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Magic labeling on graphs with ascending subgraph decomposition

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Abstract

Let t and q be positive integers that satisfy $\binom{t+1}{2} \le q < \binom{t+2}{2}$ and let G be a simple and finite graph of size q. G is said to have ascending subgraph decomposition (ASD) if G can be decomposed into t subgraphs $H_1, H_2, \ldots, H_t$ without isolated vertices such that H_i is isomorphic to a proper subgraph of H_{i+1} for $1 \le i \le t-1$, where $\{E(H_1), \ldots, E(H_t)\}$ is a partition of $E(G)$. A graph that admits an ascending subgraph decomposition is called an ASD graph.

In this paper, we introduce a new type of magic labeling based on the notion of ASD. Let G be an ASD graph and $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ be a bijection. The weight of a subgraph H_i $(1 \le i \le n)$ is $w(H_i) = \sum_{v \in V(H_i)} f(v) + \sum_{e \in E(H_i)} f(e)$. If the weight of each ascending subgraph is constant, say $w(H_i) = k$, $\forall\ 1 \le i \le t$, then f is called an ASD-magic labeling of G and G is called an ASD-magic graph. We present general properties of ASD-magic graphs and characterize certain classes of them.

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1. Introduction

Let G and H be simple and finite graphs. If each edge of G belongs to at least one subgraph isomorphic to H, then G admits a H-*covering*. In 2005, Gutiérrez and Lladó introduced the H-magic labeling or the H-magic covering of a graph G. A bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., |V(G)| + |E(G)|\}$ is called a H-*magic labeling* of G if there exists an integer k such that for any subgraph $H'(V', E')$ of G which is isomorphic to H, the weight $w(H) = \sum_{v\in V'} f(v) + \sum_{e\in E'} f(e)$ is equal to k. A graph G is said to be H-*magic* if it admits a H-magic labeling. Additionally, if $f(V) = \{1, 2, ..., |V|\}$, then G is called H-*supermagic* [5].

On the other hand, if each edge of G belongs to exactly one subgraph isomorphic to H, then G is said to admit a H-decomposition. Formally, let $\mathcal{H} = \{H_i, i = 1, 2, 3, ..., t\}$ be a collection of t subgraphs of G. If $H_i \cong H_j,\ i \neq j,\ E(H_i) \cap E(H_j) = \emptyset$, and $\bigcup_{i=1}^{t} H_i = G$, then G is *decomposable on* H or G admits a H-*decomposition* [3]. Inayah *et. al.* in [6] then defined a H-*magic decomposition* of a graph G as a bijection $f : V(G) \cup E(G) \rightarrow \{0, 1, 2, ..., |V(G)| + |E(G)|\}$ such that the weights of all subgraphs is constant. Recent results on H-magic covering and decomposition include P_h-covering of graphs [10] and K_h-decomposition of some block designs [7]. For more results, refer to Gallian's survey [4].

Note that the two previous magic labelings require that the weights be counted in subgraphs that are isomorphic to a certain graph. However, in 2023, Ashari *et al.* considered weights in subgraphs isomorphic to two nonisomorphic subgraphs when they introduced the (F, H)-simultaneously-magic labelings of graphs [2]. Here, we introduce a magic labeling in which all subgraphs are not isomorphic to each other. This labeling is based on a type of graph decomposition introduced in 1987 by Alavi *et. al.* [1].

Definition 1.1. *[1] Let t and q be two positive integers satisfying $\binom{t+1}{2} \leq q < \binom{t+2}{2}$. Let G be a simple and finite graph of size q. G admits an* ascending subgraph decomposition *if G can be decomposed into t subgraphs $H_1, H_2, \ldots, H_t$ without isolated vertices such that H_i is isomorphic to a proper subgraph of H_{i+1} for $1 \leq i \leq t-1$. In this case, G is called an* ASD graph *and $H_1, H_2, \ldots, H_t$ is the* ascending subgraphs *of G.*

It was conjectured that every graph of positive size has an ascending subgraph decomposition [1]. Until today, the conjecture remains open, although many families of graphs have been showed to be ASD. Refer to Liang and Fu survey [8] for some results on ASD graphs.

The definition of ascending subgraph decomposition motivates us to define a new type of magic labeling of a graph G admitting an ascending subgraph decomposition, where the weights are counted over the ascending subgraphs.

Definition 1.2. *Let G be an ASD graph with $\mathcal{H} = \{H_1, H_2, ..., H_t\}$ is a collection of t ascending subgraphs of G. Let f be a bijection which maps $V(G) \cup E(G)$ onto $\{1, 2, ..., |V(G)| + |E(G)|\}$. If there exists a constant k such that the weight of each subgraph is constant, that is $w(H_i) = k$ for every $1 \leq i \leq t$, then f is called an* ASD-magic labeling of G.

If a graph G admits a collection of ascending subgraphs corresponding to an ASD-magic labeling of G, then G is *ASD-magic*.

In this paper, we characterize some classes of graphs admitting ASD-magic labeling, which include stars (Section 3), paths (Section 4), and cycles (Section 5). Some general properties of ASD-magic graphs are needed in characterizing the previously mentioned classes of graphs and are presented in Section 2.

2. General Properties of ASD-Magic Graphs

We start by observing the size of the smallest subgraph of an ASD graph.

Observation 2.1. *If G is an ASD graph and $H_i, 1 \leq i \leq t$ are ascending subgraphs of G, then the size of H_1 is one or two.*

Proof. It is clear that $|E(H_i)| < |E(H_j)|$ for $1 \leq i < j \leq t$. Assume that $|E(H_1)| \geq 3$ and $|E(H_{i+1})| = |E(H_i)| + 1$. Since we have t ascending subgraphs, then $q = \sum_{i=1}^{t}|E(H_i)| \geq \sum_{i=1}^{t}(i+2) = \frac{t(t+5)}{2} \geq \binom{t+2}{2}$, a contradiction. It implies that $|E(H_1)| < 3$. □

In general, the decomposition of a graph in an ascending order is not unique. This is beneficial in the sense that in proving a graph is ASD-magic, there exist alternative decompositions to be labeled. On the other hand, this is also a drawback in verifying that a graph is not ASD-magic. To avoid checking all possible ASDs in proving that a graph is not ASD-magic, we define the following notions.

