a_{i+1} \notin N^{++}(b_i) \cup N^+(b_i)$, for all $i < k$.

Lemma 2.4. *([2]) If k is odd then $a_k \rightarrow a_1$, $b_1 \notin N^{++}(a_k) \cup N^+(a_k)$, $b_k \rightarrow b_1$ and $a_1 \notin N^{++}(b_k) \cup N^+(b_k)$. If k is even then $a_k \rightarrow b_1$, $a_1 \notin N^{++}(a_k) \cup N^+(a_k)$, $b_k \rightarrow a_1$ and $b_1 \notin N^{++}(b_k) \cup N^+(b_k)$.*

Lemma 2.5. *[2] $K(C)$ is an interval of D.*

Proof. Let $f \notin K(C)$. Then f is adjacent to every vertex in $K(C)$. If $a_1 \rightarrow f$ then $b_2 \rightarrow f$, since otherwise $b_2 \in N^{++}(a_1) \cup N^+(a_1)$ which is a contradiction. So $N^+(a_1)\backslash K(C) \subseteq N^+(b_2)\backslash K(C)$.
Applying this to every losing relation of C yields:
$N^+(a_1)\backslash K(C) \subseteq N^+(b_2)\backslash K(C) \subseteq N^+(a_3)\backslash K(C) ... \subseteq N^+(b_k)\backslash K(C) \subseteq N^+(b_1)\backslash K(C) \subseteq N^+(a_2)\backslash K(C) ... \subseteq N^+(a_k)\backslash K(C) \subseteq N^+(a_1)\backslash K(C)$ if k is even. So these inclusion are equalities. An analogous argument proves the same result for odd cycles. □

3. Main Results

3.1. Removing n stars

We recall that a vertex x in a tournament T is a king if $\{x\} \cup N^+(x) \cup N^{++}(x) = V(T)$. It is well known that every tournament has a king. However, for every natural number $n \notin \{2, 4\}$, there is a tournament T_n on n vertices, such that every vertex is a king for this tournament.

A digraph is called non trivial if it has at least one arc.

Proposition 3.1. *Let D be a digraph missing disjoint stars. If the connected components of its dependency digraph are non-trivial strongly connected, then D is a good digraph.*

Proof. Let ξ be a connected component of Δ. First, suppose that $K(\xi) = K(C)$ for some directed cycle $C = a_1b_1, a_2b_2, ..., a_nb_n$ in Δ, namely $a_i \rightarrow a_{i+1}$ and $b_{i+1} \notin N^+(a_i) \cup N^{++}(a_i)$. If the set of the missing edges $\{a_ib_i; i = 1, ..., n\}$ forms a matching, then by lemma 2.5, $K(C)$ is an interval of D.

So we will suppose that a center x of a missing star appears twice in the list $a_1, b_1, a_2, b_2, ..., a_n$, b_n and assume without loss of generality that $x = a_1$. Suppose that n is even. Set $K_1 = \{a_1, b_2, ..., a_{n-1}, b_n\}$ and $K_2 = K(C)\backslash K_1$.

Suppose that $a_n \rightarrow b_1$ and $a_1 \notin N^+(a_n) \cup N^{++}(a_n)$. Then by following the proof of lemma 2.5 we get the desired result.

Suppose $a_n \rightarrow a_1$ and $b_1 \notin N^+(a_n) \cup N^{++}(a_n)$. Then by following the proof of lemma 2.5 we get that K_1 and K_2 are intervals of D. Assume, for contradiction that $K_1 \cap K_2 = \phi$ and let $i > 1$ be the smallest index for which x is incident to a_ib_i. Clearly $i > 2$. However, $b_3 \notin K_1$ and $x = a_1 \rightarrow a_2 \rightarrow a_3$ implies that $i > 3$. Suppose that $x = a_i$. Note that i must be odd by definition of K_1. Since $b_2 \rightarrow a_1 = x = a_i$ and $a_3 \notin N^+(x) \cup N^{++}(x)$ then $a_3 \rightarrow x$. Similarly $b_4, a_5, ..., b_{i-1}$ are in-neighbors of x. However, b_{i-1} is an out-neighbor of $a_i = x$, a contradiction. Suppose that $x = b_i$. Similarly, $a_3, b_4, ..., a_{i-1}$ are in-neighbors of x. However, a_{i-1} is an out-neighbor of $b_i = x$, a contradiction. Thus $K_1 \cap K_2 \neq \phi$. Whence, $K = K_1 \cup K_2$ is an interval of D. Similar argument is used to prove it when n is odd.

