Electronic Journal of Graph Theory and Applications **12** (2) (2024), 157–167

Electronic Journal of Graph Theory and Applications

Determination of all graphs whose eccentric graphs are clusters

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Abstract

A disconnected graph G is called a cluster if G is not union of K_{2s} (1-factor) but union of complete graphs of order at least two. J. Akiyama, K. Ando and D. Avis showed in Lemma 2.1 of [2] that G is equi-eccentric if the eccentric graph G_e is a cluster or pK_2. And they also characterized all graphs whose eccentric graphs are complete graphs and pK_2 in [2]. In this paper, we determined in Theorem 2 all graphs whose eccentric graphs are clusters, which is an extension of Lemma 2.1 in [2].

Keywords: eccentricity, eccentric graph, cluster, distance, radius
Mathematics Subject Classification : 05C12
DOI: 10.5614/ejgta.2024.12.2.1

1. Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph. A disconnected graph G is called a **cluster** if it is a union of complete graphs $\bigcup_{i=1}^{n} K_{p_i}$ ($n \geqq 2, p_i \geqq 2$; Figure 1).

The **eccentricity** $e(v)$ of a vertex v in $V(G)$ is defined by $e(v) = \max_{u \in V(G)} d(u,v)$, where $d(u,v)$ stands for the length of a shortest path in G between u and v. We denote by $G_e = (V(G_e), E(G_e))$ the **eccentric graph** based on G (Figure 2), where the vertex set $V(G_e)$ is identical to $V(G)$ and $uv \in E(G_e) \Leftrightarrow d(u,v) = \min(e(u), e(v))$.
A similar graph, called the "furthest neighbor graph", was introduced by Shamos [8]. Its vertex set is a set of points in the plane, and the distance between two vertices is their Euclidean distance.

Received: 24 October 2023, Revised: 22 April 2024, Accepted: 10 May 2024.

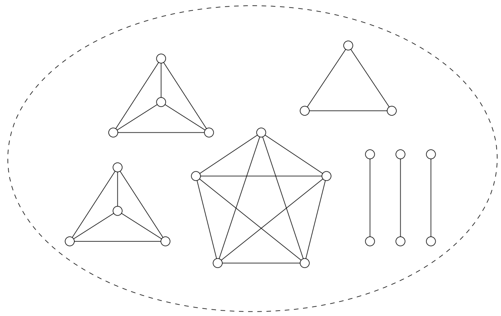
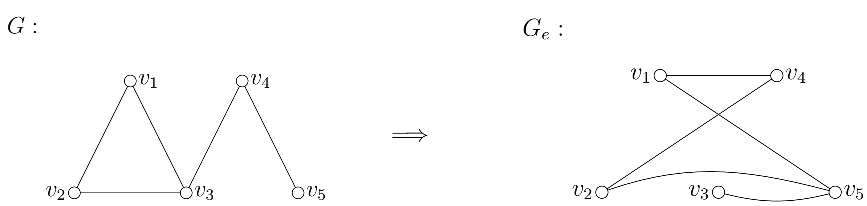


Figure 1: Example of a cluster $G = 3K_2 \cup K_3 \cup 2K_4 \cup K_5$

Two vertices are joined if either one is the "furthest neighbor" of the other. Extremal properties of this graph are studied in [5].

A central vertex of a graph G is a vertex v with the property that the maximum distance between v and any other vertex is as small as possible, this distance being called the **radius**, denoted by $r(G)$. That is, $r(G) = \min_{v \in V(G)} e(v)$. The **diameter** of G denoted $diam(G)$ is defined by $diam(G) = \max_{v \in V(G)} e(v)$. A graph is a **self-center** [4] or **r-equi-eccentric** (or briefly **r-equi**) [1] if $e(v) = r(G) = diam(G)$ for all vertices $v \in V(G)$ (see Figure 3). If $S \subset V(G)$, then we say that $e(S) = i$ if $e(v) = i$ for all $v \in S$, and we denote by $\langle S \rangle$ the subgraph induced by S. We denote by $N(v)$ the **neighborhood** of a vertex v of G consisting of the vertices in G adjacent to v. The **closed neighborhood** $N[v]$ of v is defined by $N[v] = N(v) \cup \{v\}$. All other definitions and notations used in this paper might be found in [4] or [6].



