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Stars in forbidden triples generating a finite set of graphs with minimum degree four

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Abstract

For a family $\mathcal{H}$ of graphs, a graph G is said to be $\mathcal{H}$-free if G contains no member of $\mathcal{H}$ as a induced subgraph. Let $\tilde{\mathcal{G}}_4(\mathcal{H})$ denote the family of connected $\mathcal{H}$-free graphs having minimum degree at least 4. In this paper, we characterize the families $\mathcal{H}$ of connected graphs with $|\mathcal{H}| = 3$ such that $\mathcal{H}$ contains a star and $\tilde{\mathcal{G}}_4(\mathcal{H})$ is a finite family, except for the case where $\{K_4, K_{1,n}\} \subseteq \mathcal{H}$ with $3 \leq n \leq 4$.

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1. Introduction

In this paper, we consider only finite undirected simple graphs. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For a vertex $u \in V(G)$, let $N_G(u)$ and $d_G(u)$ denote the neighborhood and the degree of u, respectively; thus $N_G(u) = \{v \in V(G) \mid uv \in E(G)\}$ and $d_G(u) = |N_G(u)|$. We let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of G, respectively. For a subset U of $V(G)$, let $N_G(U) = \bigcup_{u \in U} N_G(u)$, and let $G[U]$ denote the subgraph of G induced by U. For two subset U, U' of $V(G)$ with $U \cap U' = \emptyset$,

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$E(U,U') := \{xy \in E(G) \mid x \in U, y \in U'\}$. For $u, v \in V(G)$, let $\mathrm{dist}_G(u,v)$ denote the length of a shortest u-v path of G. Let $\mathrm{diam}(G)$ denote the maximum of $\mathrm{dist}_G(u,v)$ among all pairs $u, v \in V(G)$. For two positive integers s_1 and s_2, the Ramsey number $R(s_1, s_2)$ is the minimum positive integer R such that any graph of order at least R contains a clique of order s_1 or an independent set of cardinality s_2. For terms and symbols not defined in this paper, we refer the reader to [2].

For two graphs G and H, we write $H \prec G$ if G contains an induced copy of H, and G is said to be H-$free$ if $H \not\prec G$. For a family $\mathcal{H}$ of graphs, a graph G is said to be $\mathcal{H}$-$free$ if G is H-free for every $H \in \mathcal{H}$. In this context, the members of $\mathcal{H}$ are called $forbidden\ subgraphs$ of G. For an integer $k \geq 1$ and a family $\mathcal{H}$ of graphs, we let $\mathcal{G}_k$ (resp. $\tilde{\mathcal{G}}_k$) denote the set of all k-connected graphs (resp. connected graphs with minimum degree at least k), and let $\mathcal{G}_k(\mathcal{H})$ (resp. $\tilde{\mathcal{G}}_k(\mathcal{H})$) denote the set of all $\mathcal{H}$-free graphs belonging to $\mathcal{G}_k$(resp. $\tilde{\mathcal{G}}_k$), that is,

$$\begin{aligned}\mathcal{G}_k(\mathcal{H}) &:= \{G \mid G \text{ is a } k\text{-connected } \mathcal{H}\text{-free graph}\}\\ \tilde{\mathcal{G}}_k(\mathcal{H}) &:= \{G \mid G \text{ is a connected } \mathcal{H}\text{-free graph with minimum degree at least } k\}\end{aligned}$$

The main aim of this paper is to characterize the families $\mathcal{H}$ of connected graphs satisfying the condition that

$$|\mathcal{H}| = 3 \text{ and } \tilde{\mathcal{G}}_4(\mathcal{H}) \text{ is a finite family,} \tag{1.1}$$

in the case where $\mathcal{H}$ contains a star. If a complete graph of order at most 2 belongs to $\mathcal{H}$, then $\tilde{\mathcal{G}}_4(\mathcal{H})$ is clearly empty. Thus, in the rest of this paper, we consider the case where every graph belongs to $\mathcal{H}$ has order at least 3.

Our motivation derives from characterization of families $\mathcal{H}$ of connected graphs satisfying the condition that

$$\mathcal{G}_k(\mathcal{H}) \text{ is a finite family.} \tag{1.2}$$

If a family $\mathcal{H}$ satisfies (1.2), then for any property P on graphs, although the proposition that

$$\text{all } k\text{-connected } \mathcal{H}\text{-free graphs satisfy } P \text{ with finite exceptions} \tag{1.3}$$

holds, the proposition gives no information about P. Thus, it is important to identify families $\mathcal{H}$ satisfying (1.2) in advance. Having such a motivation, Fujisawa, Plummer and Saito [8] started a study of families $\mathcal{H}$ satisfying (1.2), and determined the families $\mathcal{H}$ satisfying (1.2) for the case where $1 \leq k \leq 6$ and $|\mathcal{H}| \leq 2$, and $(k, |\mathcal{H}|) = (2,3)$. In [1, 3, 4, 7], the research was continued by analyzing families $\mathcal{H}$ satisfying (1.2) for the case where $(k, |\mathcal{H}|) = (3,3), (4,3)$. In view of the fact that connectivity conditions can often be replaced by minimum degree condition in propositions like (1.3), it is natural to consider connected graphs with minimum degree at least k in place of k-connected graphs. Thus, we consider the characterization of families $\mathcal{H}$ of connected graphs satisfying the condition that

$$\tilde{\mathcal{G}}_k(\mathcal{H}) \text{ is a finite family.} \tag{1.4}$$

In fact, Y. Egawa and M. Furuya [5, 6] characterized the families $\mathcal{H}$ of connected graphs satisfying condition (1.4) with $(k, |\mathcal{H}|) = (3,3)$ except for a special case. Therefore, we consider the characterization of families $\mathcal{H}$ of connected graphs satisfying condition (1.4) with $(k, |\mathcal{H}|) = (4,3)$, that is, characterization of the families $\mathcal{H}$ of connected graphs satisfying condition (1.1).

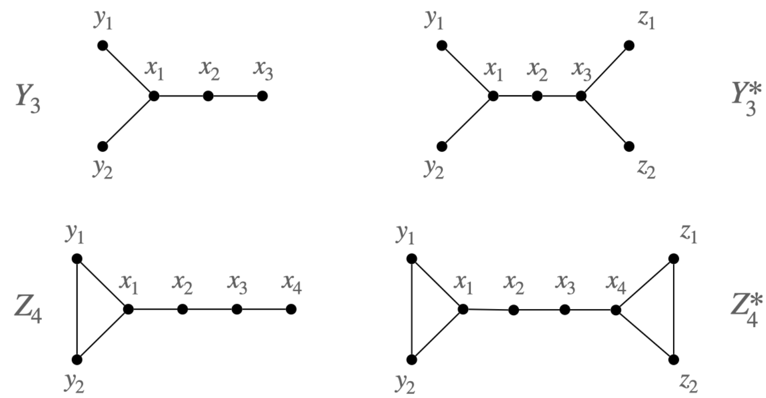

Figure 1. Graphs Y_3, Y_3^*, Z_4 and Z_4^*

Before describing several results, we define some graphs. Let n be an integer with $n > 2$. Let $P = x_1x_2...x_n$ be the path of order n, and let y_1, y_2, z_1 and z_2 be four distinct vertices different from $x_1, ..., x_n$. Let Y_n, Y_n^*, Z_n and Z_n^* denote the graphs defined by

$$V(Y_n) = V(Z_n) = V(P) \cup \{y_1, y_2\},\ V(Y_n^*) = V(Z_n^*) = V(P) \cup \{y_1, y_2, z_1, z_2\},$$
$$E(Y_n) = E(P) \cup \{x_1y_1, x_1y_2\},\ E(Y_n^*) = E(P) \cup \{x_1y_1, x_1y_2, x_nz_1, x_nz_2\},$$
$$E(Z_n) = E(Y_n) \cup \{y_1y_2\},\ \text{and } E(Z_n^*) = E(Y_n^*) \cup \{y_1y_2, z_1z_2\}.\text{(see Figure 1)}$$

Let A_1, A_2, A_3 and A_4 be the graphs depicted in Figure 2. We call vertices x and y in Figure 2 the $first\ vertex$ and the $last\ vertex$ of $A_i (i = 1, 2, 3, 4)$, respectively.