Let $f : V(G) \cup E(G) \rightarrow 1, 2, ..., |V(G)| + |E(G)|$ be a bijection. The maximum weight of the smallest subgraph, $w_{max}(H_1)$, is the weight when the vertex and edge sets of H_1 are labeled with the largest labels, that is

$$w_{max}(H_1) = \sum_{i=1}^{|V(H_1)|+|E(H_1)|} \left(|V(G)| + |E(G)| - i + 1\right). \tag{1}$$

Let $\mathfrak{H}$ be the collection of all ascending subgraph decompositions of G. The minimum number of vertices that belong to more than one subgraph over all possibilities of ascending subgraph decompositions is

$$c = \min_{\mathcal{H} \in \mathfrak{H}} \left\{ \sum_{i \neq j} |V(H_i) \cap V(H_j)| \right\} \tag{2}$$

Next, let $C = \{c_i \mid i = 1, 2, ..., c\}$ be the set of intersection vertices with minimum cardinality. We define the smallest average weight of G by considering the label set of C as

$$\overline{w}_{min}(G) = \frac{1}{t} \left[\sum_{i=1}^{|V(G)|+|E(G)|} i + \sum_{i=1}^{c} (d_i - 1) f(c_i) \right]. \tag{3}$$

where d_i is the number of subgraphs containing the intersection vertices c_i.

The previously defined weight notions lead to the following necessary condition for an ASD-magic graph.

Lemma 2.1. *Let G be an ASD graph and H_1 be the smallest ascending subgraph of G. If G admits an ASD-magic labeling, then $w_{max}(H_1) \geq \overline{w}_{min}(G)$.*

Proof. For the contrary, suppose $w_{max}(H_1) < \overline{w}_{min}(G)$. Then, there exists H_j where $w(H_1) \neq w(H_j)$, a contradiction. □

3. ASD-Magic Labelings for Stars

An ascending subgraph decomposition of stars was studied by Ma *et. al.* [9], where they proved that a star forest is an ASD graph.

Theorem 3.1. *[9] Let G be a star forest of size $\binom{t+1}{2}$ where each component has at least t edges. Then G admits an ascending subgraph decomposition where all subgraphs are stars.*

Theorem 3.2. *A star $K_{1,n-1}$ is ASD-magic if and only if $n = 2, 3, 4, 6, 10, 15$.*

Proof. Let $\{H_i \mid i = 1, 2, ..., t\}$ be a collection of t ascending subgraphs of $K_{1,n-1}$. It is straightforward that any subgraph of a star is also a star. To prove that $K_{1,n-1}$ admits an ASD-magic labeling, we have two cases to verify $H_1 \cong P_2$ and $H_1 \cong P_3$. Since the size of $K_{1,n-1}$ satisfies $\binom{t+1}{2} \leq n-1 < \binom{t+2}{2}$, then

$$t = \begin{cases} \lfloor \frac{-1+\sqrt{8n-7}}{2} \rfloor, & \text{for Case 1,} \\ \lceil \frac{-3+\sqrt{1+8n}}{2} \rceil, & \text{for Case 2.} \end{cases} \tag{4}$$

Case 1. $H_1 \cong P_2$

In this case, H_1 needs three labels. Using the three largest label $\{2n-1, 2n-2, 2n-3\}$, we have $w_{max}(H_1) = 6n-6$. Moreover, since $K_{1,n-1}$ has one center vertex which appears in each ascending subgraph, the smallest average weight of t subgraph is

$$\overline{w}_{min}(K_{1,n-1}) = \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + (t-1)(2n-3)\right] = \frac{2n^2-3n+3}{t} + 2n - 3.$$

Applying Lemma 2.1 gives an inequality

$$6n - 6 \geq \frac{2n^2-3n+3}{t} + 2n - 3 \tag{5}$$

which has solution when $n \in [0.875, 1) \cup \{2\} \cup [4, 4.5]$. Since n must be an integer, then $n \in \{2, 4\}$. So, in this case we conclude that if $K_{1,n-1}$ admits an ASD-magic labeling, then $n = 2, 4$.

Case 2. $H_1 \cong P_3$

In this case, since $H_1 \cong P_3$ has 2 edges, then $n - 1 = \binom{t+2}{2} - 1$, and it needs five labels. The five largest labels for H_1 is $\{2n-1, 2n-2, 2n-3, 2n-4, 2n-5\}$, such that $w_{max}(H_1) = 10n - 15$ and

$$\overline{w}_{min}(K_{1,n-1}) = \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + (t-1)(2n-5)\right] = \frac{2n^2-3n+5}{t} + 2n - 5.$$

Applying Lemma 2.1, we have an inequality

$$10n - 15 \geq \frac{2n^2 - 3n + 5}{t} + 2n - 5 \tag{6}$$

that has solution when $n \in [2.5,\ 3] \cup (3,\ 20.13) \cup (21,\ 24.15) \cup (28,\ 28.16)$. This implies that a star $K_{1,n-1}$ is ASD-magic only if $n = 3, 6, 10, 15$.