(a) $r(G) = 2, diam(G) = 3$ (b) G_e

Figure 2: Example of a graph and its eccentric graph

(a) C_6, 3-equi (b) 4-equi

Figure 3: Examples of a 3-equi and a 4-equi-eccentric graph

Lemma A ([2, Lemma 2.1]). *If* $G_e = \bigcup_{i=1}^{n} K_{p_i}$*, where* $\sum_{i=1}^{n} p_i = p, p_i \geqq 2 (i = 1, 2, \ldots, n)$ *and* $n \geqq 2$*, then* G *is equi-eccentric.*

Theorem A ([2, Theorem 2.2])**.** *If G is a graph of order $2p$ then $G_e = pK_2$ if and only if G is radius critical, that is, $r(G - v) = r(G) - 1$ for all $v \in V(G)$.*

Theorem B ([2, Theorem 2.1])**.** *For any connected graph G of order p, $G_e = K_p$ if and only if for all $v \in V(G)$ either $e(v) = 1$ or $e(N(v)) = 1$.*

Theorem B can be restated as follows and we give a different proof of it from one in [2]:

Theorem B′. *A graph G of order p whose eccentric graph is a complete graph K_p if and only if G is a join of a complete graph K_m and n isolated vertices $\overline{K_n}$ i.e., $G = K_m + \overline{K_n}, m+n = p, m \neq 0$.*

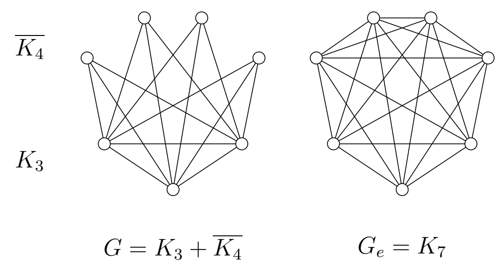


Figure 4: Example of Theorem B′

Proof. We divide our proof into two cases depending on $r(G) = 1$ or not.

Case 1. Suppose that $r(G)$=1.
Let $S(i)$ be a set of all vertices v of G with $e(v) = i$. Since $S(i) = \emptyset$ for every $i(\geqq 3)$, we have that $V(G) = S(1) \cup S(2)$ and $S(1) \cap S(2) = \emptyset$. Let $|S(1)|, |S(2)|$ be m, n, respectively. The induced subgraph $\langle S(1) \rangle$ of G is a complete subgraph K_m of G.
If there exists at least one edge $vv' \in E(G)$ where both $v, v' \in S(2)$, then we have that $vv' \notin E(G_e)$, which implies that G_e is not a complete graph (Figure 5). Therefore, $\langle S(2) \rangle$ must be a totally disconnected graph $\overline{K_n}$. That is, G must be a $K_m + \overline{K_n}$.
Conversely, if $G = K_m + \overline{K_n}$, then G_e is a complete graph K_{m+n}.

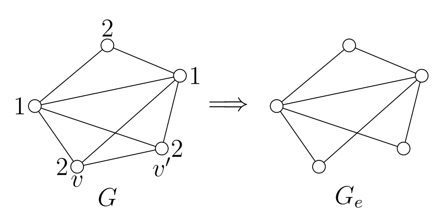


Figure 5: Example of **Case 1.** with the value of eccentricity of vertices in G

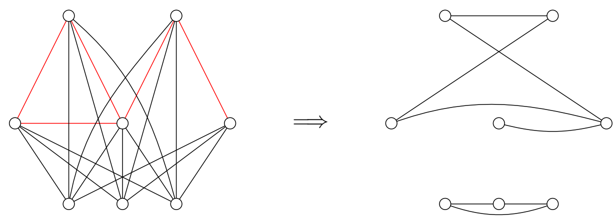
Case 2. Suppose that $r(G) \geqq 2$.
For any vertex v of G, there is a vertex v' which is adjacent to v, implying that $d(v, v') = 1$. Since $e(v) \neq 1$ and $e(v') \neq 1$, we have that vv' is not an edge of G_e. Therefore, G_e cannot be a complete graph.

