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The Alon-Tarsi number of cupolarotundas and gyroelongated rotunda

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Abstract

The *Alon-Tarsi number* of a graph G is the smallest k so that there exists an orientation D of G with max outdegree $k-1$ satisfying the number of even Eulerian subgraphs different from the number of odd Eulerian subgraphs. This paper is devoted to the study of the Alon-Tarsi number of cupolarotundas and gyroelongated rotunda.

Keywords: Alon-Tarsi number, choice number, chromatic number, Combinatorial Nullstellensatz, planar graph
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1. Introduction

We only consider simple and finite graphs in this paper. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the least positive integer k such that G has a proper vertex coloring using k colors. List coloring is a well-known variation on vertex coloring. For list coloring, a k-*list assignment* of a graph G is a mapping L which assigns to each vertex v of G a set $L(v)$ of k permissible colors. An $L\text{-}coloring$ of G is a coloring f of G such that $f(v) \in L(v)$ for each vertex v. We say G is L-*colorable* if there exists a proper L-coloring of G. A graph G is k-*choosable* if G is L-colorable for every k-list assignment L. The *choice number* of a graph G is the least positive integer k such that G is k-choosable, denoted by $ch(G)$.

A subdigraph H of a directed graph D is called *Eulerian* if $V(H) = V(G)$ and the indegree $d_H^-(v)$ of every vertex v of H in H is equal to its outdegree $d_H^+(v)$. We do not assume that H

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is connected. H is even if it has an even number of edges, otherwise, it is odd. Let $\mathcal{E}_e(D)$ and $\mathcal{E}_o(D)$ denote the family of even and odd Eulerian subgraphs of D, respectively. Let diff$(D) = |\mathcal{E}_e(D)| - |\mathcal{E}_o(D)|$. We say that D is *Alon-Tarsi* if diff$(D) \neq 0$. If an *orientation* D of G yields an *Alon-Tarsi digraph*, then we say D is an *Alon-Tarsi orientation* (*AT-orientation*, for short) of G [1].

The *Alon-Tarsi number* of G ($AT(G)$, for short) is the smallest k so that there exists an orientation D of G with max outdegree $k-1$ satisfying the number of Eulerian subgraphs with even edges different from the number of Eulerian subgraphs with odd edges. It was proposed by Alon and Tarsi [4], subsequently, they used algebraic methods to prove that $\chi(G) \leq ch(G) \leq AT(G)$. The graph G is called *chromatic-choosable* if $\chi(G) = ch(G)$. The graph G is called *chromatic-AT choosable* if $\chi(G) = AT(G)$. There are some results concerning the Alon-Tarsi number of planar graphs. Zhu [10] showed that every planar graph G has $AT(G) \leq 5$. Grytczuk and Zhu proved that every planar graph G has a matching M such that $AT(G-M) \leq 4$ in [3]. Zhu and Kim also showed that every planar graph G has a forest F such that $AT(G - E(F)) \leq 3$ in [6]. Zhu and Lu [9] used the discharging method to show that for $l \in \{5, 6, 7\}$, every graph $G \in \mathcal{P}_{4,l}$ has a matching M such that $G - M$ has the Alon-Tarsi number at most 3 which $\mathcal{P}_{4,l}$ means the family of planar graphs with no cycles of length 4 and l. The first author and Ye et al proved that a Halin graph H has the Alon-Tarsi number 4 when it is a wheel of even order and 3 otherwise in [8].

There are also some conclusions for special graphs, Zhu and Balakrishnan [11] proved that bipartite graph G has $AT(G) = max_{H \subset G} \lceil \frac{|E(H)|}{|V(H)|} \rceil + 1$, they also showed that bipartite planar graphs G have $AT(G) \leq 3$. There are some conclusions about the Cartesian product of graphs, Kaul and Mudrock proved that the Cartesian product of any cycle with a path with at least two vertices has the Alon-Tarsi number 3 in [5]. Suppose that G is a complete graph or an odd cycle with $|V(G)| \geq 3$. Suppose H is a graph on at least two vertices that contains a Hamilton path $\omega_1, \omega_2, \dots, \omega_n$, such that ω_i has at most k neighbors among $\omega_1, \omega_2, \dots, \omega_{i-1}$, Kaul, Mudrock [5] also proved the Cartesian product of G and H has $AT(G \Box H) \leq \Delta(G) + k$. The first author and Shao et al. proved that the Cartesian product of C_m and C_n has the Alon-Tarsi number 4 when n and m are both odd and 3 otherwise in [7].