This result can be easily extended to the case when $K(\xi) = K(C)$ and C is a non trivial strongly connected component of Δ, because between any two missing edges uv and zt there is directed path from uv to zt and a directed path from zt to uv. These two directed paths creat many directed cycles that are used to prove the desired result.

This also is extended to the case when $K(\xi) = \cup_{C \in \xi} K(C)$: Let u and u' be two vertices of $K(\xi)$. There are two non trivial strongly connected components of Δ such that $u \in K(C)$ and $u' \in K(C')$. Since ξ is a connected component of $\mathcal{I}_D$, there is a path $C = C_0C_1...C_n = C'$. For all $i > 0$, there is $u_i \in K(C_{i-1}) \cap K(C_i)$, by definition of edges in $\mathcal{I}_D$. Therefore, $N^+(u)\backslash K(\xi) = N^+(u_1)\backslash K(\xi) = ... = N^+(u_i)\backslash K(\xi) = ... = N^+(u_n)\backslash K(\xi) = N^+(u')\backslash K(\xi)$

and $N^-(u)\backslash K(\xi) = N^-(u_1)\backslash K(\xi) = ... = N^-(u_i)\backslash K(\xi) = ... = N^-(u_n)\backslash K(\xi) = N^-(u')\backslash K(\xi)$. □

Theorem 3.1. *Let D be a digraph obtained from a tournament by deleting the edges of disjoint stars. Suppose that, in the induced tournament by the centers of the missing stars, every vertex is a king. If $\delta^-_\Delta > 0$ then D satisfies SNC.*

Proof. Orient every missing edge of D towards the center of its star. Let L be a median order of the obtained tournament T and let f be its feed vertex. Then f has the SNP in T. We prove that f has the SNP in D as well.

First, suppose that f is a whole vertex. Then $N^+(f) = N^+_T(f)$. Let $v \in N^{++}_T(f)$. Then there $\exists u \in V(T) = V(D)$ such that $f \to u \to v \to f$ in T. Since f is whole, then (f, u) and $(v, f) \in D$. If $(u, v) \in D$ then $v \in N^{++}(f)$. Otherwise, uv is a missing edge and hence, $\exists ab$ that loses to uv, say $b \to v$ and $u \notin N^+(b) \cup N^{++}(b)$. But fb is not a missing edge, s-ince f is whole. Then $(f, b) \in D$, since otherwise, $b \to f \to u$ in D which is a contradiction. Therefore, $f \to b \to v$ in D. Whence, $v \in N^{++}(f)$. So $N^{++}_T(f) \subseteq N^{++}(f)$. Therefore, $d^+(f) = d^+_T(f) \leq d^{++}_T(f) \leq d^{++}(f)$.

Now suppose that f is the center of a missing star. Then $N^+(f) = N^+_T(f)$. Let $v \in N^{++}_T(f)$. Then there $\exists u \in V(T) = V(D)$ such that $f \to u \to v \to f$ in T. Then $(f, u) \in D$ while $(f, v) \notin D$. If $(u, v) \in D$ then $v \in N^{++}(f)$. Otherwise, uv is a missing edge and v is the center of a missing star. Then $v \in N^+(f) \cup N^{++}(f)$, because f is a king for the centers of the missing stars. Note that $v \notin N^+(f)$. So $N^{++}_T(f) \subseteq N^{++}(f)$. Therefor, f has the SNP in D.

Finally, suppose that f is not whole and not the center of a missing star. Then $\exists x$ a center of a missing star such that fx is a missing edge. We distinguish between two cases.