□

2. Main results

Theorem 1. *If* $r(G) \geqq 2$, *then* $(G+\overline{K_n})_e = \overline{(G+\overline{K_n})} = \overline{G} \cup K_n, n \geqq 2$.

Proof. Since $r(G) \geqq 2$, there is no vertex v such that $e(v) = 1$ in G. Therefore, $G+\overline{K_n}$ is 2-equi (Figure 6). Hence $uv \in E(G+\overline{K_n})$ if and only if $uv \notin E((G+\overline{K_n})_e)$, which implies that $(G+\overline{K_n})_e$ is isomorphic to the complement of $(G+\overline{K_n})$; i.e., $\overline{G+\overline{K_n}}$. Moreover, the complement $\overline{G+\overline{K_n}}$ is $\overline{G} \cup K_n$. □



$G(red) + \overline{K_3}$ $(G+\overline{K_3})_e = \overline{G+\overline{K_3}} = \overline{G} \cup K_3$

Figure 6: Example of Theorem 1

Putting $G = K(m_1, m_2, \dots, m_n)$, we obtain the Corollary 1.

Corollary 1. *For* $m_i \geqq 2$ *for* $i(1 \leqq i \leqq n)$ *and* $\ell \geqq 2$,
$(K(m_1, m_2, \dots, m_n) + \overline{K_\ell})_e = K_{m_1} \cup K_{m_2} \cup \dots \cup K_{m_n} \cup K_\ell$.

Proposition 1. $(C_{2p})_e = pK_2$ *(see Figure 7a).* $(C_{2p+1})_e = C_{2p+1}$ *(see Figure 7b).*

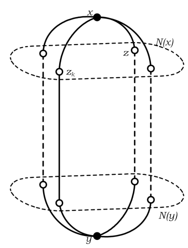
(a) C_6 and $(C_6)_e = 3K_2$ (b) C_5 and $(C_5)_e = C_5$

Figure 7: C_6 and C_5, and its eccentric graphs

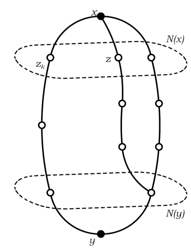
Theorem 2. *A graph whose eccentric graph is a cluster, i.e.,* $G_e = \bigcup_{i=1}^{n} K_{p_i}(n \geqq 2, p_i \geqq 2$ *for any* $i(1 \leqq i \leqq p)$*, and at least one* p_i *is not* 2, $\Sigma_{i=1}^{n} p_i = p)$*, if and only if* G *is a complete* n*-partite graph* $K(p_1, p_2, \dots, p_n)$.

Proof. Since $n \geqq 2$, $G_e = \bigcup_{i=1}^{n} K_{p_i}$ is not connected. Then, there exists no vertex $v \in V(G)$ such that $e(v) = 1$ (if not, G_e would be connected). Let $k = diam(G)$ and let x and y be any pair of vertices in G such that $d(x, y) = k$. Note that xy is an edge of G_e.
Let z_k be a vertex in $N(x)$ such that $d(z_k, y) = k-1$ (Figure 8a). For any vertex z in $N(x)$, if every path between z and y passes through x, then $d(z, y) > k$, which is a contradiction. Therefore, there exists at least one path between z and y not passing through x for any $z \in N(x)$ (Figure 8b). As

to such a path between z and y:
if $d(y,z) < k-1$ for some $z \in N(x)$, $d(x,y) = d(y,z) + d(z,x) < k$, which is a contradiction.
If $d(y,z) > k-1$ for some $z \in N(x)$, that is $d(y,z) = k(= diam(G))$, $e(z) = k$. And then, $xy \in G_e$, $yz \in G_e$, but $xz \notin G_e$, since $d(x,z) = 1 \neq e(x)$ or $e(z)$. It means that G_e is not $\bigcup K_{p_i}$ (i.e., vertices x, y and z do not form a part of a complete graph), which is a contradiction.
Then, for any vertex z in $N(x)$, there exists at least one path between z and y such that $d(y,z) = k-1$, if G_e is a cluster.