A graph G is a polyhedral graph if G is isomorphic to the 1-skeleton of a three-dimensional convex polyhedron P. According to Steinitz's theorem [2], every polyhedral graph is planar and 3-connected. Rotunda and cupola play important roles in polyhedral graphs. Moreover, they can also extend a variety of polyhedral graphs.

A *cupola* is formed by joining two parallel polygons, one as the top surface, the other as the bottom of the polygon with twice the number of edges, and its sides are formed by a combination of triangles and quadrangles. An n-gonal cupola Q_n has $3n$ vertices, $5n$ edges. The *rotunda* is similar to the *cupola* but instead of alternating quadrangles and triangles, it is composed of alternating pentagons and triangles. An n-gonal rotunda has $4n$ vertices, $7n$ edges.

There are some ways *cupola* and *rotunda* can be combined. The first *cupolarotunda* $R^I_{Q_n}$ is an infinite set of polyhedra, constructed by adjoining an n-gonal cupola to a $2n$-gonal rotunda (see Figure 1 (a) for $n = 3$). For $n \geq 3$, a $R^I_{Q_n}$ has $5n$ triangles, n squares, $2n$ pentagons, an n-gonal and a $4n$-gonal as faces. We use $|V(G)|$ and $|E(G)|$ for the number of vertices and edges in graph

G, it is easy to see $|V(R^I_{Q_n})| = 9n$ and $|E(R^I_{Q_n})| = 17n$. The second *cupolarotunda* $R^{II}_{Q_n}$ is an infinite set of polyhedra, constructed by adjoining an n-gonal rotunda to a $2n$-gonal cupola (see Figure 1 (b) for $n = 3$). For $n \geq 3$, a $R^{II}_{Q_n}$ has $4n$ triangles, $2n$ squares, n pentagons, an n-gonal and a $4n$-gonal as faces, and $|V(R^{II}_{Q_n})| = 8n$, $|E(R^{II}_{Q_n})| = 15n$.

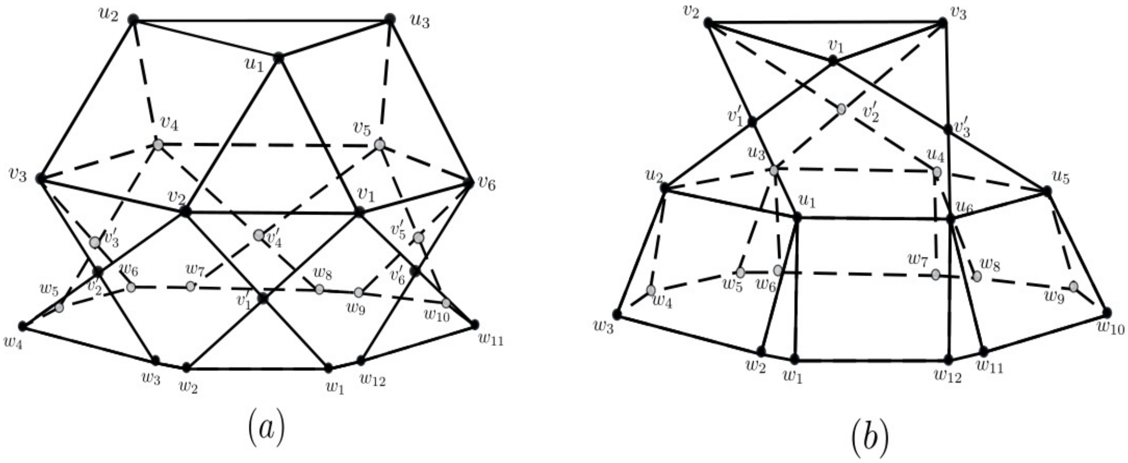


Figure 1. (a) A cupolarotunda $R^I_{Q_3}$ and (b) a cupolarotunda $R^{II}_{Q_3}$.