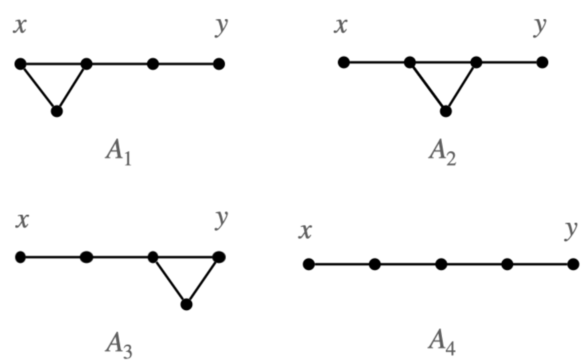

Figure 2. Graphs A_1, A_2, A_3 and A_4

For an integer $s \geq 1$, let $\mathcal{J}_s$ be the family of the graphs obtained from s pairwise vertex-disjoint copies $L_1, ..., L_s$ of A_1, A_2, A_3 or A_4 by the following construction:

- Let $x^{(i)}$ and $y^{(i)}$ be two vertices of L_i where $x^{(i)}$ and $y^{(i)}$ respectively correspond to x, y ;

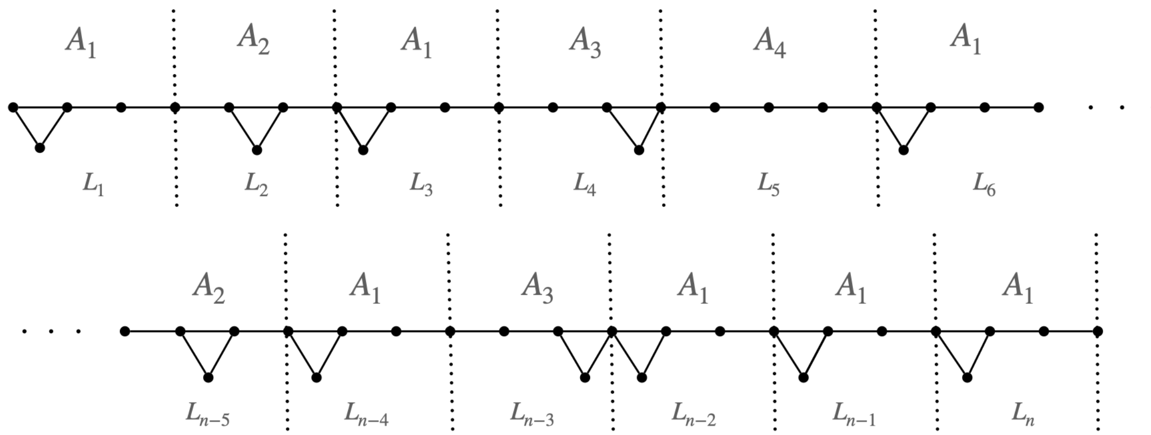

Figure 3. Graph belonging to $\mathcal{J}$

- For each integer i with $2 \leq i \leq s$, we identify $y^{(i-1)}$ with $x^{(i)}$.

Let $\mathcal{J} := \bigcup_{i \in \mathbb{N}} \mathcal{J}_i$ (see Figure 3). (Note that $\mathcal{J}$ can be expressed as the family of the graphs obtained by combining with some graphs of the form Z_t with $t \geq 1$.)

Let G be a connected graph. A vertex v of G is called a $cutvertex$ if $G - v$ is disconnected. If G has a cutvertex, G is said to be $separable$; otherwise, it is said to be $nonseparable$. Note that K_1 is a nonseparable graph. A maximal nonseparable subgraph of G is called a $block$ of G. When G is separable, the block-cutvertex graph of G is defined to be the bipartite graph Z such that Z has as its partite sets the set of all cutvertices of G and the set of all blocks of G and, for a cutvertex v and a block B, v and B are adjacent in Z if and only if v is a vertex of B in G. It is a well-known fact that the block-cutvertex graph of a connected graph is a tree.

Let K_l, K_{m_1,m_2}, P_n denote the complete graph of order l, the complete bipartite graph with partite sets having cardinalities m_1 and m_2, and the path of order n, respectively. A complete bipartite graph of the form $K_{1,m}$ with $m \geq 1$ is called a $star$. A $caterpillar$ is a tree for which the removal of all endvertices leaves a path. A $cactus$ is a connected graph every block of which is a complete graph of order two or a cycle.

We shall use the following sets in the discussion that will follow. Let $\mathcal{T}_0$ be the set of trees, are none of $K_{1,2}$, $K_{1,3}$ and $K_{1,4}$, having order greater than or equal to three and maximum degree at most 4. Note that $\mathcal{T}_0$ dose not contain a star. Let $\mathcal{T}_1$ be the set of those caterpillars belongs to $\mathcal{T}_0$ in which the vertices of degree 4 and the vertices of degree 3 or higher are not adjacent, and no three vertices of degree 3 are contiguously adjacent. Let $\mathcal{T}_2 = \{P_l, Y_m, Y_n^* \mid l \geq 4, m \geq 3, n \geq 2\}$.

Let $\mathcal{T}_0^*$ be the set of those cacti T having order greater than or equal to four such that all cycle of T are triangles. Let $\mathcal{T}_1^*$ be the set of those members of $\mathcal{T}_0^*$ whose block-cutvertex graph is a path. Let $\mathcal{T}_2^* = \{P_l, Z_m, Z_{2n}^* \mid l \geq 4, m \geq 2, n \geq 1\}$. We have $\mathcal{T}_0^* \supseteq \mathcal{T}_1^* \supseteq \mathcal{T}_2^*$.

Our main result is as follows.

Theorem 1.1. *Let l, n be integers with $l \geq 3, n \geq 2$, and let T be a connected graph of order at least 3. If $\tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, T\})$ is a finite family, then one of the following holds:*

(i) $n = 2$ *or T is a path;*

(ii) $n \geq 5$, $l = 3$, *and $T \prec Y_t^*$ for some integer $t \geq 2$ with $t \not\equiv 1 \pmod 4$;*

(iii) $3 \leq n \leq 4$, $l = 4$, *and $T \prec J$ for some graph $J \in \mathcal{J}$;*

(iv) $3 \leq n \leq 4$, $l \geq 5$, *and $T \prec Z_t$ for some integer $t \geq 2$;*

(v) $(l, n) = (3, 3), (3, 4)$.

We think (iii) of Theorem 1.1 is far from sufficient condition. Therefore, we consider the converse of Theorem 1.1 except for (iii). The result is the following.

Theorem 1.2. *Let l, n be integers with $l \geq 3, n \geq 2$, $(l, n) \neq (4, 3), (4, 4)$, and let T be a connected graph of order at least 3. Then $\tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, T\})$ is a finite family if and only if one of the following holds:*

(i) $n = 2$ *or T is a path;*

(ii) $n \geq 5$, $l = 3$, *and $T \prec Y_t^*$ for some integer $t \geq 2$ with $t \not\equiv 1 \pmod 4$;*

(iii) $3 \leq n \leq 4$, $l \geq 5$, *and $T \prec Z_t$ for some integer $t \geq 2$;*

(iv) $(l, n) = (3, 3), (3, 4)$.

We prove Theorem 1.1 in Section 3. We prove Theorem 1.2 in Section 4.

2. Preliminary resullts

The following result can be found in [2].

Lemma 2.1 (Diestel [2, Proposition 9.4.1]). *Let l, n and t be integers with $l \geq 3, n \geq 2$ and $t \geq 3$. Then $\tilde{\mathcal{G}}_1(\{K_l, K_{1,n}, P_t\})$ is a finite family.*

The following result can be found in [1].

Lemma 2.2 (Buelban et al.[1, Theorems 10-12]). *Let l, n be integers with $l \geq 3, n \geq 2$, and let T be a connected graph of order at least 3. If $\mathcal{G}_4(\{K_l, K_{1,n}, T\})$ is a finite family, then one of the following holds:*

(a) $l \geq 4, n \geq 5, T$ *is a path;*

(b) $l = 3, n \geq 5, T \in \mathcal{T}_2$;

(c) $l \geq 4$, $3 \leq n \leq 4$, *and* $\begin{cases} T \in \mathcal{T}_1^*, & l = 4, \\ T \in \mathcal{T}_2^*, & l \geq 5, \end{cases}$

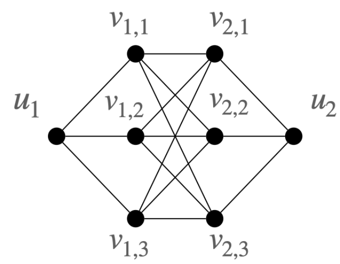

Figure 4. Graph B

(d) $n = 2$ *or* $(l, n) = (3, 3), (3, 4)$.

In [3, Case 1 in Lemma 5.1], a proposition which asserts that we have $\mathrm{diam}(G) \leq 7$ for a 3-connected $\{K_3, Y_3^*\}$-free graph G was proved. In the proof of the proposition, the 3-connectedness of G was used only to ensure that $\delta(G) \geq 3$. Thus we obtain the following lemma.

Lemma 2.3 (Egawa et al.[3, Case 1 in Lemma 5.1])**.** *Let G be a connected $\{K_3, Y_3^*\}$-free graph with $\delta(G) \geq 4$. Then* $\mathrm{diam}(G) \leq 7$.