In the first case, we suppose that fx does not lose to any missing edge. We reorient fx as (x, f). Since $(f, x) \in T$ is a backward arc with respect to L, the again L is a median order of the new tournament T' obtained by reversing the orientation of fx. Moreover, $N^+(f) = N^+_{T'}(f)$ and f has the SNP in T'. Let $v \in N^{++}_{T'}(f)$. Then there $\exists u \in V(T) = V(D)$ such that $f \to u \to v \to f$ in T'. Then $(f, u) \in D$ while $(f, v) \notin D$. If $(u, v) inD$ then $v \in N^{++}(f)$. Otherwise uv is a missing edge and v is the center of a missing star.Since Δ has no source, there is a missing edge that loses to uv. Suppose that this edge is of the form ax. Then we must have $x \to v$ and $u \notin N^+(x) \cup N^{++}(x)$, by definition of losing relation and due to the fact that $v \in N^+(x) \cup N^{++}(x)$ (x is a king for the centers of the missing stars). If $v \notin N^{++}(f)$, then fx loses uv which is a contradiction to the supposition of this case. Hence, $v \notin N^{++}(f)$. Now, suppose that the missing edge that loses to uv is of the form by with $x \notin \{b, y\}$. Suppose without loss of generality that y is the center of a missing star containing by. Then $y \to v$ and $u \notin N^+(y) \cup N^{++}(y)$, by definition of losing relation and due to the fact that $v \in N^+(y) \cup N^{++}(y)$ (y is a king for the centers of the missing stars). But $(f, u) \in D$ and fy is not a missing edge, then $(f, y) \in D$. Thus $f \to y \to v$. Whence, $v \in N^+(f) \cup N^{++}(f)$. So $N^{++}_{T'}(f) \subseteq N^{++}(f)$. Therefor, f has the SNP in D as well.

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In the second case, we suppose that fx loses to some missing edge by. We may assume without loss of generality that y is the center of a missing star containing by. Then we must have $x \rightarrow y$ and $b \notin N^+(x) \cup N^{++}(x)$. Clearly, $N^+(f) \cup \{y\} = N^+_T(f)$. We prove that $N^{++}_T(f) \subseteq N^{++}(f) \cup \{y\}$. Let $v \in N^{++}_T(f) \backslash y$. Then there $\exists u \in V(T) = V(D)$ such that $f \rightarrow u \rightarrow v \rightarrow f$ in T. Suppose that $u = x$. Since bv is not a missing edge, $x = u \rightarrow v$ and $b \notin N^+(x) \cup N^{++}(x)$ then we must have $(b, v) \in D$. Whence, $f \rightarrow b \rightarrow v$ in D. Therefore $v \in N^{++}(f)$. Now suppose that $u \neq x$. Then $(f, u) \in D$. If $(u, v) \in D$ then $v \in N^{++}(f)$. Otherwise, uv is a missing edge. Hence there is a missing edge pq that loses to uv, namely, $q \rightarrow v$ and $u \notin N^+(q) \cup N^{++}(q)$. If $q = x$, then we have $f \rightarrow x \rightarrow v \rightarrow f$ in T, which is the same as the case when $u = x$. So we may suppose that $q \neq x$. Note that q must be the center of a missing star. So $f, x \notin \{p, q\}$. Thus fq is not a missing edge, $u \notin N^+(q) \cup N^{++}(q)$ and $(f, u) \in D$. Then we must have $(f, q) \in D$, since otherwise we get $q \rightarrow f \rightarrow u$ in D which is a contradiction. Thus $f \rightarrow q \rightarrow v$ in D. Whence $v \in N^{++}(f)$. So $N^{++}_T(f) \subseteq N^{++}(f) \cup \{y\}$. Therefore $d^+(f) + 1 = d^+_T(f) \leq d^{++}_T(f) \leq d^{++}(f) + 1$. Whence f has the SNP in D.