For the sufficiency, Figure 1 provides the ASD-magic labelings of a star $K_{1,n-1}$ of order $n = 2, 3, 4, 6, 10, 15$. □

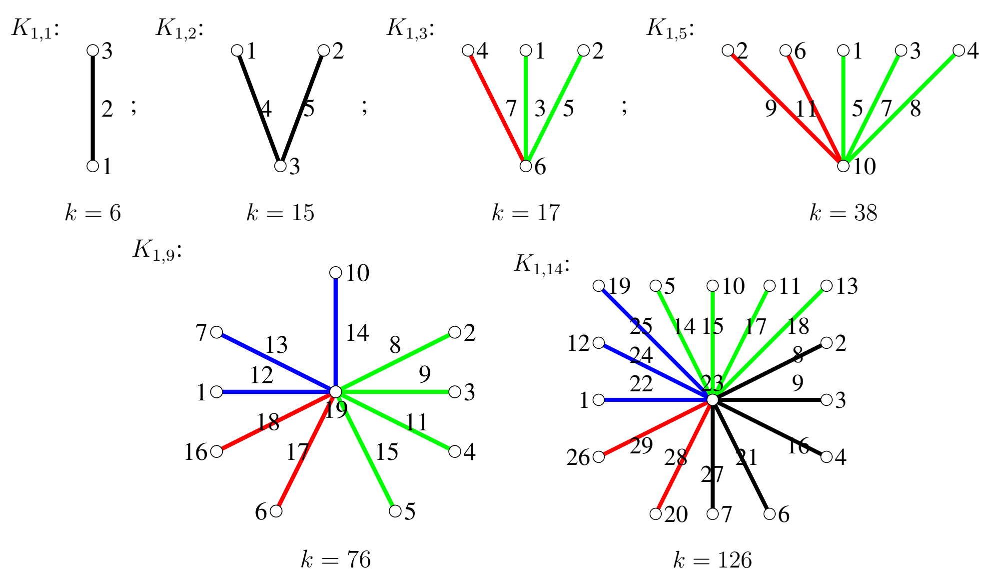

Figure 1. ASD-magic labelings of the star $K_{1,n-1}$

4. ASD-Magic Labelings for Paths

Theorem 4.1. *A path P_n admits an ASD-magic labeling if and only if $n = 1, 2, 3, 4, 6, 7, 10, 15, 21, 28,$ and 36.*

Proof. Let $\{H_i \mid i = 1, 2..., t\}$ be a collection of t ascending subgraphs of P_n. It is easy to see that P_n is ASD-magic for $n = 1, 2, 3$, but for $n \geq 4$ we have the following seven cases based on H_1 and whether it contains an end vertex:

1. $H_1 \cong P_2$ contains an end vertex;
2. $H_1 \cong P_2$ does not contain an end vertex;
3. $H_1 \cong P_3$ contains an end vertex;
4. $H_1 \cong P_3$ does not contain an end vertex;
5. $H_1 \cong 2P_2$ contains two end vertices;
6. $H_1 \cong 2P_2$ does not contain an end vertex; and

7. $H_1 \cong 2P_2$ contains exactly one end vertex.

From formula (1), we count the maximum weight of H_1 for each case.

$$w_{max}(H_1) = \begin{cases} 6n-6, & \text{for Cases 1 and 2,} \\ 10n-15, & \text{for Cases 3 and 4,} \\ 12n-21, & \text{for Cases 5, 6 and 7.} \end{cases} \tag{7}$$

Since the size of P_n satisfies $\binom{t+1}{2} \leq n-1 < \binom{t+2}{2}$,

$$t = \begin{cases} \lfloor \frac{-1+\sqrt{8n-7}}{2} \rfloor, & \text{for Cases 1 and 2,} \\ \lceil \frac{-3+\sqrt{1+8n}}{2} \rceil, & \text{for Cases 3, 4, 5, 6, and 7.} \end{cases} \tag{8}$$

Case 1. $H_1 \cong P_2$ contains an end vertex

Let $q_1, q_2, ..., q_t$ be a sequence of size of H_i for $1 \leq i \leq t$ and $q_1 = 1$. To minimize the number of intersection vertices, $H_i \cong P_{q_i+1}$ $\forall i = 1, 2, ..., t$, and so $c = t-1$. To find the smallest average weight, the label set of C must contain $\{1, 2, ..., t-2, 2n-3\}$, where $2n-3$ is a label of a vertex of H_1. Hence, the minimum average weight for P_n is

$$\overline{w}_{min}(P_n) \geq \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + \sum_{i=0}^{t-2} i + 2n-3\right] = \frac{2n^2+n-2}{t} + \frac{t-3}{2}.$$

By Lemma 2.1,

$$6n-6 \geq \frac{2n^2+n-2}{t} + \frac{t-3}{2}$$

which gives the integer solution $n \in \{4, 7\}$. The ASD-magic labelings of P_4 and P_7 can be seen in Figure 2.

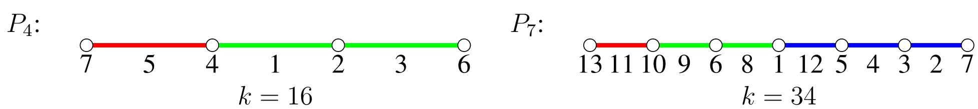

Figure 2. ASD-magic labelings of paths in Case 1

Case 2 $H_1 \cong P_2$ does not contain an end vertex

Let $q_1, q_2, ..., q_t$ be a sequence of the size of H_i for $1 \leq i \leq t$ and $q_1 = 1$. To minimize the number of intersection vertices, we divide the order set into two subcases: when $4 \leq n \leq 6$ and $n \geq 7$. For $n = 4$, the ascending subgraphs of P_4 are $H_1 \cong P_2$ and $H_2 \cong 2P_2$. For $n = 5$, the ascending subgraphs of P_5 are $H_1 \cong P_2$ and $H_2 \cong P_2 \cup P_3$. While for $n = 6$, the ascending subgraphs of P_6 are $H_1 \cong P_2$ and $H_2 \cong 2P_3$ or $H_1 \cong P_2$ and $H_2 \cong P_2 \cup P_4$. Therefore, $c = 2$ when $n = 4, 5, 6$. Next, for each $n = 4, 5,$, and 6, we apply the largest labels for H_1, the smallest labels for H_2, and the two smallest labels of H_1 are labels for the two intersection vertices. Then we derive

$$w_{max}(H_1) = \begin{cases} 18, & \text{if } n = 4, \\ 24, & \text{if } n = 5, \\ 30, & \text{if } n = 6. \end{cases} \tag{9}$$

and

$$\overline{w}_{min}(P_n) = \begin{cases} 19.5, & \text{if } n = 4, \\ 30, & \text{if } n = 5, \\ 42.5, & \text{if } n = 6. \end{cases} \tag{10}$$

From (9) and (10), we see that $w_{max}(H_1) < \overline{w}_{min}(P_n)$ for $n = 4, 5, 6$, which means P_4, P_5, and P_6 do not admit an ASD-magic labeling.