(a) Vertices in $N(x)$ and $N(y)$



(b) Paths between z and y

Figure 8

Due to the above results, G includes a k-equi-eccentric graph G' which is composed of all paths $xz_i \cup z_iy$ where $z_i \in N(x)$. Note that the number of $N(y)$ is more than 1. Otherwise, for a unique point $w \in N(y)$, $e(w) = d(x,w)$ and $d(w,y) = 1$. Then, $xy \in G_e, xw \in G_e$ but $yw \notin G_e$, which means that vertices x, y and w do not form a part of complete graph, that is, G_e cannot be a cluster.

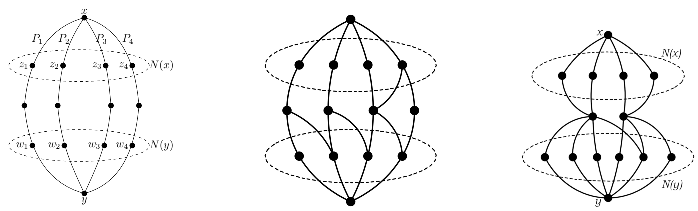
Any G' can be constructed by applying one of or combination of following operations to G_L, which is a union of n disjoint paths $P_i = xz_i \cup z_iy$ $(i = 1, 2, \ldots, n)$ where $n = |N(x)|$, $|V(G_L)| = |V(G')|$ and the length of P_i is k (Figure 9):

Operation 1. Add an edge between points p_i and p_j where $d(p_i, x) + 1 = d(p_j, x)$.

Operation 2. A point p_i and some points $p_j s$ coincide, where $1 < d(p_i, x) = d(p_j, x) = \ell < k-1$, but at least two of them for each ℓ must be distinct. And add m points $p_k s$, edges $p_kp_q s$ and $p_kp_r s$ so that m is just the reduced number above, and $d(p_k, x) \neq \ell$, $d(p_k, x) - 1 = d(p_q, x)$ and $d(p_k, x) + 1 = d(p_r, x)$.

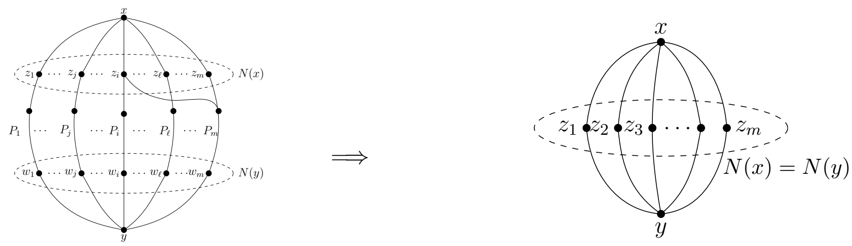
For any $z \in N(x)$, there must exist at least one point $w \in N(y)$ such that $e(z) = d(z,w) = k$, if G_e is a cluster. We can classify the situation into the following two cases by the maximum number of w for all $z \in N(x)$ where $d(z,w) = k$.

Case 1. The maximum number of w is more than 1 for all $z \in N(x)$ where $d(z,w) = k$.
In this case, $|N(x)| \geqq 3$ and $|N(y)| \geqq 3$. For z_i in $N(x)$, there exist at least two vertices w_j and w_ℓ in $N(y)$ such that $d(z_i, w_j) = k, d(z_i, w_\ell) = k$. That is, any paths z_iy with $d(z_i, y) = k-1$ never pass through w_j or w_ℓ (Figure 10a).
Then, both z_iw_j and z_iw_ℓ are edges of G_e, thus w_jw_ℓ must also be an edge of G_e =



(a) G_L (b) G' (c) G'