Antiprism also plays an important role in polyhedral graphs. The *gyroelongated rotunda* G_{R_n} is an infinite set of polyhedra, constructed by adjoining an n-gonal rotunda to a $2n$-gonal antiprism (see Figure 2 for $n = 3$). For $n \geq 3$, a *gyroelonged rotunda* has $6n$ triangles, n pentagons, an n-gonal and a $2n$-gonal as faces, and $|V(G_{R_n})| = 6n$, $|E(G_{R_n})| = 13n$.

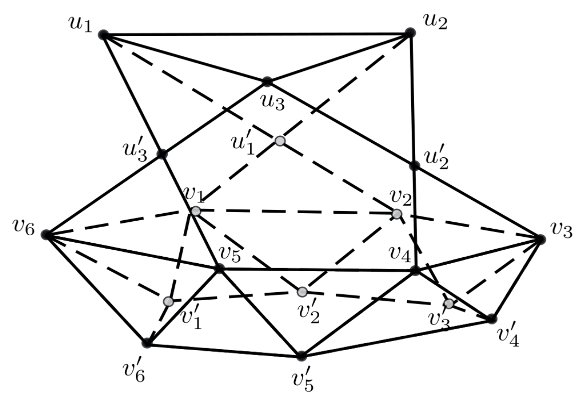


Figure 2. Gyroelongated rotunda G_{R_3}.

Lemma 1.1. *Given an orientation D of graph G, if D has no odd directed cycle, then D is an AT-orientation of G.*

Our goal is to compute the exact value of the Alon-Tarsi number of cupolarotundas $R^I_{Q_n}$, $R^{II}_{Q_n}$, and gyroelongated rotunda G_{R_n}. The main results are the following theorems:

Theorem 1. For $cupolarotundas$ $R^I_{Q_n}$ and $R^{II}_{Q_n}$, we have

$$AT(R^I_{Q_n}) = AT(R^{II}_{Q_n}) = 3.$$

Theorem 2. For a $gyroelongated\ rotunda$ G_{R_n}, we have

$$AT(G_{R_n}) = 4.$$

2. The Proof of the Theorem 1

Assume $V(R^I_{Q_n}) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{2n}, v'_1, v'_2, \ldots, v'_{2n}, w_1, w_2, \ldots, w_{4n}\}$. The top cycle $u_1u_2\cdots u_nu_1$ is called C_1, the middle cycle $v_1v_2\cdots v_{2n}v_1$ is called C_2 and the bottom cycle $w_1w_2\cdots w_{4n}w_1$ is called C_3; the vertices $v'_1, v'_2, \ldots, v'_{2n}$ are points located between C_2 and C_3. For $i = 1, \ldots, n$, the vertices $u_iv_{2i}v_{2i-1}$ form a triangle and $u_iu_{i+1}v_{2i+1}v_{2i}$ form a quadrangle. For $j = 1, \ldots, 2n$, the vertices $v^{'}_jv_jv_{j+1}$ form a triangle, $v^{'}_jw_{2j}w_{2j-1}$ form a triangle and $v^{'}_jv_{j+1}v^{'}_{j+1}w_{2j+1}w_{2j}$ form a pentagon (see Figure 3).

The proof of Theorem 1 will be completed by the following lemmas.

Lemma 2.1. *For a cupolarotunda* $R^I_{Q_n}$,

$$\chi(R^I_{Q_n}) = 3.$$

Proof. It is easily seen that $R^I_{Q_n}$ contains a triangle as its subgraph, hence $\chi(R^I_{Q_n}) \geq 3$. It remains to show that $\chi(R^I_{Q_n}) \leq 3$. It suffices to show that ϕ: $V(R^I_{Q_n}) \to \{0, 1, 2\}$ is a proper 3-coloring of $R^I_{Q_n}$.

Case 1. n is even.

For the vertices in C_1, let $\phi(u_1) = 2$. For $i = 2, \ldots, n$, let $\phi(u_i) = 1$ when i is an odd number and $\phi(u_i) = 0$ when i is an even number.