Subsequently, the ascending subgraphs for $P_n, n \geq 7$ are $H_i \cong P_{q_i+1}$, which implies $c = t-1$. Since two of elements in C are two vertices in H_1, then the label set for C must contain $\{1, 2, ..., t-3, 2n-2, 2n-3\}$ and the minimum average weight for P_n is

$$\overline{w}_{min}(P_n) \quad \geq \quad \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + \sum_{i=0}^{t-3} i + (2n-2) + (2n-3)\right] = \frac{2n^2+3n-2}{t} + \frac{t-5}{2}.$$

By Lemma 2.1,

$$6n - 6 \geq \frac{2n^2+3n-2}{t} + \frac{t-5}{2},$$

with solution in $n \in [0.875, 0.999)$, or no integer solution.

Case 3. $H_1 \cong P_3$ contains an end vertex

Let $q_1, q_2, ..., q_t$ be a sequence of the size of subgraph H_i for $1 \leq i \leq t$ and $q_1 = 2$. To minimize the number of intersection vertices, $H_i \cong P_{q_i+1}\ \forall i = 1, 2, ..., t$, and so $c = t-1$. Consider that one element of C is a vertex of H_1, then the labels for C must contain $\{1, 2, ..., t-2, 2n-5\}$ and the minimum average weight for P_n is

$$\overline{w}_{min}(P_n) \quad \geq \quad \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + \sum_{i=0}^{t-2} i + 2n - 5\right] = \frac{2n^2+n-4}{t} + \frac{t-3}{2}.$$

By Lemma 2.1,

$$10n - 15 \geq \frac{2n^2+n-4}{t} + \frac{t-3}{2}$$

with solution in $n \in [2, 2.5] \cup (3, 27.79] \cup (28, 32.74] \cup (36, 37.69]$. Using the fact that $q = \binom{t+2}{2} - 1 = \frac{t(t+3)}{2}$ and $n \geq 6$, if P_n admits an ASD-magic labeling, then $n = 6, 10, 15, 21$. The ASD-magic labelings on paths in this case can be seen in Figure 3.

Case 4. $H_1 \cong P_3$ does not contain an end vertex

Let $q_1, q_2, ..., q_t$ be a sequence of size of H_i for $1 \leq i \leq t$ and $q_1 = 2$. To minimize the number of intersection vertices, we separate into two subcases: $t = 2$ and $t \geq 3$. When $t = 2$ ($n = 6$), P_6 admits an ASD-magic labeling as demonstrated in the Figure 4.

Subsequently, when $t \geq 3$ the ascending subgraphs of P_n are $H_i \cong P_{q_i+1}$, $\forall i = 1, 2, ..., t$, and so $c = t-1$. The label set for C must contain $\{1, 2, ..., t-3, 2n-5, 2n-4\}$ where two of its elements are the labels for two vertices in H_1. Thus,

$$\overline{w}_{min}(P_n) \quad = \quad \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + \sum_{i=1}^{t-3} i + (2n-5) + (2n-4)\right] = \frac{2n^2+3n-6}{t} + \frac{t-5}{2}.$$

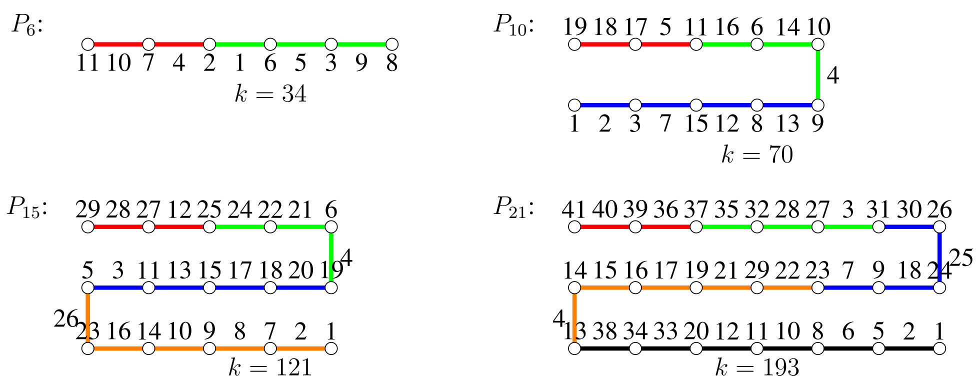

Figure 3. ASD-magic labelings of paths in Case 3

And by Lemma 2.1,

$$10n - 15 \geq \frac{2n^2 + 3n - 6}{t} + \frac{t-5}{2}$$

with solution $n \in (3, 26.88] \cup (28, 31.84] \cup (36, 36.79]$. Using the fact that $q = \binom{t+2}{2} - 1 = \frac{t(t+3)}{2}$ and $n \geq 6$, if a path with $t \geq 3$ admits an ASD-magic labeling, then its order is $n = 6, 10, 15,$ and 21. The ASD-magic labelings of P_n in Case 4 are shown in Figure 4.

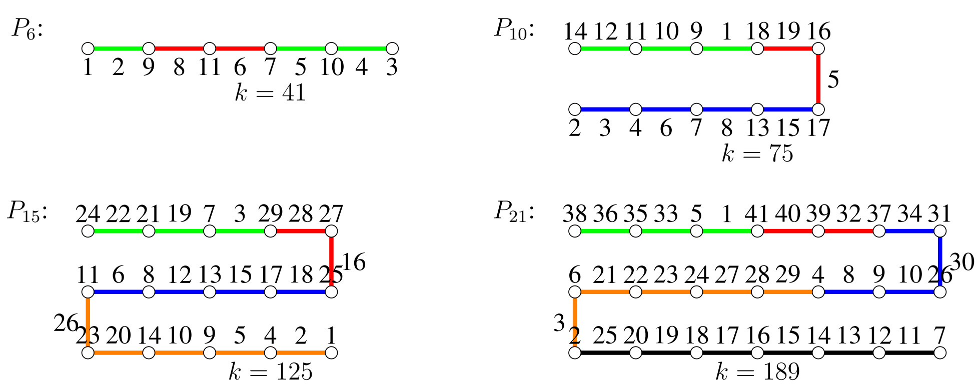

Figure 4. ASD-magic labelings of paths in Case 4

Case 5. $H_1 \cong 2P_2$ contains two end vertices
The minimal number of intersection vertices can be derived utilizing the ascending subgraphs of P_{n-1} in Case 1. Here we add one more intersection vertex while preserving the number of ascending subgraphs. Thus, $c = t$ that includes two vertices of H_1. The label set for C contains $\{1, 2, ..., t-2, 2n-6, 2n-5\}$, thus we obtain