Figure 9: Examples of G_L and G' in the case of $k = 4$



(a) $d(z_i, w_j) = d(z_i, w_\ell) = k$ (b) G: 2-equi-eccentric graph

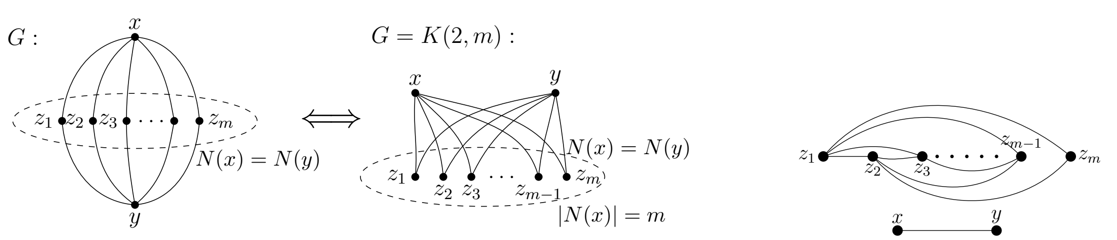
Figure 10

$\bigcup_{i=1}^{n} K_{p_i}$. Since $d(w_j, w_\ell) = 2$ and $e(w_j) = e(w_\ell) = k$, k must be 2, which implies that $N(x) = N(y)$ and G is 2-equi-eccentric (Figure 10b).

Case 1-1. G' includes all vertices of G.

There are two cases (i) and (ii) depending on whether some pairs of $N(x)$ are joined by edges of G or not.

(i) If no pair of $N(x)$ is joined by an edge of G, G must be a complete bipartite graph $K(2, m)$ (Figure 11a) and G_e is $K_2 \cup K_m$ (Figure 11b).



(a) G is isomorphic to $K(2, m)$ (b) $G_e = K_2 \cup K_m$

Figure 11: Example of G and G_e of Case 1-1 (i)

(ii) If some pairs of $N(x)$ are joined by edges of G, G_e is a cluster if and only if $N(x)$ induces a complete n-partite graph $K(m_1, m_2, \ldots, m_n), m_i \geqq 2$ for $1 \leqq i \leqq n$ by Corollary 1.

That is, G is $K(m_1, m_2, \ldots, m_n) + \overline{K_2} = K(2, m_1, m_2, \ldots, m_n)$ and G_e is $K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_n} \cup K_2$. Figure 12 shows that $N(x)$ induces a complete bipartite graph $K(2, 3)$ and $G = K(2, 3) + \overline{K_2} = K(2, 2, 3)$, respectively.

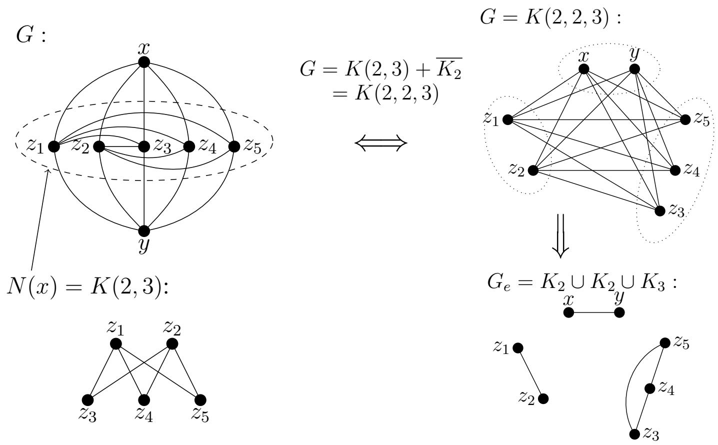
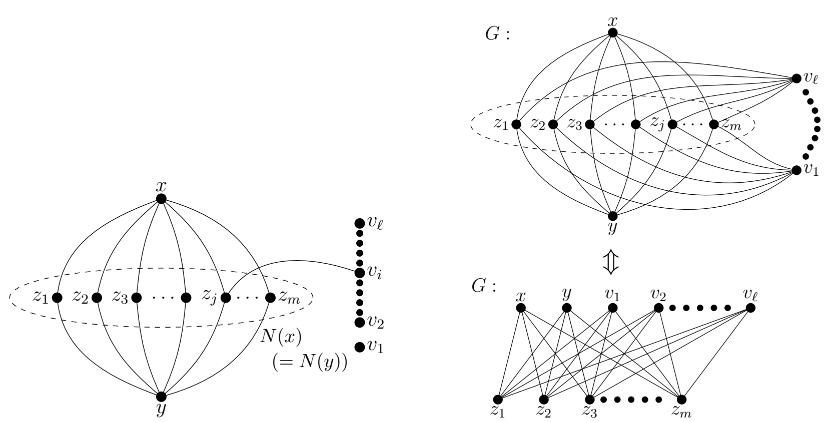


Figure 12: $N(x)$ induces a complete bipartite graph $K(2,3)$

Case 1-2. G' does not include all vertices of G.