For the vertices in C_2, for $j = 1, \ldots, 2n$, let $\phi(v_j) = 0$ or 1 when j is an even number. Therefore, for $i = 2, \ldots, n$, if $\phi(u_i) = 1$, then $\phi(v_{2i}) = 0, \phi(v_{2i-1}) = 2$; if $\phi(u_i) = 0$, then $\phi(v_{2i}) = 1, \phi(v_{2i-1}) = 2$. Since n is even, we have $\phi(u_n) = 0, \phi(v_{2n}) = 1$; the neighbours of v_1 are u_1, v_{2n}, and $u_1v_1v_2$ form a triangle, it is easy to know that $\phi(v_1) = 0, \phi(v_2) = 1$.

Then $\phi(v'_j)$ can be determined in a unique way.

For the vertices in C_3, for $j = 1, \ldots, 2n$, the neighbours of v'_j are $v_j, v_{j+1}, w_{2j}, w_{2j-1}$. Let $\phi(v_j) = \phi(w_{2j})$, $\phi(v_{j+1}) = \phi(w_{2j-1})$ when j is an even number and let $\phi(v_j) = \phi(w_{2j-1})$, $\phi(v_{j+1}) = \phi(w_{2j})$ when j is an odd number. We can know that ϕ is a proper coloring of $R^I_{Q_n}$.

Case 2. n is odd.

This is similar to what happens in case 1. Since n is odd, we have $\phi(u_n) = 1, \phi(v_{2n}) = 0$; the neighbours of v_1 are u_1, v_{2n}, and $u_1v_1v_2$ form a triangle, it is easy to know that $\phi(v_1) = 1$, $\phi(v_2) = 0$. The other points are colored in the same way in case 1, we can know that ϕ is a proper coloring of $R^I_{Q_n}$. (See Figure 3 (a) for $n = 4$,(b) for $n = 5$)

Hence $\chi(R^I_{Q_n}) \leq 3$.

□

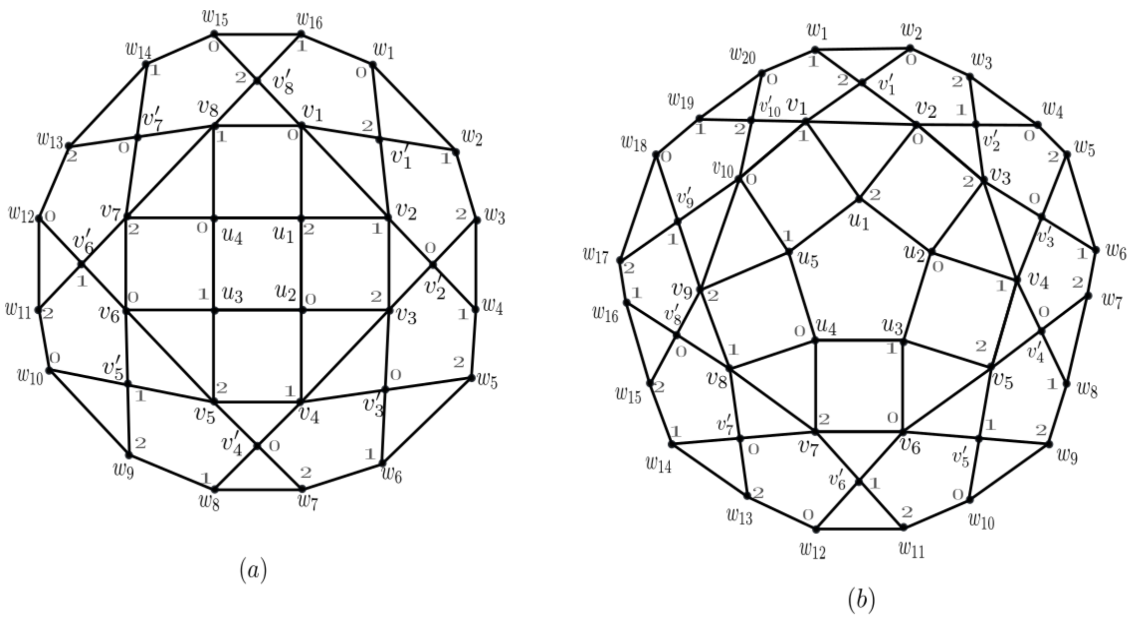


Figure 3. (a) A proper 3-coloring of $R^I_{Q_4}$ and (b) for $R^I_{Q_5}$.