□

3.2. Removing a star

A more general statement to the following theorem is proved in [4] . Here we introduce another prove that uses the sedimentation technique of a median order.

Theorem 3.2. *[2] Let D be an oriented graph missing a star. Then D satisfies SNC.*

Proof. Orient all the missing edges of D towards the center x of the missing star. The obtained digraph is a tournament T. Let L be a median order of T that maximizes α, the index of x in L, and let f denote its feed vertex. Reorient the missing edges incident to f towards f (if any). L is also a median order of the new tournament T'. Note that $N^+(f) = N^+_{T'}(f)$ and we have $d^+_{T'}(f) \leq |G^{T'}_L|$. If $x \in G^{T'}_L$ and $d^+_{T'}(f) = |G^{T'}_L|$ then $sed(L)$ is a median order of T' in which the index of x is greater than α, and also greater than the index of f. So we can give the missing edge incident to f (if it exists then it is xf) its initial orientation (as in T) such that $sed(L)$ is a median order of T, a contradiction to the fact that L maximizes α. So $x \notin G^{T'}_L$ or $d^+T'(f) < |G^{T'}_L|$. If $f = x$ then, clearly, $d^+(f) = d^+_{T'}(f) \leq |G^{T'}_L| \leq d^{++}_{T'}(f) = d^{++}(f)$. Now suppose that $f \neq x$. We have that x is the only possible gained second out-neighbor vertex for f. If $x \notin G^{T'}_L$ then $G^{T'}_L \subseteq N^{++}(f)$, whence the result follows. If $d^+_{T'}(f) < |G^{T'}_L|$ then $d^+(f) = d^+_{T'}(f) \leq |G^{T'}_L| - 1 \leq d^{++}(f)$. So f has the SNP in D. □

3.3. Removing 2 disjoint stars

In this section, let D be a digraph obtained from a tournament by deleting the edges of 2 disjoint stars and let Δ denote its dependency digraph. Let S_x and S_y be the two missing disjoint stars with centers x and y respectively, $A = V(S_x)\backslash x$, $B = V(S_y)\backslash y$, $K = V(S_x) \cup V(S_y)$ (the set of non whole vertices) and assume without loss of generality that $x \rightarrow y$. In [4] it is proved that if the dependency digraph of any digraph consists of isolated vertices only then it satisfies SNC. Here we consider the case when the Δ has no isolated vertices.

Theorem 3.3. *Let D be an oriented graph missing 2 disjoint stars. If Δ has no isolated vertex, then D satisfies SNC.*

Proof. Assume without loss of generality that $x \rightarrow y$. We note that the condition Δ has no isolated vertex, implies that for every $a \in A$ and $y \in B$ we have $y \rightarrow a$ and $b \rightarrow x$. We shall orient all the missing edges of D. First, we give every good edge a convenient orientation. For the other missing edges, let the orientation be towards the center of the 2 missing stars S_x or S_y. The obtained digraph is a tournament T. Let L be a median order of T such that the index k of x is maximum and let f denote its feed vertex. We know that f has the SNP in T. We have only 5 cases:

Suppose that f is a whole vertex. In this case $N^+(f) = N^+_T(f)$. Suppose $f \rightarrow u \rightarrow v$ in T. Clearly $(f, u) \in D$. If $(u, v) \in D$ or is a convenient orientation then $v \in N^+(f) \cup N^{++}(f)$. Otherwise there is a missing edge zt that loses to uv with $t \rightarrow v$ and $u \notin N^+(f) \cup N^{++}(f)$. But $f \rightarrow u$, then $f \rightarrow t$, whence $f \rightarrow t \rightarrow v$ in D. Therefore, $N^{++}(f) = N^{++}_T(f)$ and f has the SNP in D as well.

Suppose $f = x$. Orient all the edges of S_x towards the center x. L is a median order of the modified completion T' of D. We have $N^+(f) = N^+_{T'}(f)$. Suppose $f \rightarrow u \rightarrow v$ in T'. If $(u, v) \in D$ or is a convenient orientation then $v \in N^+(f) \cup N^{++}(f)$. Otherwise $(u, v) = (b, y)$ for some $b \in B$, but $f = x \rightarrow y$. Thus, $N^{++}(f) = N^{++}_{T'}(f)$ and f has the SNP in T' and D.