Let $v_1, v_2, \ldots, v_n$ be vertices which are included in none of G'. Since (1) G is connected, (2) $diam(G) = 2$, and (3) v_i belongs to neither $N(x)$ or $N(y)$, $v_i z_j$ must be an edge of G for at least one vertex of $N(x)$ (say z_j) (Figure 13a).

(i) If no pair of $N(x)$ has an edge of G, $v_i (i = 1, 2, \ldots, \ell)$ is joined by an edge with every vertex in $N(x)$ (Figure 13b), since otherwise $e(v_i) = 3 > diam(G)(= 2)$, which is a contradiction (Figure 13a). Therefore, G must be $K(m, \ell + 2)$ (Figure 13b), and G_e is $K_m \cup K_{\ell+2}$.



(a) Example of **Case 1-2.**

(b) $G = K(m, \ell + 2)$

Figure 13: Example of G of **Case 1-2.**(i)

(ii) If some pairs of $N(x)$ are joined by edges of G, $N(x)$ must form $K(m_1, m_2, \ldots, m_n)$ to make G_e be a cluster by Corollary 1 (Figure 14). Especially, $K(2, 2, \ldots, 2)$ is the power $(C_{2n})^{n-1}$ of C_{2n}. If $N(x)$ is $(C_{2n})^{n-1}$, G is also a power $(C_{2(n+1)})^n$ of $C_{2(n+1)}$ and G_e is $(n + 1)K_2$, which is not a cluster (Figure 15 is the case of $n = 4$).

Let $v_1, v_2, \ldots, v_\ell$ be vertices not in $N(x) \cup \{x, y\}$. For $v_i (i = 1, 2, \ldots, \ell)$, since (1) v_i does not belong to $N(x)$, (2) $diam(G) = 2$, and (3) if there exists $z \in N(x)$ such that $v_i z \notin G, v_i z \in G_e, v_i x \in G_e$ but $xz \notin G_e$, then $v_i z$ must be an edge of G for all z in $N(x)$ (Figure 16a). Therefore, G is $K(m_1, m_2, \ldots, m_n) + \overline{K_{2+\ell}}$, i.e., $K(m_1, m_2, \ldots, m_n, 2+\ell)$ (Figure 16b), and G_e is $K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_n} \cup K_{2+\ell}$ (Figure 16c).