Lemma 2.2. *For a cupolarotunda $R^I_{Q_n}$,*

$$AT(R^I_{Q_n}) = 3.$$

Proof. By Lemma 2.1, $AT(R^I_{Q_n}) \geq \chi(R^I_{Q_n}) = 3$. What is left is to show that $AT(R^I_{Q_n}) \leq 3$. We denote $L(n)$ as the set of edges belonging to quadrangles connecting C_1 and C_2.

We give $R^I_{Q_n}$ an orientation D. The rules of orientation D are as follows:

R1: For the cycles C_2, C_3 are clockwise. For the cycle C_1, u_1u_2 is oriented from u_2 to u_1, u_iu_{i+1} is oriented from u_i to u_{i+1} $(i = 2, \ldots, n)$.

R2: The edges belonging to $L(n)$ are oriented from C_2 to C_1.

R3: For $j = 1, 2, \ldots, 2n$, let $v_jv'_j, v_{j+1}v'_j$ are oriented from v'_j to v_j, v_{j+1} and let the edges v'_jw_{2j}, v'_jw_{2j-1} are oriented from w_{2j}, w_{2j-1} to v'_j.

Since the orientation D has no odd directed cycle, by Lemma 1.1, D is an AT-orientation, and it is easy to see that every vertex $x \in V(R^I_{Q_n})$ has outdegree at most 2, so $AT(R^I_{Q_n}) \leq 3$ (see Figure 4 (a) for $n = 4$,(b) for $n = 5$).

□

Assume $V(R^{II}_{Q_n}) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n, u_1, u_2, \ldots, u_{2n}, w_1, w_2, \ldots, w_{4n}\}$. The top cycle $v_1v_2 \ldots v_nv_1$ is called C_1, the middle cycle $u_1u_2 \ldots u_{2n}u_1$ is called C_2 and the bottom cycle $w_1w_2 \ldots w_{4n}w_1$ is called C_3; the vertices $v'_1, v'_2, \ldots, v'_n$ are points located between C_1 and C_2. For $i = 1, \ldots, n$, the vertices $v'_iv_iv_{i+1}$ form a triangle, $v'_iu_{2i}u_{2i-1}$ form a triangle and $v'_iv_{i+1}v'_{i+1}u_{2i+1}u_{2i}$ form a pentagon. For $j = 1, \ldots, 2n$, $u_jw_{2j}w_{2j-1}$ form a triangle and $u_ju_{j+1}w_{2j+1}$ w_{2j} form a quadrangle (see Figure 5).