$$\overline{w}_{min}(P_n) \quad \geq \quad \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + \sum_{i=1}^{t-2} i + (2n-6) + (2n-5)\right] = \frac{2n^2+3n-10}{t} + \frac{t-3}{2}.$$

Due to (7), since $H_1 \cong 2P_2$, then $w_{max}(H_1) = 12n - 21$. Using Lemma 2.1, solution of $w_{max}(H_1) \geq \overline{w}_{min}(P_n)$ or $12n - 21 \geq \frac{2n^2+3n-10}{t} + \frac{t-3}{2}$ is $n \in [2, 2.5] \cup (3, 44.5] \cup (45, 50.46] \cup (55, 56.41]$. Using the fact that $q = \binom{t+2}{2} - 1 = \frac{t(t+3)}{2}$ and $n \geq 6$, if a path P_n admits an ASD-magic labeling, then its order is $n = 6, 10, 15, 21, 28,$ and 36. Conversely, the ASD-magic labelings of the paths in Case 5 are shown in Figure 5.

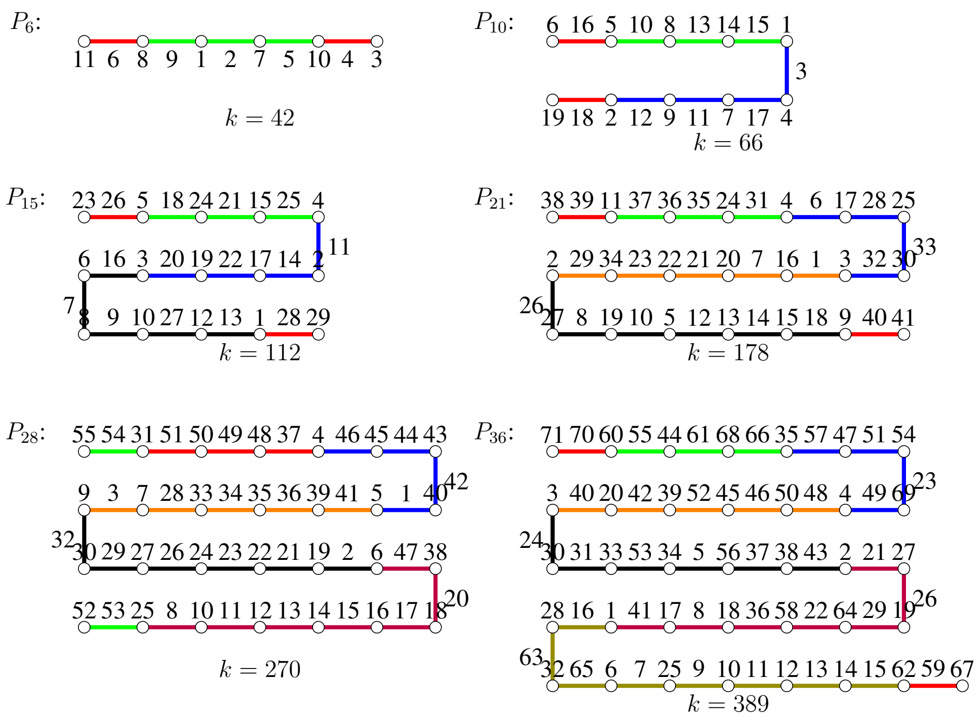

Figure 5. ASD-magic labelings of paths in Case 5

Case 6. $H_1 \cong 2P_2$ does not contain an end vertex

Let $q_1, q_2, ..., q_t$ be a sequence of size of H_i for $1 \leq i \leq t$ and $q_1 = 2$. To minimize the number of intersection vertices, the ascending subgraphs of the paths for $t = 2, 3,$ and 4 are as follows. For $t = 2$ or $n = 6$, the ascending subgraphs of P_6 are $H_1 \cong 2P_2$ and $H_1 \cong 3P_2$. For $t = 3$ or $n = 10$, the ascending subgraphs of P_{10} are $H_1 \cong 2P_2$, $H_2 \cong P_4$, and $H_3 \cong P_2 \cup P_4$. For $t \geq 4$, the ascending subgraphs of P_n are $H_1 \cong 2P_2$, and for $i \geq 2$, $H_i \cong P_{q_i}$. Hence

$$c = \begin{cases} 4, & \text{if } t = 2, 3, 4 \\ t, & \text{if } t \geq 5. \end{cases}$$

For $t = 2, 3$ or 4 $(n = 6, 10, 15)$, to determine $\overline{w}_{min}(P_n)$, the label set of all intersection vertices should contain the smallest labels of H_1, that is $\{2n-6, 2n-5, 2n-4, 2n-3\}$. Consider that those labels are counted twice, so

$$\overline{w}_{min}(P_n) \geq \frac{1}{t}\left[\sum_{i=1}^{2n} i + (2n-6) + (2n-5) + (2n-4) + (2n-3)\right] = \frac{2n^2 + 7n - 18}{t}.$$

Combining Formula (7) and Lemma 2.1, the solution of $w_{max}(H_1) \geq \overline{w}_{min}(P_n)$ or $12n - 21 \geq \frac{2n^2+7n-18}{t}$ is $n \in [1, 1.5] \cup (3, 42.74] \cup (45, 48.74]$. Using the fact that $q = \binom{t+2}{2} - 1 = \frac{t(t+3)}{2}$ and $n \geq 6$, if a path P_n admits an ASD-magic labeling, then its order is $n = 6, 10$, and 15.