Suppose $f = b \in B$. Orient the missing edge by towards b. Again, L is a median order of the modified tournament T' and $N^+(f) = N^+_{T'}(f)$. Suppose $f \rightarrow u \rightarrow v$ in T'. If $(u, v) \in D$ or is a convenient orientation then $v \in N^+(f) \cup N^{++}(f)$. Otherwise $(u, v) = (b', y)$ for some $b' \in B$ or $(u, v) = (a, x)$ for some $a \in A$, however $x, y \in N^{++}(f) \cup N^+(f)$ because $f = b \rightarrow x \rightarrow y$ in D. Thus, $N^{++}(f) = N^{++}_{T'}(f)$ and f has the SNP in T' and D.

Suppose $f = y$. Orient the missing edges towards y and let T' denote the new tournament. We note that $B \subseteq N^{++}(y) \cap N^{++}_{T'}(y)$ due to the condition $\delta_\Delta > 0$. Also, x is the only possible new second neighbor of y in T'. If $B \cup \{x\} \nsubseteq G_L$ or $d^+_{T'}(y) < d^{++}_{T'}(y)$, then $d^+(y) = d^+_{T'}(y) \leq d^{++}_{T'}(y) - 1 \leq d^{++}(y)$. Otherwise, $B \cup \{x\} \nsubseteq G_L$ and $d^+_{T'}(y) = |G_L|$. In this case we consider the median order $Sed(L)$ of T'. Now the feed vertex of $sed(L)$ is different from y, the index of x had increased, and the index of y became less than the index of any vertex of B which makes $Sed(L)$ a median order of T also, in which the index of x is greater than k, a contradiction.

Suppose $f = a \in A$. Orient the missing edge ax as (x, a) and let T' denote the new tournament. Note that y is the only possible new second neighbor of a in T' and not in D. Also $x \in N^{++}_T(a) \cap N^{++}(a)$. If $d^+_{T'}(a) < d^{++}_{T'}(a)$, then $d^+(a) = d^+_{T'}(a) \leq d^{++}_{T'}(a) - 1 \leq d^{++}(a)$, hence a has the SNP in D. Otherwise, $d^+_{T'}(a) = |G_L| = d^{++}_{T'}(a)$ and in particular $x \in G_L$. In this case we consider $sed(L)$ which is a median order of T'. Note that the feed vertex of $Sed(L)$ is different from a and the index of a is less than the index of x in the new order $Sed(L)$. Hence $Sed(L)$ is a median of T as well, in which the index of x is greater than k, a contradiction.
So in all cases f has the SNP in D. Therefore D satisfies SNC. □

Theorem 3.4. *Let D be a digraph obtained from a tournament by deleting the edges of 2 disjoint*

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stars. If Δ has neither a source nor a sink and D has no sink, then D has at least two vertices with the SNP.

Proof.
claim 1: Suppose $K = V(D)$. If Δ has no isolated vertex, then D has at least two vertices with the SNP.
Proof of claim 1: The condition Δ has no isolated vertex implies that for every $a \in A$ and $b \in B$ we have $y \to a$ and $b \to x$. Clearly, $N^+(x) = \{y\}$, $N^+(y) = A$, $d^+(x) \leq 1 \leq |A| \leq d^{++}(x)$, thus x has the SNP. Let H be the tournament $D - \{x, y\}$. Then H has a vertex v with the SNP in H. If $v \in A$, then $d^+(v) = d^+_H(v) \leq d^{++}_H(v) = d^{++}(v)$. If $v \in B$, then $d^+(v) = d^+_H(v) + 1 \leq d^{++}_H(v) + 1 = d^{++}(v)$. Whence, v also has the SNP in D.