Lemma 2.3. *For a cupolarotunda $R^{II}_{Q_n}$,*

$$\chi(R^{II}_{Q_n}) = 3.$$

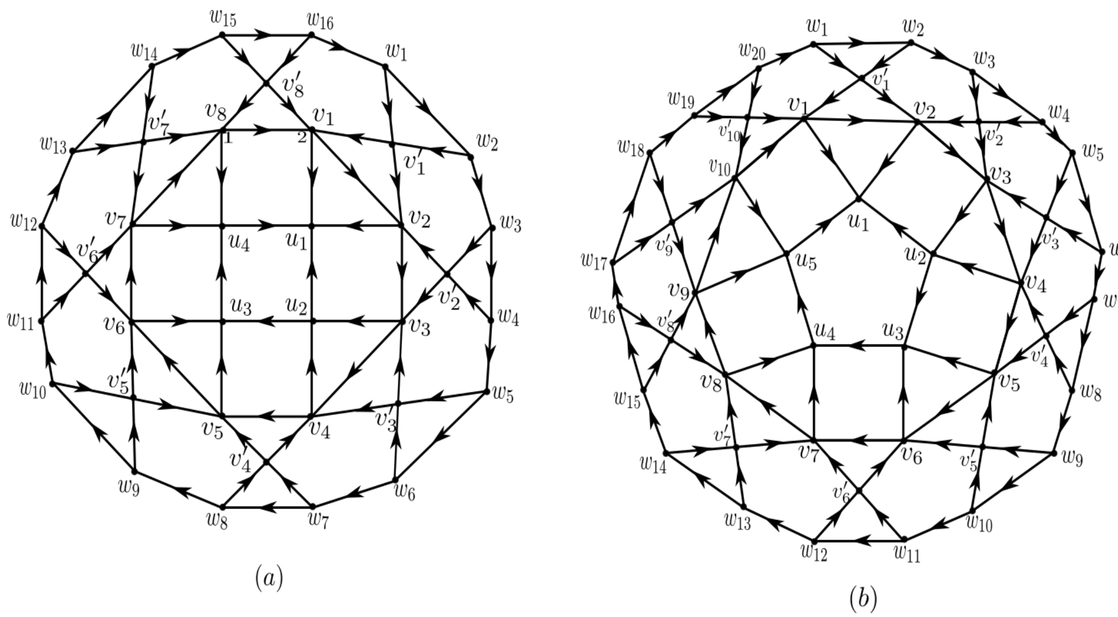


Figure 4. (a) An orientation of $R^I_{Q_4}$ and (b) for $R^I_{Q_5}$.

Proof. It is easily seen that $R^{II}_{Q_n}$ contains a triangle as its subgraph, hence $\chi(R^{II}_{Q_n}) \geq 3$. It remains to show that $\chi(R^{II}_{Q_n}) \leq 3$. It suffices to show that ϕ: $V(R^{II}_{Q_n}) \to \{0, 1, 2\}$ is a proper 3-coloring of $R^{II}_{Q_n}$.

Case 1. $n \equiv 0 \pmod 3$.

For the vertices in C_3. For $k = 1, 2, \ldots, 4n$, let $\phi(w_k) = 0$ if $k \equiv 1 \pmod 3$; let $\phi(w_k) = 1$ if $k \equiv 2 \pmod 3$ and let $\phi(w_k) = 2$ if $k \equiv 0 \pmod 3$. The coloring of the remaining vertices can be uniquely determined.

For the vertices in C_2. For $j = 1, 2, \ldots, 2n$, since $u_j w_{2j} w_{2j-1}$ form a triangle, we have $\phi(u_j) = 2$ if $j \equiv 1 \pmod 3$; $\phi(u_j) = 1$ if $j \equiv 2 \pmod 3$ and $\phi(u_j) = 0$ if $j \equiv 0 \pmod 3$.

For $i = 1, 2, \ldots, n$, since $v_i' u_{2i} u_{2i-1}$ form a triangle, we have $\phi(v_i') = 0$ if $i \equiv 1 \pmod 3$; $\phi(v_i') = 1$ if $i \equiv 2 \pmod 3$ and $\phi(v_i') = 2$ if $i \equiv 0 \pmod 3$.

For the vertices in C_1. Since v_i is adjacent v_i', v_{i-1}', hence $\phi(v_i) = 1$ if $i \equiv 1 \pmod 3$; $\phi(v_i) = 2$ if $i \equiv 2 \pmod 3$ and $\phi(v_i) = 0$ if $i \equiv 0 \pmod 3$ (see Figure 5 (a) for $n = 3$).

Case 2. $n \equiv 1 \pmod 3$.

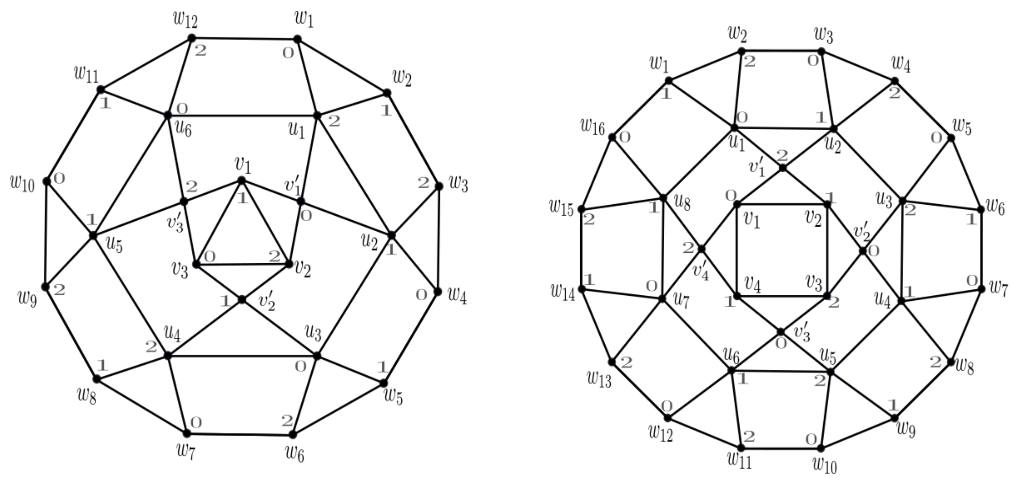
For the vertices in C_1, let $\phi(v_n) = 1$. For $i = 1, 2, \ldots, n-1$, let $\phi(v_i) = 0$ if $i \equiv 1 \pmod 3$; $\phi(v_i) = 1$ if $i \equiv 2 \pmod 3$ and let $\phi(v_i) = 2$ if $i \equiv 0 \pmod 3$.