The following result can be found in [5].

Lemma 2.4 (Egawa and Furuya [5, Proposition 3.6])**.** *Let t be an integer with $t \geq 4$ and $t \not\equiv 1 \pmod 4$, and let*

$$d = \begin{cases} 13, & t = 4, \\ 2t + 3, & t \geq 6. \end{cases}$$

Let G be a connected $\{K_3, Y_t^*\}$-free graph with $\delta(G) \geq 4$. Then $\mathrm{diam}(G) \leq d$.

3. Proof of Theorem 1.1

Let B be the graph depicted in Figure 4. For an integer $s \geq 2$, let $H_s^{(1)}$ be the obtained from s pairwise vertex-disjoint copies $B_1, B_2, ..., B_s$ of B such that $V(B_i) = \{u_j^{(i)}, v_{j,h}^{(i)} \mid 1 \leq j \leq 2, 1 \leq h \leq 3\}$ where $u_j^{(i)}$ and $v_{j,h}^{(i)}$ respectively correspond to u_j and $v_{j,h}$, by adding edges $u_2^{(i)} u_1^{(i+1)}$ where indices i and $i+1$ are read modulo s.

Lemma 3.1. *Let s be an integer with $s \geq 2$, and let T be a graph such that $T \prec H_s^{(1)}$. If $T \in \mathcal{T}_2$, then $T \prec Y_t^*$ for some integer $t \geq 2$ with $t \not\equiv 1 \pmod 4$.*

Proof. First assume that $T \simeq P_l$ $(l \geq 4)$. If $l - 2 \not\equiv 1 \pmod 4$, then letting $t = l - 2$, we get $t \not\equiv 1 \pmod 4$ and $T \simeq P_{t+2} \prec Y_t^*$; if $l - 2 \equiv 1 \pmod 4$, then letting $t = l - 1$, we get $t \not\equiv 1 \pmod 4$ and $T \simeq P_{t+1} \prec P_{t+2} \prec Y_t^*$. Next assume that $T \simeq Y_m$ $(m \geq 3)$. If $m - 1 \not\equiv 1 \pmod 4$, then letting $t = m - 1$, we get $t \not\equiv 1 \pmod 4$ and $T \simeq Y_{t+1} \prec Y_t^*$; if $m - 1 \equiv 1 \pmod 4$, then letting $t = m$, we get $t \not\equiv 1 \pmod 4$ and $T \simeq Y_t \prec Y_{t+1} \prec Y_t^*$. Therefore, we may assume $T \simeq Y_{n'}^*$ for $n' \geq 2$. Then $\Delta(T) = 3$. Now, we divide the proof into the following two cases:

Case 1. $u_j^{(i)} \in V(T)$ and $d_T(u_j^{(i)}) = 3$ for some $1 \leq i \leq s$ and $1 \leq j \leq 2$.

Without loss of generality, we may assume $u_2^{(s)} \in V(T)$ and $d_T(u_2^{(s)}) = 3$. First, we assume $\{v_{2,1}^{(s)}, v_{2,2}^{(s)}, v_{2,3}^{(s)}\} \subseteq V(T)$. Since T has no cycle, $v_{1,p}^{(s)} \notin V(T)$ for any $1 \leq p \leq 3$. Then $T \simeq K_{1,3}$, which contradicts $T \simeq Y_{n'}^*$. Hence, we may assume $\{v_{2,1}^{(s)}, v_{2,2}^{(s)}, u_1^{(1)}\} \subseteq V(T)$. Since T has no cycle, we have $v_{1,p}^{(s)} \notin V(T)$ for any $1 \leq p \leq 3$. Let i_1 be the maximum integer such that $1 \leq i_1 \leq s$ and $u_1^{(i_1)} \in V(T)$. Since T has no cycle, $|V(T) \cap \{v_{j,1}^{(i)}, v_{j,2}^{(i)}, v_{j,3}^{(i)}\}| = 1$ for any $1 \leq i \leq i_1 - 1$ and $1 \leq j \leq 2$. Therefore, we may assume $\{v_{j,1}^{(i)} \mid 1 \leq i \leq i_1 - 1, 1 \leq j \leq 2\} \subseteq V(T)$. If $u_1^{(s)} \in V(T)$, then $T \simeq Y_{4s-2}$. Thus we may assume $i_1 \neq s$. If $V(T) \cap \{v_{2,1}^{(i_1)}, v_{2,2}^{(i_1)}, v_{2,3}^{(i_1)}\} = \emptyset$, then $V(T) \subseteq \{u_j^{(i)}, v_{j,1}^{(i)} \mid 1 \leq i \leq i_1 - 1, 1 \leq j \leq 2\} \cup \{u_2^{(s)}, v_{2,1}^{(s)}, v_{2,2}^{(s)}, u_1^{(i_1)}, v_{1,1}^{(i_1)}, v_{1,2}^{(i_1)}, v_{1,3}^{(i_1)}\}$, which implies that $T \prec Y_{4i_1-2}^*$. Thus we may assume $V(T) \cap \{v_{2,1}^{(i_1)}, v_{2,2}^{(i_1)}, v_{2,3}^{(i_1)}\} \neq \emptyset$, say $v_{2,1}^{(i_1)} \in V(T)$. Since T has no cycle, we have $|V(T) \cap \{v_{1,1}^{(i_1)}, v_{1,2}^{(i_1)}, v_{1,3}^{(i_1)}\}| = 1$, say $v_{1,1}^{(i_1)} \in V(T)$. Since T has no cycle and $\Delta(T) = 3$, we have $|V(T) \cap \{v_{2,2}^{(i_1)}, v_{2,3}^{(i_1)}, u_2^{(i_1)}\}| \leq 1$. Then

$$V(T) \subseteq \{u_j^{(i)}, v_{j,1}^{(i)} \mid 1 \leq i \leq i_1 - 1, 1 \leq j \leq 2\} \cup \{u_2^{(s)}, v_{2,1}^{(s)}, v_{2,2}^{(s)}, u_1^{(i_1)}, v_{1,1}^{(i_1)}, v_{2,1}^{(i_1)}, v_{2,2}^{(i_1)}\},$$
$$V(T) \subseteq \{u_j^{(i)}, v_{j,1}^{(i)} \mid 1 \leq i \leq i_1 - 1, 1 \leq j \leq 2\} \cup \{u_2^{(s)}, v_{2,1}^{(s)}, v_{2,2}^{(s)}, u_1^{(i_1)}, v_{1,1}^{(i_1)}, v_{2,1}^{(i_1)}, v_{2,3}^{(i_1)}\}, \text{ or}$$
$$V(T) \subseteq \{u_j^{(i)}, v_{j,1}^{(i)} \mid 1 \leq i \leq i_1, 1 \leq j \leq 2\} \cup \{u_2^{(s)}, v_{2,1}^{(s)}, v_{2,2}^{(s)}\},$$

which implies that $T \prec Y_{4i_1-1}^*$ or $T \prec Y_{4i_1+1} (\prec Y_{4i_1}^*)$. Consequently, we obtain the desired conclusion.