Claim 2: D is a good digraph.
Proof of claim 2: Let $\mathcal{I}_D$ be the interval graph of D. Let C_1 and C_2 be two distinct connected components of Δ. Then the centers x and y appear in each of the these two connected components, whence $K(C_1) \cap K(C_2) \neq \phi$. Therefore, $\mathcal{I}_D$ is a connected graph, having only one connected component ξ. Then, $K = K(\xi)$.
So if Δ is composed of non trivial strongly connected components, the result holds by lemma 3.1. Due to the condition Δ has neither a source nor a sink, Δ has a non trivial strongly connected component, hence $N^+(x)\backslash K = N^+(y)\backslash K$. Now let $v \in K$ and assume without loss of generality that xv is a missing edge. Due to the condition Δ has neither a source nor a sink, we have that xv belongs to a non trivial strongly connected component of Δ, and in this case $v \in R$ and $N^+(v)\backslash K = N^+(x)\backslash K = N^+(y)\backslash K$, or xv belongs to a directed path $P = xa_1, yb_1, \cdots, xa_p, yb_p$ joining 2 non trivial strongly connected components C_1 and C_2 with $xa_1 \in C_1$ and $yb_p \in C_2$. There is $i > 1$ such that $v = a_i$. $L = xa_{i-1}, yb_{i-1}, xa_i, yb_i$ is a path in Δ. By the definition of losing cycles we have $N^+(x)\backslash K \subseteq N^+(b_{i-1})\backslash K \subseteq N^+(a_i)\backslash K \subseteq N^+(y)\backslash K = N^+(x)\backslash K$. Hence $N^+(x)\backslash K = N^+(v)\backslash K$ for all $v \in K$. Since every vertex outside K is adjacent to every vertex in K we also have $N^-(x)\backslash K = N^-(v)\backslash K$ for all $v \in K$. This proves the second claim.

Since D is a good digraph, then it has a good median order $L = x_1x_2...x_n$. If $J(x_n) = K$, then the result follows by claim 1 and theorem 2.1. Otherwise, x_n is whole, that is $J(x_n) = \{x_n\}$. By theorem 2.1, x_n has the SNP in D. So we need to find another vertex with the SNP in D. Consider the good median order $L' = x_1x_2...x_{n-1}$ of the good digraph $D' = D[\{x_1, ..., x_{n-1}\}]$. Suppose first that L' is stable. There is q for which $Sed^q(L') = y_1...y_{n-1}$ and $| \; N^+(y_{n-1})\backslash J(y_{n-1}) \; |<| \; G_{Sed^q(L')}\backslash J(y_{n-1}) \; |$ $(*)$. Note that $y_1...y_{n-1}x_n$ is also a good median order of D. By theorem 2.1 and claim 1, there is $y \in J(y_{n-1})$ that has the SNP in D', more precisely $|N^+(y)| < |N^{++}(y)|$ due to $(*)$. Since $y \in J(y_{n-1})$ and $y_{n-1} \to x_n$ then $y \to x_n$. So $| \; N^+(y) \; |=| \; N^+_{D'}(y) \; | +1 \leq| \; N^{++}(y) \; |$.

Now suppose that L' is periodic. Since D has no sink then x_n has an out-neighbor x_j. Choose j to be the greatest (so that it is the last vertex of its corresponding interval). Note that for every q, x_n is an out-neighbor of the feed vertex of $Sed^q(L')$. So x_j is not the feed vertex of any $Sed^q(L')$. Since L' is periodic, x_j must be a bad vertex of $Sed^q(L')$ for some integer q, otherwise the index of x_j would always increase during the sedimentation process. Let q be such an integer and set $Sed^q(L') = y_1...y_{n-1}$. By theorem 2.1 and claim 1, there is $y \in J(y_{n-1})$ that has the SNP in D', more precisely $|N^+_{D'}(y)\backslash J(y_{n-1})| < |G_{Sed^q(L')}\backslash J(y_{n-1})|$ due to $(*)$. Since $y \in J(y_{n-1})$ and $y_{n-1} \to x_n$ then $y \to x_n$. Note that $y \to x_n \to x_j$, $G_{Sed^q(L')} \cup \{x_j\}\backslash J(y_{n-1}) \subseteq N^{++}(y)\backslash J(y_{n-1})$ and $|\ N^+_{D'}(y)\backslash J(y_{n-1})\ | = |\ G_{Sed^q(L')}\backslash J(y_{n-1})\ |$.