Let $\phi(v_{n-1}') = 0, \phi(v_n') = 2$. For $i = 1, 2, \ldots, n-2$, let $\phi(v_i') = 2$ if $i \equiv 1 \pmod 3$; $\phi(v_i') = 0$ if $i \equiv 2 \pmod 3$ and let $\phi(v_i') = 1$ if $i \equiv 0 \pmod 3$.

For the vertices in C_2, note that v_i' is adjacent $v_i, v_{i+1}, u_{2i}, u_{2i-1}$. When n is an even number, for $i = 1, 2, \ldots, n$, let $\phi(v_i) = \phi(u_{2i})$, $\phi(v_{i+1}) = \phi(u_{2i-1})$ when i is an even number and $\phi(v_{i+1}) = \phi(u_{2i})$, $\phi(v_i) = \phi(u_{2i-1})$ when i is an odd number. When n is an odd number, let $\phi(u_{2n}) = 1, \phi(u_{2n-1}) = 0$ and for $i = 1, 2, \ldots, n-1$, let $\phi(v_i) = \phi(u_{2i})$, $\phi(v_{i+1}) = \phi(u_{2i-1})$ when i is an even number and $\phi(v_{i+1}) = \phi(u_{2i})$, $\phi(v_i) = \phi(u_{2i-1})$ when i is an odd number.

For the vertices in C_3, we know that u_{2i} is adjacent w_{4i}, w_{4i-1} and u_{2i-1} is adjacent w_{4i-2}, w_{4i-3}.

When n is an odd number, let $\phi(w_{4n-7}) = 2, \phi(w_{4n-6}) = \phi(w_{4n-4}) = 0, \phi(w_{4n-5}) = 1$. For $i = 1, 2, \ldots, n-2, n$, let $\phi(v'_i) = \phi(w_{4i-1}) = \phi(w_{4i-3})$ when i is an even number and $\phi(v'_i) = \phi(w_{4i}) = \phi(w_{4i-2})$ when i is an odd number. When n is an even number, for $i = 1, 2, \ldots, n$, let $\phi(v'_i) = \phi(w_{4i-1}) = \phi(w_{4i-3})$ when i is an even number and $\phi(v'_i) = \phi(w_{4i}) = \phi(w_{4i-2})$ when i is an odd number (see Figure 5 (b) for $n = 4$, Figure 7 (a) for $n = 7$).



(a) (b)

Figure 5. (a) A proper 3-coloring of $R^{II}_{Q_3}$ and (b) for $R^{II}_{Q_4}$.

Case 3. $n \equiv 2 \pmod 3$.

This is similar to what happens in case 2. For the vertices in C_1, let $\phi(v_{n-1}) = 1, \phi(v_n) = 2$. The other points are colored in the same way as C_1 in case 2, it is easy to know that $\phi(v'_i)$ is colored in a unique way $(i = 1, \ldots, n)$.

For the vertices in C_2, let $\phi(u_{2n}) = 2, \phi(u_{2n-1}) = 0$. The other points are colored in the same way as C_2 in case 2.