Case 1. $d_T(u_j^{(i)}) \leq 2$ for any $1 \leq i \leq s$ and $1 \leq j \leq 2$ with $u_j^{(i)} \in V(T)$.
We may assume $v_{2,1}^{(s)} \in V(T)$ and $d_T(v_{2,1}^{(s)}) = 3$. First, we assume $\{v_{1,1}^{(s)}, v_{1,2}^{(s)}, v_{1,3}^{(s)}\} \subseteq V(T)$. Since T has no cycle, $u_1^{(s)}, v_{2,2}^{(s)}, v_{2,3}^{(s)} \notin V(T)$. Then $T \simeq K_{1,3}$, which contradicts the assumption that $T \simeq Y_{n'}^*$. Hence, we may assume $\{v_{1,1}^{(s)}, v_{1,2}^{(s)}, u_2^{(s)}\} \subseteq V(T)$. Since T has no cycle, we have $u_1^{(s)}, v_{2,2}^{(s)}, v_{2,3}^{(s)} \notin V(T)$. Let i_2 be the minimum integer such that $1 \leq i_2 \leq s$ and $u_1^{(i_2)} \notin V(T)$. If $i_2 = 1$, then $T \simeq K_{1,3}$, which contradicts the assumption that $T \simeq Y_{n'}^*$. Thus we may assume that $i_2 \geq 2$. Since T has no cycle, $|V(T) \cap \{v_{j,1}^{(i)}, v_{j,2}^{(i)}, v_{j,3}^{(i)}\}| = 1$ for any $1 \leq i \leq i_2 - 2$ and $1 \leq j \leq 2$. Therefore, we may assume $\{v_{j,1}^{(i)} \mid 1 \leq i \leq i_2 - 2, 1 \leq j \leq 2\} \subseteq V(T)$. If $V(T) \cap \{v_{1,1}^{(i_2-1)}, v_{1,2}^{(i_2-1)}, v_{1,3}^{(i_2-1)}\} = \emptyset$, $V(T) = \{u_j^{(i)}, v_{j,1}^{(i)} \mid 1 \leq i \leq i_2 - 2, 1 \leq j \leq 2\} \cup \{v_{1,1}^{(s)}, v_{1,2}^{(s)}, v_{2,1}^{(s)}, u_2^{(s)}, u_1^{(i_2-1)}\}$, which implies that $T \simeq Y_{4i_2-5} (\prec Y_{4i_2-6}^*)$. Thus we assume $V(T) \cap \{v_{1,1}^{(i_2-1)}, v_{1,2}^{(i_2-1)}, v_{1,3}^{(i_2-1)}\} \neq \emptyset$. Since $d_T(u_1^{(i_2-1)}) \leq 2$, $|V(T) \cap \{v_{1,1}^{(i_2-1)}, v_{1,2}^{(i_2-1)}, v_{1,3}^{(i_2-1)}\}| = 1$, say $v_{1,1}^{(i_2-1)} \in V(T)$. If $V(T) \cap \{v_{2,1}^{(i_2-1)}, v_{2,2}^{(i_2-1)}, v_{2,3}^{(i_2-1)}\} = \emptyset$, $V(T) = \{u_j^{(i)}, v_{j,1}^{(i)} \mid 1 \leq i \leq i_2 - 2, 1 \leq j \leq 2\} \cup \{v_{1,1}^{(s)}, v_{1,2}^{(s)}, v_{2,1}^{(s)}, u_2^{(s)}, u_1^{(i_2-1)}, v_{1,1}^{(i_2-1)}\}$, which implies that $T \simeq Y_{4i_2-4} (\prec Y_{4i_2-5}^*)$. Thus we may assume $V(T) \cap \{v_{2,1}^{(i_2-1)}, v_{2,2}^{(i_2-1)}, v_{2,3}^{(i_2-1)}\} \neq \emptyset$, say $v_{2,1}^{(i_2-1)} \in V(T)$. Since T has no cycle and

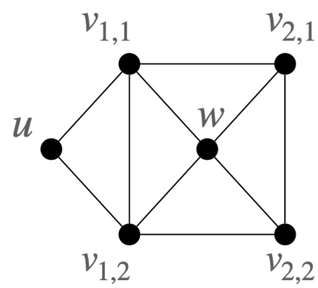

Figure 5. Graph C

$\Delta(T)=3$, we have $|V(T)\cap\{v_{2,2}^{(i_2-1)},v_{2,3}^{(i_2-1)},u_2^{(i_2-1)}\}|\le 1$. Then

$$V(T)\subseteq\{u_j^{(i)},v_{j,1}^{(i)}\mid 1\le i\le i_2-2,1\le j\le 2\}\cup\{v_{1,1}^{(s)},v_{1,2}^{(s)},v_{2,1}^{(s)},u_2^{(s)},u_1^{(i_2-1)},v_{1,1}^{(i_2-1)},v_{2,1}^{(i_2-1)},v_{2,2}^{(i_2-1)}\},$$
$$V(T)\subseteq\{u_j^{(i)},v_{j,1}^{(i)}\mid 1\le i\le i_2-2,1\le j\le 2\}\cup\{v_{1,1}^{(s)},v_{1,2}^{(s)},v_{2,1}^{(s)},u_2^{(s)},u_1^{(i_2-1)},v_{1,1}^{(i_2-1)},v_{2,1}^{(i_2-1)},v_{2,3}^{(i_2-1)}\},$$

or

$$V(T)\subseteq\{u_j^{(i)},v_{j,1}^{(i)}\mid 1\le i\le i_2-1,1\le j\le 2\}\cup\{v_{1,1}^{(s)},v_{1,2}^{(s)},v_{2,1}^{(s)},u_2^{(s)}\},$$

which implies that $T\prec Y^*_{4i_2-4}$ or $T\prec Y_{4i_2-2}(\prec Y^*_{4i_2-3})$. Consequently, we obtain the desired conclusion. □

Let C be the graph depicted in Figure 5. For an integer $s\ge 2$, let $H_s^{(2)}$ be the graph obtained from s pairwise vertex-disjoint copies $C_1,C_2,...,C_s$ of C such that $V(C_i)=\{u^{(i)},w^{(i)},v_{j,h}^{(i)}\mid 1\le j\le 2,1\le h\le 2\}$ where $u^{(i)},w^{(i)}$ and $v_{j,h}^{(i)}$ respectively correspond to u,w and $v_{j,h}$, by adding edges $v_{2,1}^{(i)}u^{(i+1)},v_{2,2}^{(i)}u^{(i+1)}$ where indices i and $i+1$ are read modulo s.

Lemma 3.2. *Let s be an integer with $s\ge 2$, and let T be a graph such that $T\prec H_s^{(2)}$. If $T\in\mathcal{T}_0^*$ and $3(s-1)>|V(T)|$, then $T\prec J$ for some graph $J\in\mathcal{J}$.*

Proof. Since $T\in\mathcal{T}_0^*$ (note that $|V(T)|\ge 4$), there exists the integer i with $1\le i\le s$ such that $u^{(i)}\in V(T)$. Hence, since $3(s-1)>|V(T)|$, there exists the integer i_3 with $1\le i_3\le s$ such that $u^{(i_3)}\notin V(T)$ and $u^{(i_3+1)}\in V(T)$. Without loss of generality, we may assume $i_3=s$. Let i_4 be the maximum integer such that $1\le i_4\le s-1$ and $u^{(i_4)}\in V(T)$. For each $1\le i\le i_4$ and $i=s$, let $T^{(i)}:=T\cap G[V(C_i)\cup\{u_{i+1}\}]$. Note that $T=T^{(s)}\cup(\bigcup_{1\le i\le i_4}T^{(i)})$.

Claim 1. *One of the following holds:*

(i) $T^{(s)}\simeq K_1$*;*

(ii) $T^{(s)}\prec A_4$ *and* $u^{(1)}$ *is the last vertex of* A_4*;*

(iii) $T^{(s)}\prec A_2$ *and* $u^{(1)}$ *is the last vertex of* A_2*; or*

(iv) $T^{(s)}\prec A_3$ *and* $u^{(1)}$ *is the last vertex of* A_3*.*

Proof. If $|V(T^{(s)}) \cap \{v_{2,1}^{(s)}, v_{2,2}^{(s)}\}| = 0$, then $V(T^{(s)}) = \{u^{(1)}\}$, and hence (i) holds. Thus, we may assume $1 \le |V(T^{(s)}) \cap \{v_{2,1}^{(s)}, v_{2,2}^{(s)}\}| \le 2$.

Case 1. $|V(T^{(s)}) \cap \{v_{2,1}^{(s)}, v_{2,2}^{(s)}\}| = 1$.
Without loss of generality, we may assume $v_{2,1}^{(s)} \in V(T^{(s)})$.

In the case where $|V(T^{(s)}) \cap \{v_{1,1}^{(s)}, w^{(s)}\}| = 0$, we have $V(T^{(s)}) = \{v_{2,1}^{(s)}, u^{(1)}\}$,which implies that $T^{(s)} \prec A_4$. Thus (ii) holds.

In the case where $|V(T^{(s)}) \cap \{v_{1,1}^{(s)}, w^{(s)}\}| = 1$, without loss of generality, we may assume $v_{1,1}^{(s)} \in V(T^{(s)})$. Then $V(T^{(s)}) = \{v_{1,1}^{(s)}, v_{2,1}^{(s)}, u^{(1)}\}$ or $V(T^{(s)}) = \{v_{1,1}^{(s)}, v_{1,2}^{(s)}, v_{2,1}^{(s)}, u^{(1)}\}$,which implies that $T^{(s)} \prec A_4$. Thus (ii) holds.

In the case where $|V(T^{(s)}) \cap \{v_{1,1}^{(s)}, w^{(s)}\}| = 2$, we have $v_{1,2}^{(s)} \notin V(T^{(s)})$ since $T \in \mathcal{T}_0^*$. Hence $V(T^{(s)}) = \{v_{1,1}^{(s)}, w^{(s)}, v_{2,1}^{(s)}, u^{(1)}\}$, which implies that $T^{(s)} \prec A_2$. Thus (iii) holds.