Furthermore, for $t \geq 5$, the label set for C contains $\{1, 2, ..., t-5, 2n-6, 2n-5, 2n-4, 2n-3\}$ which includes the four smallest labels of H_1. Therefore,

$$\begin{aligned} \overline{w}_{min}(P_n) &\geq \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + \sum_{i=1}^{t-5} i + (2n-6) + (2n-5) + (2n-4) + (2n-3)\right] \\ &= \frac{2n^2+7n-8}{t} + \frac{t-9}{2}. \end{aligned}$$

Solution of the inequality $w_{max}(H_1) \geq \overline{w}_{min}(P_n)$ or $12n - 21 \geq \frac{2n^2+7n-8}{t} + \frac{t-9}{2}$ is $n \in (3, 42.67] \cup (45, 48.64]$. Using the fact that $q = \binom{t+2}{2} - 1 = \frac{t(t+3)}{2}$ and $n \geq 6$, if $t \geq 5$, if P_n admits an ASD-magic labeling, then its order is $n = 6, 10, 15, 21, 28, 36$. On the other hand, the ASD-magic labelings on paths in Case 6 can be seen in Figure 6.

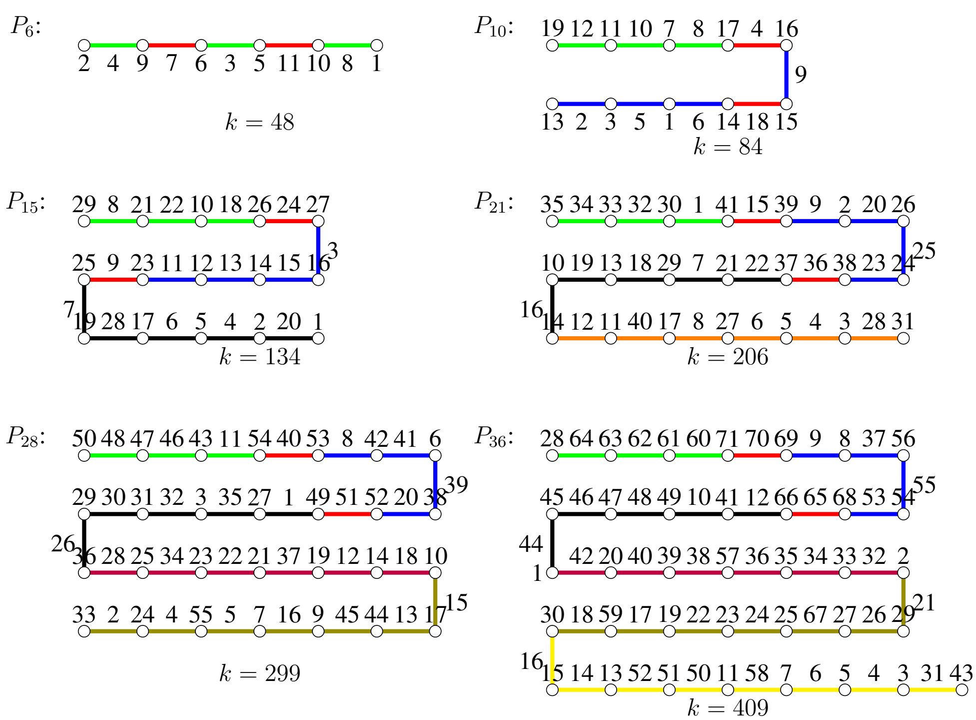

Figure 6. ASD-magic labelings of paths in Case 6

Case 7. $H_1 \cong 2P_2$ contains exactly one end vertex

The minimal number of intersection vertices can be derived utilizing the ascending subgraphs for P_{n-1} in Case 2. Here we add one more intersection vertex while preserving the number of ascending subgraphs. Thus, $c = t$ that includes three vertices of H_1. Now, consider two subcases for $t = 2$ and $t \geq 3$. When $t = 2$ or $n = 6$, P_6 admits an ASD-magic labeling as illustrated in

Figure 7. For $t \geq 3$ or $n \geq 10$, the label set for all the intersection vertices contains $\{1, 2, ..., t-3, 2n-6, 2n-5, 2n-4, 2n-3\}$ and we obtain

$$\overline{w}_{min}(P_n) \;\geq\; \frac{1}{t}\left[\sum_{i=1}^{2n-1} i + \sum_{i=1}^{t-3} i + (2n-6) + (2n-5) + (2n-4)\right] = \frac{2n^2+5n-12}{t} + \frac{t-5}{2}.$$

By Lemma 2.1, the solution of the inequality $w_{max}(H_1) \geq \overline{w}_{min}(P_n)$ or $12n-21 \geq \frac{2n^2+5n-12}{t} + \frac{t-5}{2}$ is $n \in (3, 43.57] \cup (45, 49.53] \cup (55, 55.49]$. Using the fact that $q = \binom{t+2}{2} - 1 = \frac{t(t+3)}{2}$ and $n \geq 6$, if a path in case 7 admits an ASD-magic labeling, then $n = 6, 10, 15, 21, 28,$ and 36. To complete the proof, we present Figure 7 to illustrate the ASD-magic labelings of all paths in Case 7. □