Therefore, $|N^+(y)| = |N^+_{D'}(y)| + 1 = |N^+_{D'}(y)\backslash J(y_{n-1})| + 1 + |N^+_{D'}(y) \cap J(y_{n-1})| = |G_{Sed^q(L')}\backslash J(y_{n-1})| + 1 + |N^+_{D'}(y) \cap J(y_{n-1})\backslash J(y_{n-1})| = |G_{Sed^q(L')} \cup \{x_j\}\backslash J(y_{n-1})| + |N^+_{D'}(y) \cap J(y_{n-1})| \leq |N^{++}_D(y)\backslash J(y_{n-1})| + |N^{++}_D(y) \cap J(y_{n-1})| \leq |N^{++}(y)|$.

□

3.4. Removing 3 disjoint stars

In this section, D is an oriented graph missing three disjoint stars S_x, S_y and S_z with centers x, y and z respectively. Set $A = V(S_x) - x$, $B = V(S_y) - x$, $C = V(S_z) - z$ and $K = A \cup B \cup C \cup \{x, y, z\}$. Let Δ denote the dependency digraph of D. The triangle induced by the vertices x, y and z is either a transitive triangle or a directed triangle.
First we will deal with the case when this triangle is directed, and assume without loss of generality that $x \to y \to z \to x$. This is a particular case of the case when the missing graph is a disjoint union of stars such that, in the induced tournament by the centers of the missing stars, every vertex is a king.

Theorem 3.5. *Let D be an oriented graph missing 3 disjoint stars whose centers form a directed triangle. If Δ has no isolated vertices, then D satisfies SNC.*

Proof.
Claim: The only possible arcs in Δ have the forms $xa \to yb$ or $yb \to zc$ or $zc \to xa$, where $a \in A$, $b \in B$ and $c \in C$.
Proof of the claim: xa can not lose to zc because $z \to x$ and $z \in N^{++}(x)$. Similarly yb can not lose to xa and zc can not lose to yb.

Orient the good missing edges in a convenient way and orient the other edges toward the centers. The obtained digraph T is a tournament. Let L be a median order of T such that the sum of the indices of x, y and z is maximum. Let f denote the feed vertex of L. Due to symmetry, we may assume that f is a whole vertex or $f = x$ or $f = a \in A$.

Suppose f is a whole vertex. Clearly, $N^+(f) = N^+_T(f)$. Suppose $f \to u \to v$ in T. If $(u, v) \in E(D)$ or uv is a good missing edge then $v \in N^+(f) \cup N^{++}(f)$. Otherwise, there is missing edge rs that loses to uv with $r \to v$ and $u \notin N^{++}(r) \cup N^+(r)$. But $f \to u$, then $f \to r$, whence $f \to r \to v$ and $v \in N^+(f) \cup N^{++}(f)$. Thus, $N^{++}_T(f) = N^{++}(f)$ and f has the SNP in D.

Suppose $f = x$. Reorient all the missing edges incident to x toward x. In the new tournament T' we have $N^+(x) = N^+_{T'}(x)$ and x has the SNP in T'. Since $y \in N^+(x)$ and $z \in N^{++}(x)$ we

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have that $N^{++}(x) = N^{++}_{T'}(x)$. Thus x has the SNP in D.