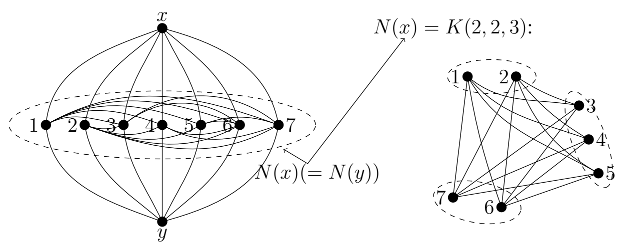


Figure 14: Example of G: $N(x) + K_2(|N(x)| = 7)$

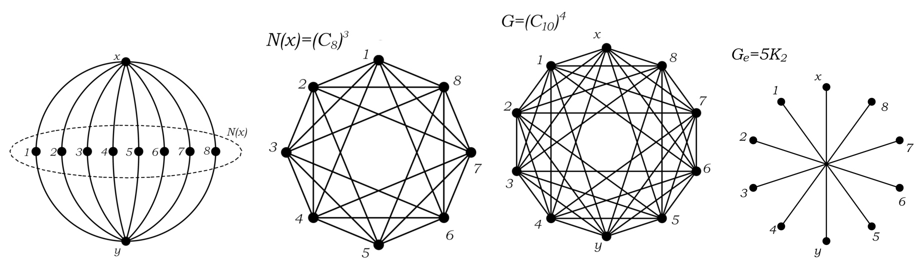
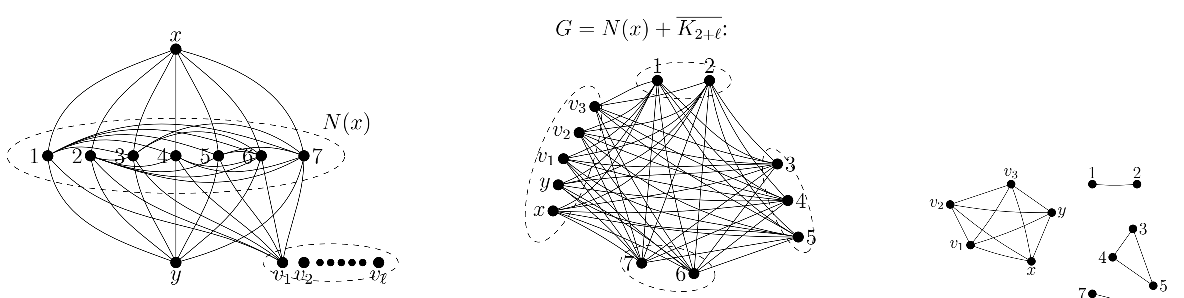


Figure 15: Example of **Case 1-2.**(ii) for $N(x) = C_{2n}$ with $n = 4$



(a) $V(G) = N(x) \cup \{x, y\} \cup \{v_1, \ldots, v_\ell\}$, where $N(x) = K(2, 2, 3)$

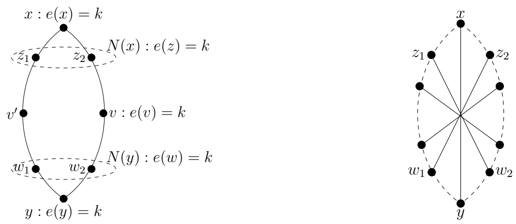
(b) Example with $\ell = 3, G = K(2, 2, 3) + \overline{K_{2+3}} = K(2, 2, 3, 5)$

(c) $G_e = K_2 \cup K_3 \cup K_2 \cup K_5$

Figure 16: z_i is simply stated by i.

Case 2. The number of w for each $z \in N(x)$ where $d(z, w) = k$ is 1.
This case can be also classified into the following four cases:

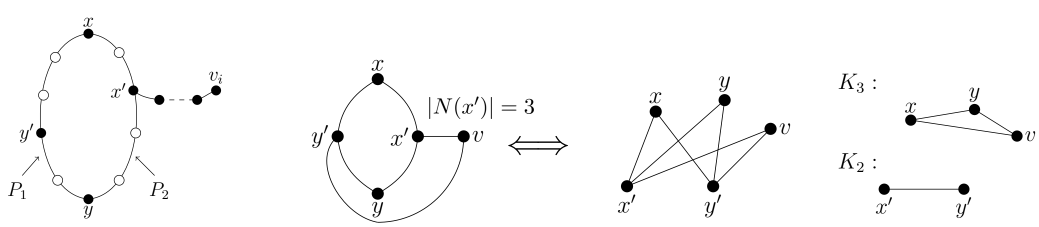
Case 2-1. $|N(x)| = |N(y)| = 2$ and G' includes all vertices of G.
In this case, if G is a C_{2k} (where $2k = p$), G is k-equi-eccentric and G_e is a kK_2 by Proposition 1 (Figure 17), which is not a cluster. There exists no G such that G_e is a cluster.



(a) $G = C_{2k}$ (b) $G_e = kK_2$

Figure 17: Example of G and G_e of **Case 2-1.**

Case 2-2. $|N(x)| = |N(y)| = 2$ and G' does not include all vertices of G.
Let v_i be a vertex which is not included in G'. Since G is connected, v_i is connected to some vertex of G'. Let x' be a vertex on G' such that $d(v_i, x')$ is the shortest among $d(v_i, v_c)$ for any vertex v_c on G' (Figure 18a). Let y' be a vertex on G' such that $d(x', y') = k$. For x' and y', we repeat the analogous argument as one for x and y. Then we see that there are at least three paths of length k from x' to y'. If $k \neq 2$, G_e cannot be a cluster as seen in **Case 1.**. If $k = 2$ and there is only one vertex v other than x and y in $N(x')$ (Figure 18b where $p = 5$), G is $K(3, 2)$ which is a 2-equi-eccentric graph, and thus G_e is $K_3 \cup K_2$ (Figure 18c).