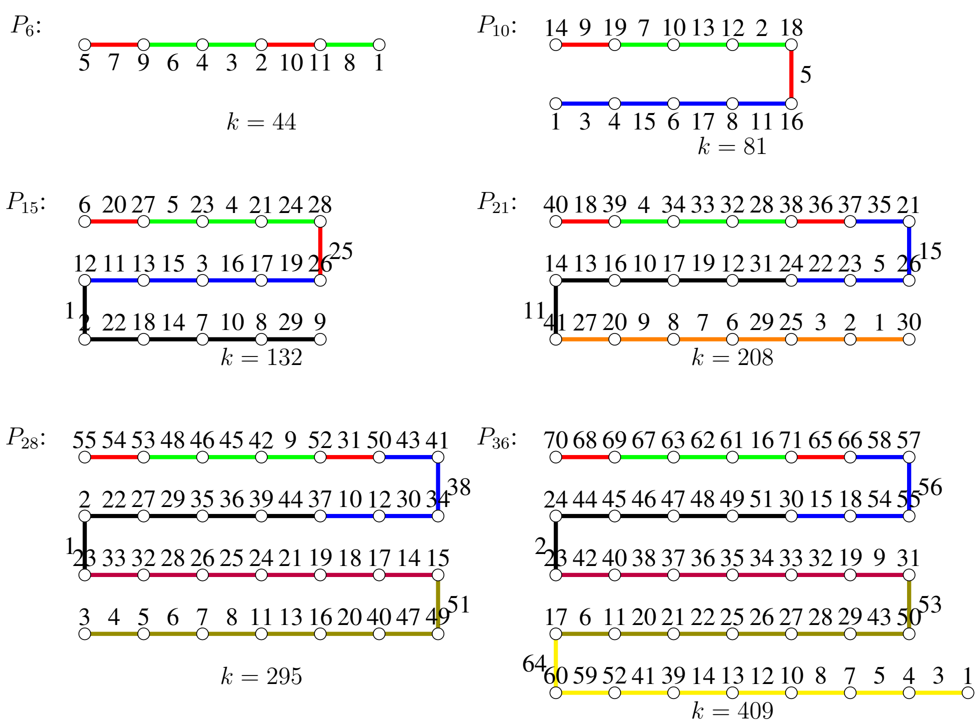

Figure 7. ASD-magic labelings of paths in Case 7

5. ASD-Magic Labelings for Cycles

In this last section, we characterize ASD-magic cycles.

Theorem 5.1. *Cycle C_n is ASD-magic if and only if $n = 3, 5, 9, 14, 20, 27,$ and 35.*

Proof. The three cases to be considered are

1. $H_1 \cong P_2$;
2. $H_1 \cong P_3$; and

3. $H_1 \cong 2P_2$.

As a consequence, we have

$$w_{min}(H_1) = \begin{cases} 6n-3, & \text{for Case 1,} \\ 10n-10, & \text{for Case 2,} \\ 12n-15, & \text{for Case 3.} \end{cases}$$

Since the size of C_n satisfy $\binom{t+1}{2} \leq n < \binom{t+2}{2}$, we obtain

$$t = \begin{cases} \lfloor \frac{-1+\sqrt{1+8n}}{2} \rfloor, & \text{for Case 1,} \\ \lceil \frac{-3+\sqrt{9+8n}}{2} \rceil, & \text{for Cases 2 and 3.} \end{cases}$$

Next, we shall count $\overline{w}_{min}(C_n)$ for each case so that we can apply Lemma 2.1.

Case 1. $H_1 \cong P_2$

The minimal number of intersection vertices can be derived utilizing the ascending subgraphs as in Case 1 of the proof of Theorem 4.1 by joining the two ends of the path. Thus, we obtain $c = t$. To obtain the smallest average weight of a cycle, the set of labels for the intersection vertices must contain $\{1, 2, ..., t-2, 2n-2, 2n-1\}$. This set involves two labels of H_1, so we have

$$\overline{w}_{min}(C_n) \geq \frac{1}{t}\left[\sum_{i=1}^{2n} i + \sum_{i=1}^{t-2} i + (2n-2) + (2n-1)\right] = \frac{2n^2+5n-2}{t} + \frac{t-3}{2}.$$

By Lemma 2.1, the integer solution for $w_{max}(H_1) \geq \overline{w}_{min}(C_n)$ or $6n-3 \geq \frac{2n^2+5n-2}{t} + \frac{t-3}{2}$ is $n = 3$. Therefore, the only cycle admitting an ASD-magic labeling in case 1 is C_3. Moreover, the ASD-magic labeling of C_3 is given in Figure 8.

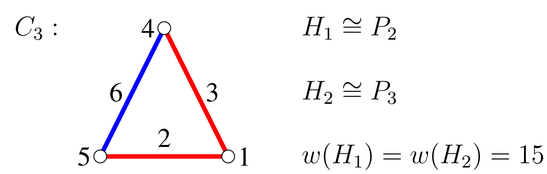

Figure 8. ASD-magic labeling of C_3

Case 2. $H_1 \cong P_3$

The minimal number of intersection vertices can be derived utilizing the ascending subgraphs as in Case 3 of the proof of Theorem 4.1 by joining the two ends of the path. Thus, we get $c = t$. Subsequently, to obtain the smallest average weight of a cycle, the set of labels for the intersection vertices must contain $\{1, 2, ..., t-2, 2n-4, 2n-3\}$, which include two labels of H_1: $2n-4$ and $2n-3$. Hence,

$$\overline{w}_{min}(C_n) \geq \frac{1}{t}\left[\sum_{i=1}^{2n} i + \sum_{i=1}^{t-2} i + (2n-4) + (2n-3)\right] = \frac{2n^2+5n-6}{t} + \frac{t-3}{2}.$$

By Lemma 2.1 the solution for $w_{max}(H_1) \geq \overline{w}_{min}(C_n)$ or $10n - 10 \geq \frac{2n^2+5n-2}{t} + \frac{t-3}{2}$ is $n \in (1, 1.5) \cup (2, 26.3) \cup (27, 31.25) \cup (35, 36.2)$. Using the fact that $q = \binom{t+2}{2} - 1 = \frac{t(t+3)}{2}$ and $n \geq 5$, if a cycle admits an ASD-magic labeling, then the order is $n = 5, 9, 14, 20$. Moreover, the ASD-magic labelings of C_n in Case 2 are given in Figure 9.