Case 2. $|V(T^{(s)}) \cap \{v_{2,1}^{(s)}, v_{2,2}^{(s)}\}| = 2$.
In this case, we have $w^{(s)} \notin V(T^{(s)})$ and $|V(T^{(s)}) \cap \{v_{1,1}^{(s)}, v_{1,2}^{(s)}\}| \le 1$ since $T \in \mathcal{T}_0^*$. In the case where $|V(T^{(s)}) \cap \{v_{1,1}^{(s)}, v_{1,2}^{(s)}\}| = 0$, we have $V(T^{(s)}) = \{v_{2,1}^{(s)}, v_{2,2}^{(s)}, u^{(1)}\}$,which implies that $T^{(s)} \prec A_3$. If $|V(T^{(s)}) \cap \{v_{1,1}^{(s)}, v_{1,2}^{(s)}\}| = 1$, we may assume $v_{1,1}^{(s)} \in V(T^{(s)})$. Then $V(T^{(s)}) = \{v_{1,1}^{(s)}, v_{2,1}^{(s)}, v_{2,2}^{(s)}, u^{(1)}\}$,which implies that $T^{(s)} \prec A_3$. Consequently, (iv) holds. □

Claim 2. *One of the following holds:*

(i) $T^{(i_4)} \simeq K_1$*;*

(ii) $T^{(i_4)} \prec A_4$ *and* $u^{(i_4)}$ *is the first vertex of* A_4*;*

(iii) $T^{(i_4)} \prec A_2$ *and* $u^{(i_4)}$ *is the first vertex of* A_2*; or*

(iv) $T^{(i_4)} \prec A_1$ *and* $u^{(i_4)}$ *is the first vertex of* A_1*.*

Proof. If $|V(T^{(i_4)}) \cap \{v_{1,1}^{(i_4)}, v_{1,2}^{(i_4)}\}| = 0$, then $V(T^{(i_4)}) = \{u^{(i_4)}\}$, and hence (i) holds. Thus, we may assume that $1 \le |V(T^{(i_4)}) \cap \{v_{1,1}^{(i_4)}, v_{1,2}^{(i_4)}\}| \le 2$.

Case 1. $|V(T^{(i_4)}) \cap \{v_{1,1}^{(i_4)}, v_{1,2}^{(i_4)}\}| = 1$.
Without loss of generality, we may assume $v_{1,1}^{(i_4)} \in V(T^{(i_4)})$.

In the case where $|V(T^{(i_4)}) \cap \{v_{2,1}^{(i_4)}, w^{(i_4)}\}| = 0$, we have $V(T^{(i_4)}) = \{u^{(i_4)}, v_{1,1}^{(i_4)}\}$, which implies $T^{(i_4)} \prec A_4$. Thus, (ii) holds.

In the case where $|V(T^{(i_4)}) \cap \{v_{2,1}^{(i_4)}, w^{(i_4)}\}| = 1$, without loss of generality, we may assume $v_{2,1}^{(i_4)} \in V(T^{(i_4)})$. Then $V(T^{(i_4)}) = \{u^{(i_4)}, v_{1,1}^{(i_4)}, v_{2,1}^{(i_4)}\}$ or $V(T^{(i_4)}) = \{u^{(i_4)}, v_{1,1}^{(i_4)}, v_{2,1}^{(i_4)}, v_{2,2}^{(i_4)}\}$, which implies $T^{(i_4)} \prec A_4$. Thus, (ii) holds.

In the case where $|V(T^{(i_4)}) \cap \{v_{2,1}^{(i_4)}, w^{(i_4)}\}| = 2$, Then we have $v_{2,2}^{(s)} \notin V(T^{(i_4)})$ since $T \in \mathcal{T}_0^*$. Thus $V(T^{(i_4)}) = \{u^{(i_4)}, v_{1,1}^{(i_4)}, v_{2,1}^{(i_4)}, w^{(i_4)}\}$, which implies $T^{(i_4)} \prec A_2$. Thus (iii) holds.

Case 2. $|V(T^{(i_4)}) \cap \{v_{1,1}^{(i_4)}, v_{1,2}^{(i_4)}\}| = 2$.

In this case, we have $w^{(i_4)} \notin V(T^{(i_4)})$ and $|V(T^{(i_4)}) \cap \{v_{2,1}^{(i_4)}, v_{2,2}^{(i_4)}\}| \leq 1$ since $T \in \mathcal{T}_0^*$. In the case where $|V(T^{(i_4)}) \cap \{v_{2,1}^{(i_4)}, v_{2,2}^{(i_4)}\}| = 0$, we have $V(T^{(i_4)}) = \{u^{(i_4)}, v_{1,1}^{(i_4)}, v_{2,1}^{(i_4)}\}$, which implies $T^{(i_4)} \prec A_1$. Thus, (iv) holds.

In the case where $|V(T^{(i_4)}) \cap \{v_{2,1}^{(i_4)}, v_{2,2}^{(i_4)}\}| = 1$, we may assume $v_{2,1}^{(i_4)} \in V(T^{(i_4)})$. Then $V(T^{(i_4)}) = \{u^{(i_4)}, v_{1,1}^{(i_4)}, v_{2,1}^{(i_4)}\}$, which implies $T^{(i_4)} \prec A_1$. Consequently, (iv) holds. □

Claim 3. *For each i with $1 \leq i \leq i_4 - 1$, one of the following holds:*

(i) $T^{(i)} \simeq A_4$, *$u^{(i)}$ is the first vertex of A_4 and $u^{(i+1)}$ is the last vertex of A_4;*

(ii) $T^{(i)} \prec A_3$, *$u^{(i)}$ is the first vertex of A_3 and $u^{(i+1)}$ is the last vertex of A_3;*

(iii) $T^{(i)} \simeq A_2$, *$u^{(i)}$ is the first vertex of A_2 and $u^{(i+1)}$ is the last vertex of A_2; or*

(iv) $T^{(i)} \simeq A_1$, *$u^{(i)}$ is the first vertex of A_1 and $u^{(i+1)}$ is the last vertex of A_1.*

Proof. Since T is connected, we have $u^{(i)}, u^{(i+1)} \in V(T^{(i)})$ and $1 \leq |V(T^{(i)}) \cap \{v_{1,1}^{(i)}, v_{1,2}^{(i)}\}| \leq 2$.

Case 1. $|V(T^{(i)}) \cap \{v_{1,1}^{(i)}, v_{1,2}^{(i)}\}| = 1$.
We may assume that $v_{1,1}^{(i)} \in T^{(i)}$. Since $T^{(i)}$ is connected, we have $1 \leq |V(T^{(i)}) \cap \{w^{(i)}, v_{2,1}^{(i)}\}| \leq 2$.

Suppose that $|V(T^{(i)}) \cap \{w^{(i)}, v_{2,1}^{(i)}\}| = 1$. In the case where $w^{(i)} \in T^{(i)}$, we have $V(T^{(i)}) = \{u^{(i)}, v_{1,1}^{(i)}, w^{(i)}, v_{2,2}^{(i)}, u^{(i+1)}\}$, which implies $T^{(i)} \simeq A_4$. Thus, (i) holds. In the case where $v_{2,1}^{(i)} \in T^{(i)}$, we have $V(T^{(i)}) = \{u^{(i)}, v_{1,1}^{(i)}, v_{2,1}^{(i)}, u^{(i+1)}\}$ or $V(T^{(i)}) = \{u^{(i)}, v_{1,1}^{(i)}, v_{2,1}^{(i)}, v_{2,2}^{(i)}, u^{(i+1)}\}$, which implies $T^{(i)}$ is a path with order 4 or $T^{(i)} \simeq A_3$. Thus (ii) holds.

Suppose that $|V(T^{(i)}) \cap \{w^{(i)}, v_{2,1}^{(i)}\}| = 2$. Since $T^{(i)} \in \mathcal{T}_0^*$, we have $v_{2,2}^{(i)} \notin T^{(i)}$. Hence we have $V(T^{(i)}) = \{u^{(i)}, v_{1,1}^{(i)}, w^{(i)}, v_{2,1}^{(i)}, u^{(i+1)}\}$, which implies $T^{(i)} \simeq A_2$. Thus, (iii) holds.

Case 2. $|V(T^{(i)}) \cap \{v_{1,1}^{(i)}, v_{1,2}^{(i)}\}| = 2$.
Since $T^{(i)}$ is connected and $T^{(i)} \in \mathcal{T}_0^*$, we have $w^{(i)} \notin T^{(i)}$ and $|V(T^{(i)}) \cap \{v_{2,1}^{(i)}, v_{2,2}^{(i)}\}| = 1$. We may assume $v_{2,1}^{(i)} \in V(T^{(i)})$. Then $V(T^{(i)}) = \{u^{(i)}, v_{1,1}^{(i)}, v_{1,2}^{(i)}, v_{2,1}^{(i)}, u^{(i+1)}\}$, which implies $T^{(i)} \simeq A_1$. Consequently, (iv) holds. □

By the definition of $\mathcal{J}$ and Claims 1-3, we obtain $T(= T^{(s)} \cup (\bigcup_{1 \leq i \leq i_4} T^{(i)})) \prec J$ for some graph $J \in \mathcal{J}$. Consequently, the proof of Lemma 3.2 is complete. □

For an integer $s \geq 2$, let $H_s^{(3)}$ be the graph obtained from s pairwise vertex-disjoint copies $D_1, D_2, ..., D_s$ of K_4 such that $V(D_i) = \{u^{(i)}, v_j^{(i)} \mid 1 \leq j \leq 3\}$ by adding edges $u^{(i)}v_1^{(i+1)}$, $u^{(i)}v_2^{(i+1)}$, $u^{(i)}v_3^{(i+1)}$ where indices i and $i+1$ are read modulo s.