Suppose that $f = a \in A$. Reorient ax toward a. Suppose $a \to u \to v$ in the new tournament T' with $v \neq y$. If $(u, v) \in E(D)$ or uv is a good missing edge then $v \in N^+(a) \cup N^{++}(a)$. Otherwise, there is $b \in B$ and $c \in C$ such that $(u, v) = (c, z)$ and by loses to cz, then $f \to c$ implies that $a \to y$, but $y \to z$, whence $z \in N^{++}(a) \cup N^+(a)$. So the only possible new second out-neighbor of a is y, hence if $y \notin N^{++}_{T'}(a)$ then a has the SNP in D. Suppose $y \in N^{++}_{T'}(a)$. If $d^+_{T'}(a) < d^{++}_{T'}(a)$ then $d^+(a) = d^+_{T'}(a) \leq d^{++}_{T'}(a) = d^{++}(a)$, hence a has the SNP in D. Otherwise, $d^+_{T'}(a) = |G_L|$ and $G_L = N^{++}_{T'}(a)$. So x, y and z are not bad vertices, hence the index of each increases in the median order $Sed(L)$ of T'. But the index of a is less than the index of x, then we can give ax its initial orientation as in T nd the same order $Sed(L)$ is a median order of T. However, the sum of indices of x, y and z has increased. A contradiction. Thus f has the SNP in D and D satisfies SNC. □

Theorem 3.6. *Let D be an oriented graph missing 3 disjoint stars whose centers form a directed triangle. If Δ has neither a source nor a sink and D has no sink, then D has at least two vertices with the SNP.*

Proof. **Claim 1:** For every $a \in A$, $b \in B$ and $c \in C$ we have:
$b \to x \to c \to y \to a \to z \to b$.
Proof of Claim 1: This is due to the claim in the previous proof and the condition that Δ has neither a source nor a sink.

Claim 2: If $K = V(D)$ then D has at least 3 vertices with the SNP.
Proof of Claim 2: Let $H = D - \{x, y, z\}$. H is a tournament with no sink (dominated vertex). Then H has 2 vertices u and v with SNP in H. Without loss of generality we may assume that $u \in A$. But $y \to u \to z$, the adding the vertices x, y and z makes u gains only one vertex to its first out-neighborhood and x to its second out-neighborhood. Thus, also u has the SNP in D. Similarly, v has the SNP in D. Suppose, without loss of generality, that $|A| \geq |C|$. We have $C \cup \{y\} = N^+(x)$ and $A \cup \{z\} = N^{++}(x)$. Hence, $d^+(x) = |C| + 1 \leq |A| + 1 \leq d^{++}(x)$, whence, x has the SNP in D.

Claim 3: D is a good oriented graph.
Proof of Claim 3: Let $\mathcal{I}_\mathcal{D}$ be the interval graph of D. Let C_1 and C_2 be two distinct connected components of Δ. The three centers of the missing disjoint stars must appear in each of the these two connected components, whence $K(C_1) \cap K(C_2) \neq \phi$. Therefore, $\mathcal{I}_\mathcal{D}$ is a connected graph, having only one connected component ξ. Then, $K = K(\xi)$.
So if Δ is composed of non trivial strongly connected components, the result holds by proposition 3.1.
Due to the condition that Δ has neither a source nor a sink, Δ has a non trivial strongly connected component C.

Since x, y and z must appear in C, we have $N^+(x)\backslash K = N^+(y)\backslash K = N^+(z)\backslash K$. Now let $v \in K$. If v appears in a non trivial strongly connected component of Δ then $N^+(v)\backslash K = N^+(x)\backslash K = N^+(y)\backslash K = N^+(z)\backslash K$.

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Otherwise,due to the condition that Δ has neither a source nor a sink, v appears in a directed path P of Δ joining two non trivial strongly connected components C_1 and C_2 of Δ. By the definition of losing relations we can prove easily that for all $a \in K(C_1)$, $b \in K(P)$ and $c \in K(C_2)$ we have $N^+(a)\backslash K(\xi) \subseteq N^+(b)\backslash K(\xi) \subseteq N^+(c)\backslash K(\xi)$. In particular, for $a = x = c$ and $b = v$. So $N^+(v)\backslash K = N^+(x)\backslash K$. Similarly, $N^-(v)\backslash K = N^-(x)\backslash K$. This proves claim 3.

To conclude we apply the same argument of the proof of theorem 3.4. □

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