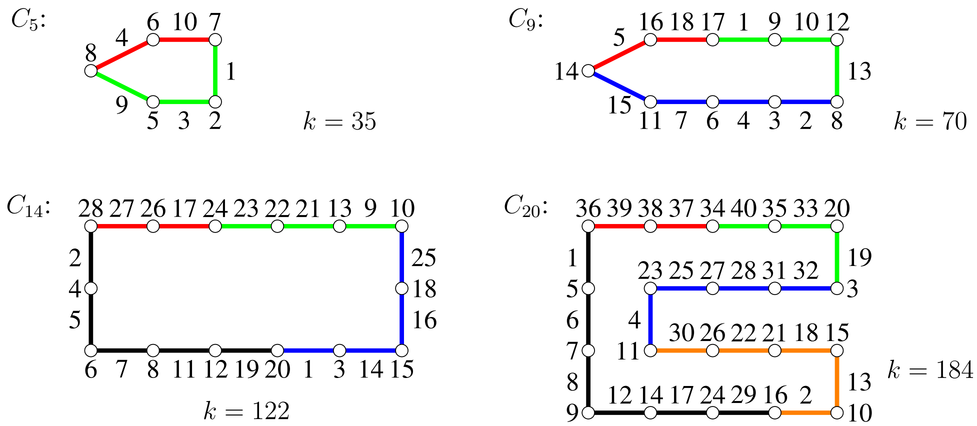

Figure 9. ASD-magic labelings of cycles in Case 2

Case 3. $H_1 \cong 2P_2$
The minimal number of intersection vertices can be derived utilizing the ascending subgraphs as in Cases 6 or 7 of the proof of Theorem 4.1. Thus, for $t = 2$ we have $c = 2$ and for $t \geq 3$ we have $c = t + 1$. For $t = 2$ or $n = 5$, C_5 admits an ASD-magic labeling as shown in the Figure 10. For $t \geq 3$, to find the smallest average weight of cycle, the set of labels for the intersection vertices must contain $\{1, 2, ..., t-3, 2n-5, 2n-4, 2n-3, 2n-2\}$, where $2n-5, 2n-4, 2n-3$, and $2n-2$ are labels for four vertices in H_1. Hence,

$$\begin{aligned} \overline{w}_{min}(C_n) &\geq \frac{1}{t}\left[\sum_{i=1}^{2n} i + \sum_{i=1}^{t-3} i + (2n-5) + (2n-4) + (2n-3) + (2n-2)\right] \\ &= \frac{2n^2+9n-11}{t} + \frac{t-5}{2}. \end{aligned}$$

By Lemma 2.1, the solution of inequality $w_{max}(H_1) \geq \overline{w}_{min}(C_n)$ or $12n - 15 \geq \frac{2n^2+9n-13}{t} + \frac{t-3}{2}$ is $n \in [0.5, 1] \cup (2, 41.98] \cup (44, 47.95]$. Using the fact that $q = \binom{t+2}{2} - 1 = \frac{t(t+3)}{2}$ and $n \geq 5$, if a cycle C_n admits an ASD-magic labeling then its order is $n = 5, 9, 14, 20, 27$, and 35. To complete the proof, Figure 10 presents the ASD-magic labelings of cycles in case 3. □

6. Remark and Open Problems

In this paper, we introduce the notion of ASD-magic labeling, a natural magic labeling arising from the ascending subgraph decomposition. From our preliminary results on characterizing ASD-magic stars, paths, and cycles, few graphs seem to be ASD-magic. However, we can still ask several general questions regarding ASD-magic labeling.

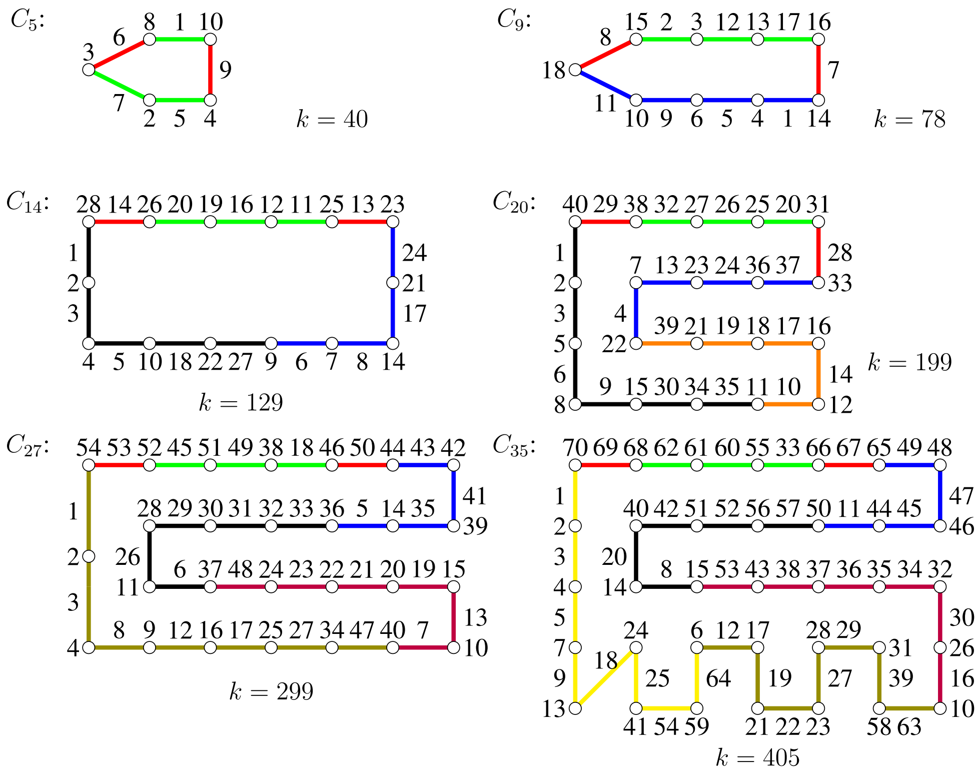

Figure 10. ASD-magic labelings of cycles in Case 3

Problem 1. *If G is an ASD-magic graph of order n, what are the upper and lower bounds of the magic constant k, as functions of n?*

Problem 2. *Does there exist an infinite graph class where most of the graphs in that class are ASD-magic?*

Problem 3. *If G and H are two ASD-magic graphs, which binary graph operation $\circ$ preserves the ASD-magicness of $G \circ H$?*

Problem 4. *If we relax the bijection condition in the ASD-magic labeling to injection, is it possible to have ASD-magic injection for all graphs?*

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