Lemma 3.3. *Let s be an integer with $s \geq 2$, and let T be a graph such that $T \prec H_s^{(3)}$. If $T \in \mathcal{T}_2^*$, then $T \prec Z_t$ for some integer $t \geq 2$.*

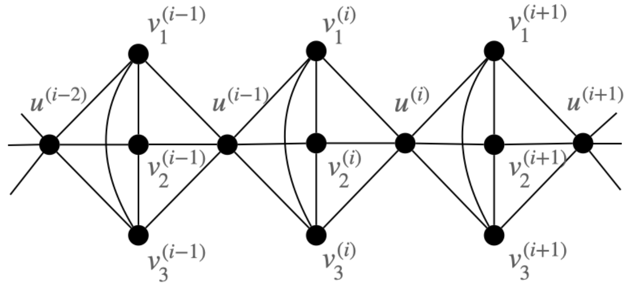

Figure 6. Graph $H_s^{(3)}$

Proof. If T is a path, then $T \prec Z_t$, where t is the minimum integer such that $t \geq |V(T)| - 1$, as desired. If $T \simeq Z_m$ for some integer m with $m \geq 2$, the assertion clearly holds. Thus we may assume $T \simeq Z^*_{n'}$ for $n' \geq 2$. Then T contains a triangle. Without loss of generality, we may assume $\{v_1^{(1)}, v_2^{(1)}, u^{(1)}\} \subseteq V(T)$. Note that $u^{(s)}, v_3^{(1)} \notin V(T)$. Let i_5 be the maximum integer such that $1 \leq i_5 \leq s-1$ and $u^{(i_5)} \in V(T)$. Since $T \simeq Z^*_n$ for some $n \geq 2$, we have $|V(T) \cap \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}\}| = 1$ for any $2 \leq i \leq i_5$. Therefore we may assume $\{v_1^{(i)} \mid 2 \leq i \leq i_5\} \subseteq V(T)$. Since $T \simeq Z^*_n$ for some $n \geq 2$, we have $|V(T) \cap \{v_1^{(i_5+1)}, v_2^{(i_5+1)}, v_3^{(i_5+1)}\}| = 2$, say $\{v_1^{(i_5+1)}, v_2^{(i_5+1)}\} \subseteq V(T)$. Therefore we have $V(T) = \{u^{(i)}, v_1^{(i)} \mid 2 \leq i \leq i_5\} \cup \{u^{(1)}, v_1^{(1)}, v_2^{(1)}, v_1^{(i_5+1)}, v_2^{(i_5+1)}\}$. Hence $T \simeq Z^*_{2i_5-1}$, which contradicts $T \in \mathcal{T}^*_2$. Consequently, we obtain the desired conclusion. □

Proof of Theorem 1.1. Let l, n and T be as in Theorem 1.1, and suppose that $\tilde{\mathcal{G}}_4\{K_l, K_{1,n}, T\}$ is a finite family. Since $\mathcal{G}_4(\{K_l, K_{1,n}, T\}) \subseteq \tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, T\})$, $\mathcal{G}_4(\{K_l, K_{1,n}, T\})$ is also a finite family. It follows from Lemma 2.2 that one of $(a) - (d)$ in Lemma 2.2 holds. If either (a) or (d) in Lemma 2.2 holds, then one of (i) and (v) holds. Thus we may assume that either (b) or (c) in Lemma 2.2 holds.

Case 1. (b) in Lemma 2.2 holds.

By (b) in Lemma 2.2, $T \in \mathcal{T}_2$. For each $s \geq 2$, $H_s^{(1)}$ is a connected $\{K_3, K_{1,5}\}$-free graph with $\delta(H_s^{(1)}) = 4$. Since $\tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, T\})$ is finite, there exists $s_1 \geq 2$ such that $T \prec H_{s_1}^{(1)}$. Then by Lemma 3.1, $T \prec Y^*_t$ for some integer $t \geq 2$ with $t \not\equiv 1 \pmod 4$. Consequently (ii) holds.

Case 2. (c) in Lemma 2.2 holds.

By (c) in Lemma 2.2, $T \in \mathcal{T}^*_1$ or $T \in \mathcal{T}^*_2$.

Subcase 2.1. $T \in \mathcal{T}^*_1$.

Note that $T \in \mathcal{T}^*_0$ since $\mathcal{T}^*_1 \subseteq \mathcal{T}^*_0$. For each $s > \frac{|V(T)|}{3} + 1$, $H_s^{(2)}$ is a connected $\{K_4, K_{1,3}\}$-free graph with $\delta(H_s^{(2)}) = 4$. Since $\tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, T\})$ is finite, there exists $s_2 > \frac{|V(T)|}{3} + 1$ such that $T \prec H_{s_2}^{(2)}$. Then by Lemma 3.2, $T \prec J$ for some graph $J \in \mathcal{J}$. Consequently (iii) holds.

Subcase 2.2. $T \in \mathcal{T}^*_2$.

For each $s \geq 2$, $H_s^{(3)}$ is a connected $\{K_5, K_{1,3}\}$-free graph with $\delta(H_s^{(1)}) \geq 4$. Since $\tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, T\})$ is finite, there exists $s_3 \geq 2$ such that $T \prec H_{s_3}^{(3)}$. Then by Lemma 3.3, $T \prec Z_t$ for some integer $t \geq 2$. Consequently (iv) holds. □

4. Proof of Theorem 1.2

By Theorem 1.1, the "only if" part of Theorem 1.2 holds. In this section, we prove the "if" part of Theorem 1.2. It is suffices to show that

(A1) $\tilde{\mathcal{G}}_4(\{K_l, K_{1,2}, T\})$ $(l \geq 3,\ T$ is a connected graph$)$ are finite families;

(A2) $\tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, P_t\})$ $(l \geq 3,\ n \geq 2,\ t \geq 3)$ are finite families;

(A3) $\tilde{\mathcal{G}}_4(\{K_3, K_{1,n}, Y_t^*\})$ $(n \geq 5,\ t \geq 2$ with $t \not\equiv 1 \pmod 4)$ are finite families;

(A4) $\tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, Z_t\})$ $(l \geq 5,\ 3 \leq n \leq 4,\ t \geq 2)$ are finite families; and

(A5) $\tilde{\mathcal{G}}_4(\{K_3, K_{1,n}, T\})$ $(3 \leq n \leq 4,\ T$ is a connected graph$)$ are finite families.

Since $\tilde{\mathcal{G}}_4(\{K_l.K_{1,2}, T\}) \subseteq \tilde{\mathcal{G}}_1(\{K_l, K_{1,2}\}) = \tilde{\mathcal{G}}_1(\{K_l, K_{1,2}, P_3\})$ and $\tilde{\mathcal{G}}_4(\{K_l, K_{1,n}, P_t\}) \subseteq \tilde{\mathcal{G}}_1(\{K_l, K_{1,n}, P_t\})$, it follows from Lemma 2.1 that (A1) and (A2) hold. Let G be a connected K_3-free graph with $\delta(G) \geq 4$, and let $w \in V(G)$. Then $N_G(w)$ is an independent set of G and there are 4 vertices $x_1, x_2, x_3, x_4 \in N_G(w)$, and $G[\{w, x_1, x_2, x_3, x_4\}] \simeq K_{1,4}$. In particular, for any integer n with $3 \leq n \leq 4$, $\tilde{\mathcal{G}}_4(\{K_3, K_{1,n}\}) = \emptyset$, which proves (A5). Thus, in the remainder of this section, we show that (A3) and (A4) hold.

Let l and n be integers with $l \geq 3$ and $n \geq 3$, and let G be a graph having a vertex w of degree at least $R(l-1, n)$. Then by the definition of $R(l-1, n)$, $N_G(w)$ contains a clique C of order $l-1$ or an independent set I of size n. If the former holds, then $\{w\} \cup C$ induces a copy of K_l in G; if the latter holds, then $\{w\} \cup I$ induces a copy of $K_{1,n}$ in G. This implies that the maximum degree of a $\{K_l, K_{1,n}\}$-free graph is bounded by a constant $R(l-1, n) - 1$ (which depends on l and n). Furthermore, it follows from Lemmas 2.3 and 2.4 that for each $t \geq 3$ with $t \not\equiv 1 \pmod 4$, the diameter of a connected $\{K_3, Y_t^*\}$-free graph is bounded by the constant which depends on t. Since it is known that every connected graph G satisfies $|V(G)| \leq \Delta(G)^{\mathrm{diam}(G)} + 1$ (see, for example,[2]), the following propositions will complete the proof of (A3) and (A4).