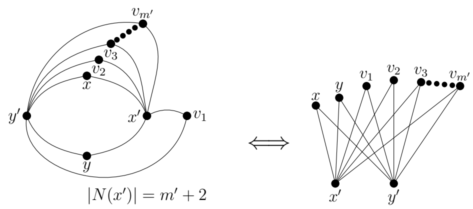


(a) C_{2k} (b) G (c) $G = K(3, 2)$ (d) $G_e = K_2 \cup K_3$

Figure 18: Example of **Case 2-2.** where $|N(x')| = 3$

If $k = 2$ and there are $m' (\geqq 2)$ vertices $v_1, v_2, \ldots, v_{m'}$ other than x', y', x and y in $N(x')$ (where $m' = p - 4$) (Figure 19a), G is $K(m' + 2, 2)$ (Figure 19b) and G is a 2-equi-eccentric graph. Thus G_e is $K_2 \cup K_{m'+2}$.

Case 2-3. $|N(x)| = 2$ and $|N(y)| > 2$.
If the number of w for each $z_i \in N(x)$ where $d(z_i, w) = k$ is 1, there must exist just one $w_{z_i} \in N(y)$ such that no path $z_i y$ with distance $k-1$ passes through w_{z_i} and every path $z_i y$ passes through all $w_j \in N(y)$ other than w_{z_i} (Figure 20). But such a graph G' cannot be a k-equi-eccentric, so G_e is not a cluster. We do not need to consider this case.



(a) G (b) $G = K(m' + 2, 2)$

Figure 19: Example of **Case 2-2.** where $|N(x')| > 3$

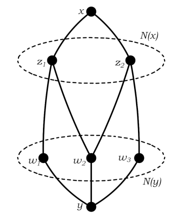
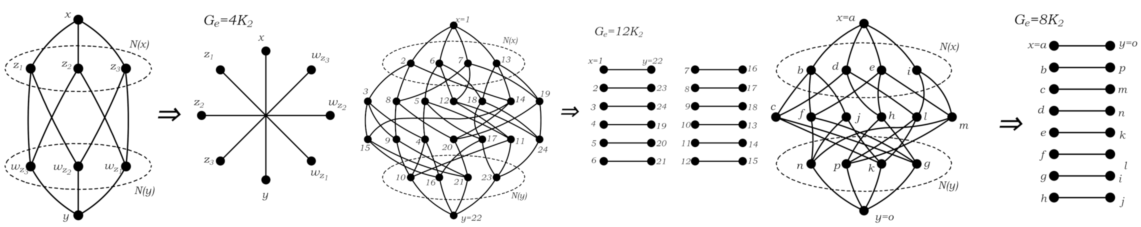


Figure 20: Example of **Case 2-3.**

Case 2-4. $|N(x)| > 2$.
If the number of w for each $z_i \in N(x)$ where $d(z_i, w) = k$ is 1, there must exist just one $w_{z_i} \in N(y)$ such that no path $z_i y$ with distance $k - 1$ passes through w_{z_i} and every path $z_i y$ passes through all $w_j \in N(y)$ other than w_{z_i}. If G satisfies both this condition and $G_e = \bigcup_{i=1}^{n} K_{p_i}$, G is radius critical (Figure 21). Therefore, G_e is $\frac{p}{2}K_2$ by Theorem A, which is not a cluster.



(a) G = 3Cube (3-equi) (b) $G = C_4 \times C_4 (= K_{2,2} \times K_{2,2})$; 5-equi (c) G = 4Cube (4-equi)

Figure 21: Examples of **Case 2-4.**

□

3. Further research

In this article, we determined that all graphs whose eccentric graphs are clusters if and only if the graph is a complete n-partite graph $K(p_1, p_2, \ldots, p_n)$, $p \geqq 2$. For further research, one can try to determine all graphs that have the same eccentric graph. One can also try to find more graph operations which preserve eccentricity like Mycielski's operation, or to find practical applications of eccentric graphs.

Acknowledgement

The authors would like to thank Kiyoshi Ando for his valuable comments.

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