- The diameter of all connected $\{K_3, Y_2^*\}$-free graphs G with $\delta(G) \geq 4$ is bounded by a constant (Proposition 4.4).
- For a fixed integer t with $t \geq 2$, the diameter of all connected $\{K_{1,4}, Z_t\}$-free graphs G with $\delta(G) \geq 4$ is bounded by a constant (Proposition 4.5).

We start with the following fundamental lemmas.

Lemma 4.1. *Let G be a connected graph. Let u and v be two vertices of G, and let $Q = u_0u_1 \ldots u_l$ be a shortest u-v path of G, where $l = \mathrm{dist}_G(u, v)$, $u_0 = u$ and $u_l = v$. Let t, t' be integers with $0 \leq t' < t \leq l$. Then the following hold.*

(i) If $t - t' \geq 2$, then $u_tu_{t'} \notin E(G)$.

(ii) If $t - t' \geq 3$, then $N_G(u_t) \cap N_G(u_{t'}) - V(Q) = \emptyset$.

Proof. This lemma immediately follows from the assumption that Q is a shortest u-v path. □

Lemma 4.2. *Let G be a connected K_3-free graph. Let u and v be two vertices of G, and let $Q = u_0u_1 \dots u_l$ be a u-v path of G, where $u_0 = u$ and $u_l = v$. Then the following hold.*

(i) *For any integers t and t' with $0 \le t' < t \le l$, if $t - t' = 1$, then $N_G(u_t) \cap N_G(u_{t'}) - V(Q) = \emptyset$.*

(ii) *For any integer t with $1 \le t \le l-1$, $E(N_G(u_t) - V(Q), \{u_{t-1}, u_{t+1}\}) = \emptyset$.*

Proof. Statement (i) immediately follows from the assumption that G is K_3-free. Statement (ii) immediately holds by (i). □

The following lemma is used in the proof of Proposition 4.4

Lemma 4.3. *Let l be an integer with $l \ge 7$, let G be a connected $\{K_3, Y_2^*\}$-free graph with $\delta(G) \ge 4$ and $\mathrm{diam}(G) \ge l$. Let u and v be two vertices of G such that $l = \mathrm{dist}_G(u, v)$, and let $Q = u_0u_1...u_l$ be a shortest u-v path of G, where $u_0 = u$ and $u_l = v$. Then for any i with $0 \le i \le l-7$, $N_G(u_j) \cap N_G(u_{j+2}) - V(Q) \ne \emptyset$ for some j with $i \le j \le i+5$.*

Proof. Let i be an integer with $0 \le i \le l-7$. By way of contradiction, suppose that

$$N_G(u_j) \cap N_G(u_{j+2}) - V(Q) = \emptyset \tag{4.1}$$

for any j with $i \le j \le i+5$.

Since $\delta(G) \ge 4$, $|N_G(u_t) - V(Q)| \ge 2$. For each t with $0 \le t \le l$, we take two vertices a_t, $b_t \in N_G(u_t) - V(Q)$. Since G is K_3-free,

$$a_tb_t \notin E(G) \tag{4.2}$$

for $0 \le t \le l$. For t, t' with $i \le t < t' \le i+7$,

$$\{a_t, b_t\} \cap \{a_{t'}, b_{t'}\} = \emptyset \tag{4.3}$$

by Lemma 4.1(ii), Lemma 4.2(i) and (4.1).

Let k be an integer with $i+1 \le k \le i+5$. Take two vertices $x \in \{a_k, b_k\}$ and $y \in \{a_{k+1}, b_{k+1}\}$ (note that $x \ne y$ by (4.3)). Since G is Y_2^*-free, $G[u_{k-1}, u_k, u_{k+1}, u_{k+2}, x, y] \not\simeq Y_2^*$, which implies that

$$\text{for any vertices } x \in \{a_k, b_k\} \text{ and } y \in \{a_{k+1}, b_{k+1}\}, xy \in E(G)$$

by Lemma 4.1(i), Lemma 4.2(ii) and (4.1). Since k is arbitrary,

$$\text{for any vertices } x \in \{a_t, b_t\} \text{ and } y \in \{a_{t+1}, b_{t+1}\}, xy \in E(G) \tag{4.4}$$

for any t with $i+1 \le t \le i+5$. Since G is K_3-free,

$$\text{there is no edge between } \{a_t, b_t\} \text{ and } \{a_{t+2}, b_{t+2}\} \tag{4.5}$$

for any t with $i+1 \le t \le i+4$. Since G is Y_2^*-free, $G[a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5}, b_{i+2}, b_{i+5}] \not\simeq Y_2^*$, which implies that there exists an edge between $\{a_{i+2}, b_{i+2}\}$ and $\{a_{i+5}, b_{i+5}\}$ by (4.2), (4.4) and (4.5). Without loss of generality, we may assume that

$$a_{i+2}a_{i+5} \in E(G) \text{ (see Figure 7).}$$

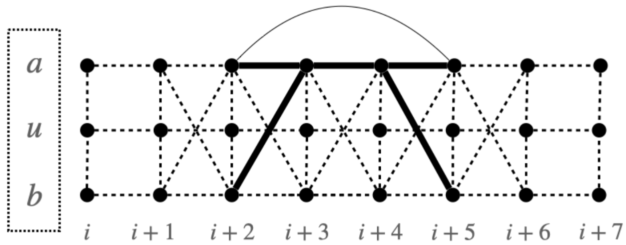

Figure 7. $G[a_{i+2}, a_{i+3}, a_{i+4}, a_{i+5}, b_{i+2}, b_{i+5}]$

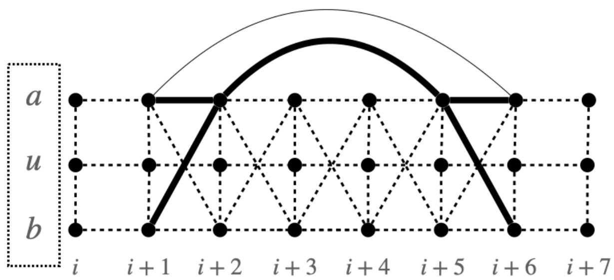

Figure 8. $G[a_{i+2}, a_{i+5}, a_{i+1}, b_{i+1}, a_{i+6}, b_{i+6}]$

Since G is K_3-free,

$$a_{i+1}a_{i+5}, b_{i+1}a_{i+5}, a_{i+6}a_{i+2}, b_{i+6}a_{i+2} \notin E(G). \tag{4.6}$$

Since G is Y_2^*-free, $G[a_{i+2}, a_{i+5}, a_{i+1}, b_{i+1}, a_{i+6}, b_{i+6}] \not\simeq Y_2^*$, which implies that there exists an edge between $\{a_{i+1}, b_{i+1}\}$ and $\{a_{i+6}, b_{i+6}\}$ by (4.2), (4.4) and (4.6). Without loss of generality, we may assume that

$$a_{i+1}a_{i+6} \in E(G) \text{ (see Figure 8).}$$

Then $u_0u_1...u_{i+1}a_{i+1}a_{i+6}u_{i+6}...u_l$ is a shortest u-v path of G having length $(i+1)+3+\{l-(i+6)\} = l-2$, which contradicts the fact that $\mathrm{dist}_G(u, v) = l$. □

Proposition 4.4. *Let G be a connected $\{K_3, Y_2^*\}$-free graph with $\delta(G) \geq 4$. Then* $\mathrm{diam}(G) \leq 15$.

Proof. By way of contradiction, suppose that $\mathrm{diam}(G) \geq 16$. Take two vertices u and v of G with $\mathrm{dist}_G(u, v) = 16$, and let $Q = u_0u_1...u_{16}$ be a shortest u-v path of G, where $u_0 = u$ and $u_{16} = v$. By Lemma 4.3, $N_G(u_{j_1}) \cap N_G(u_{j_1+2}) - V(Q) \neq \emptyset$ for some j_1 with $0 \leq j_1 \leq 5$. Let a_{j_1+2} be a vertex of G with $a_{j_1+2} \in N_G(u_{j_1}) \cap N_G(u_{j_1+2}) - V(Q)$.

By Lemma 4.3, $N_G(u_i) \cap N_G(u_{i+2}) - V(Q) \neq \emptyset$ for some i with $j_1 + 3 \leq i \leq j_1 + 8$. Let j_2 be a minimum integer with $j_1 + 3 \leq j_2 \leq j_1 + 8$ such that $N_G(u_{j_2}) \cap N_G(u_{j_2+2}) - V(Q) \neq \emptyset$. Hence, for any j' with $j_1 + 3 \leq j' \leq j_2 - 1$,

$$N_G(u_{j'}) \cap N_G(u_{j'+2}) - V(Q) = \emptyset. \tag{4.7}$$

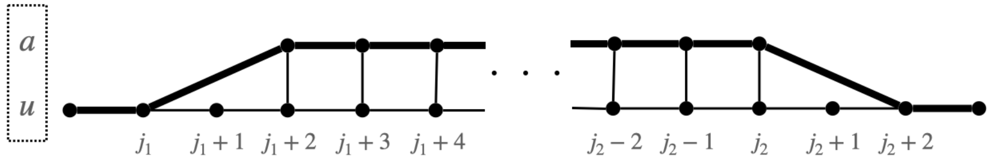

Figure 9. u-v path of G

Let a_{j_2} be a vertex of G with $a_{j_2} \in N_G(u_{j_2}) \cap N_G(u_{j_2+2}) - V(Q)$. Since $\delta(G) \geq 4$, $N_G(u_t) - V(Q) \neq \emptyset$ for all t with $j_1 + 3 \leq t \leq j_2 - 1$. For each t with $j_1 + 3 \leq t \leq j_2 - 1$, take a vertex $a_t \in N_G(u_t) - V(Q)$.

Since G is a Y_2^*-free, $G[u_m, u_{m+1}, u_{m+2}, u_{m+3}, a_{m+1}, a_{m+2}] \not\cong Y_2^*$ for any m with $j_1 + 3 \leq m \leq j_2 - 2$. Hence

$$\{a_{j_1+4}a_{j_1+5}, a_{j_1+5}a_{j_1+6}, ..., a_{j_2-1}a_{j_2}\} \subseteq E(G) \tag{4.8}$$

by Lemma 4.1(i), Lemma 4.2(i)(ii), and (4.7).

If $u_{j_1+2}a_{j_1+4} \in E(G)$, then $u_0u_1...u_{j_1+2}a_{j_1+4}a_{j_1+5}...a_{j_2-1}a_{j_2}u_{j_2+2}u_{j_2+3}...u_{16}$ is a u-v path of G having length $(j_1 + 2) + 1 + \{j_2 - (j_1 + 4)\} + 1 + \{16 - (j_2 + 2)\} = 14$, which contradicts the fact that $\mathrm{dist}_G(u, v) = 16$. Thus we may assume that $u_{j_1+2}a_{j_1+4} \notin E(G)$. Therefore, since G is a Y_2^*-free, $G[u_{j_1+2}, u_{j_1+3}, u_{j_1+4}, u_{j_1+5}, a_{j_1+3}, a_{j_1+4}] \not\cong Y_2^*$, we obtain

$$a_{j_1+3}a_{j_1+4} \in E(G) \tag{4.9}$$

by Lemma 4.1(i), Lemma 4.2(i)(ii), and (4.7).

If $u_{j_1+1}a_{j_1+3} \in E(G)$, then $u_0u_1...u_{j_1+1}a_{j_1+3}a_{j_1+4}...a_{j_2-1}a_{j_2}u_{j_2+2}u_{j_2+3}...u_{16}$ is a u-v path of G having length $(j_1 + 1) + 1 + \{j_2 - (j_1 + 3)\} + 1 + \{16 - (j_2 + 2)\} = 14$, which contradicts the fact that $\mathrm{dist}_G(u, v) = 16$. Thus, we may assume that

$$u_{j_1+1}a_{j_1+3} \notin E(G).$$

If $a_{j_1+2}u_{j_1+4} \in E(G)$, then $u_0u_1...u_{j_1}a_{j_1+2}u_{j_1+4}u_{j_1+5}...u_{15}u_{16}$ is a u-v path of G having length $j_1 + 1 + 1 + \{16 - (j_1 + 4)\} = 14$, which contradicts the fact that $\mathrm{dist}_G(u, v) = 16$. Thus, we may assume that

$$a_{j_1+2}u_{j_1+4} \notin E(G).$$

Therefore, since G is a Y_2^*-free, $G[u_{j_1+1}, u_{j_1+2}, u_{j_1+3}, u_{j_1+4}, a_{j_1+2}, a_{j_1+3}] \not\cong Y_2^*$, and hence we obtain

$$a_{j_1+2}a_{j_1+3} \in E(G) \tag{4.10}$$

by Lemma 4.1(i), Lemma 4.2(i)(ii), and (4.7).

Consequently, we obtain $u_0u_1...u_{j_1}a_{j_1+2}a_{j_1+3}...a_{j_2-1}a_{j_2}u_{j_2+2}u_{j_2+3}...u_{16}$ by (4.8), (4.9) and (4.10) is a u-v path of G having length $j_1 + 1 + \{j_2 - (j_1 + 2)\} + 1 + \{16 - (j_2 + 2)\} = 14$, which contradicts the fact that $\mathrm{dist}_G(u, v) = 16$ (see Figure 9). □

Proposition 4.5. *Let t be an integer with $t \geq 2$, and let G be a connected $\{K_{1,4}, Z_t\}$-free graph with $\delta(G) \geq 4$. Then* $\mathrm{diam}(G) \leq 2t - 1$.

Proof. Suppose that $\mathrm{diam}(G) \geq 2t$. Take two vertices u and v of G with $\mathrm{dist}_G(u, v) = 2t$, and let $u_0u_1...u_{2t}$ be a shortest u-v path of G, where $u_0 = u$ and $u_{2t} = v$. By $\delta(G) \geq 4$, we can take two distinct vertices $a_t, a'_t \in N_G(u_t) - \{u_{t-1}, u_{t+1}\}$. Since G is $K_{1,4}$-free, $G[u_t, u_{t-1}, u_{t+1}, a_t, a_{t'}] \not\simeq K_{1,4}$, i.e. $\{u_{t-1}, u_{t+1}, a_t, a_{t'}\}$ is not an independent set of G. With Lemma 4.1(i) in mind, we divide the proof into two cases.

Case 1. $E(\{u_{t-1}, u_{t+1}\}, \{a_t, a_{t'}\}) \neq \emptyset$. Without loss of generality, we may assume that $u_{t-1}a_t \in E(G)$. Since G is Z_t-free, neither $\{u_t, a_t, u_{t-1}, ..., u_0\}$ nor $\{u_{t-1}, a_t, u_t, ..., u_{2t-1}\}$ induces a copy of Z_t in G. Thus, there exist indices i_1 and i_2 with $0 \leq i_1 \leq t-2$ and $t+1 \leq i_2 \leq 2t-1$ such that $u_{i_1}a_t, u_{i_2}a_t \in E(G)$ by Lemma 4.1(i). Then $u_0u_1...u_{i_1}a_tu_{i_2}u_{i_2+1}...u_{2t}$ is a u-v path of G having length $i_1 + 2 + (2t - i_2)$ $(\leq (t-2) + 2 + 2t - (t+1) = 2t - 1)$, which contradicts the fact that $\mathrm{dist}_G(u, v) = 2t$.

Case 2. $a_ta'_t \in E(G)$ Since G is Z_t-free, neither $\{a_t, a'_t, u_t, ..., u_1\}$ nor $\{a_t, a'_t, u_t, ..., u_{2t-1}\}$ induces a copy of Z_t in G. Thus, $E(\{u_1, u_2, \ldots, u_{t-2}\}, \{a_t, a_{t'}\}) \neq \emptyset$ and $E(\{u_{t+2}, u_{t+3}, \ldots, u_{2t-1}\}, \{a_t, a_{t'}\}) \neq \emptyset$ by Lemma 4.1(i). Let $b \in \{a_t, a'_t\}$ and $u_{j_1} \in \{u_1, u_2, \ldots, u_{t-2}\}$ be two vertices such that $bu_{j_1} \in E(G)$, and let $b' \in \{a_t, a'_t\}$ and $u_{j_2} \in \{u_{t+2}, u_{t+3}, ..., u_{2t-1}\}$ be two vertices such that $b'u_{j_2} \in E(G)$. Note that if $b \neq b'$ then $bb' \in E(G)$.

If $b = b'$, then $u_0u_1...u_{j_1}bu_{j_2}u_{j_2+1}...u_{2t}$ is a u-v path of G having length $j_1 + 2 + (2t - j_2)$ $(\leq (t-2) + 2 + 2t - (t+2) = 2t - 2)$, which contradicts the fact that $\mathrm{dist}_G(u, v) = 2t$. If $b \neq b'$, then $u_0u_1...u_{j_1}bb'u_{j_2}u_{j_2+1}...u_{2t}$ is a u-v path of G having length $j_1 + 3 + (2t - j_2)$ $(\leq (t-2) + 3 + 2t - (t+2) = 2t - 1)$, which contradicts the fact that $\mathrm{dist}_G(u, v) = 2t$. □

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