For the vertices in C_3. When n is an even number, let $\phi(w_{4n-9}) = 1, \phi(w_{4n-8}) = \phi(w_{4n-10}) = 0, \phi(w_{4n-11}) = 2$ and for $i = 1, 2, \ldots, n-3, n-1, n$, $\phi(w_{4i}), \phi(w_{4i-1}), \phi(w_{4i-2}), \phi(w_{4i-3})$ are similar to C_3 in case 2. When n is an odd number, let $\phi(w_{4n}) = 0, \phi(w_{4n-1}) = \phi(w_{4n-3}) = 1, \phi(w_{4n-2}) = 2$ and let $\phi(w_{4n-4}) = \phi(w_{4n-6}) = 0, \phi(w_{4n-5}) = 2, \phi(w_{4n-7}) = 1$ and for $i = 1, 2, \ldots, n-2$, $\phi(w_{4i}), \phi(w_{4i-1}), \phi(w_{4i-2}), \phi(w_{4i-3})$ are similar to C_3 in case 2 (see Figure 6 (a) for $n = 5$, Figure 7 (b) for $n = 8$).

□

Lemma 2.4. *For a cupolarotunda $R^{II}_{Q_n}$,*

$$AT(R^{II}_{Q_n}) = 3.$$

Proof. By Lemma 2.3, $AT(R^{II}_{Q_n}) \geq \chi(R^{II}_{Q_n}) = 3$. What is left is to show that $AT(R^{II}_{Q_n}) \leq 3$, the orientation method is similar to $R^{I}_{Q_n}$. Since the orientation D has no odd directed cycle, by Lemma 1.1, D is an AT-orientation and it is easy to see that every vertex $x \in V(R^{II}_{Q_n})$ has outdegree at most 2, so $AT(R^{II}_{Q_n}) \leq 3$ (see Figure 6 (b) for $n = 3$). □

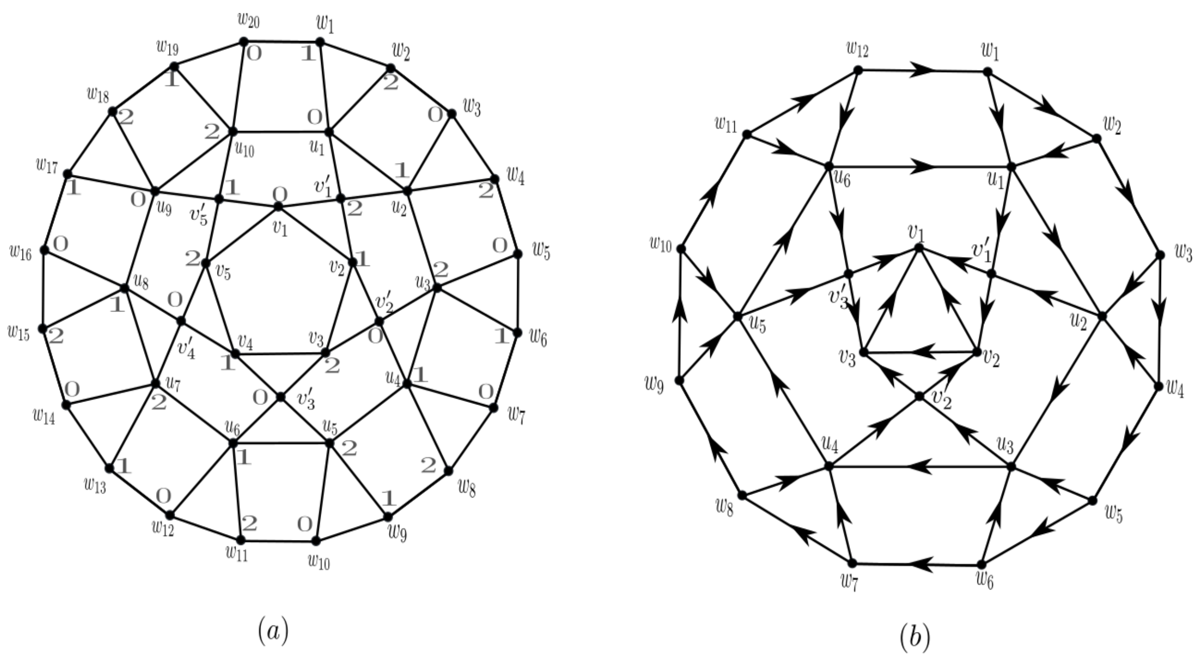


Figure 6. (a) A proper 3-coloring of $R^{II}_{Q_5}$ and (b) an orientation of $R^{II}_{Q_3}$.

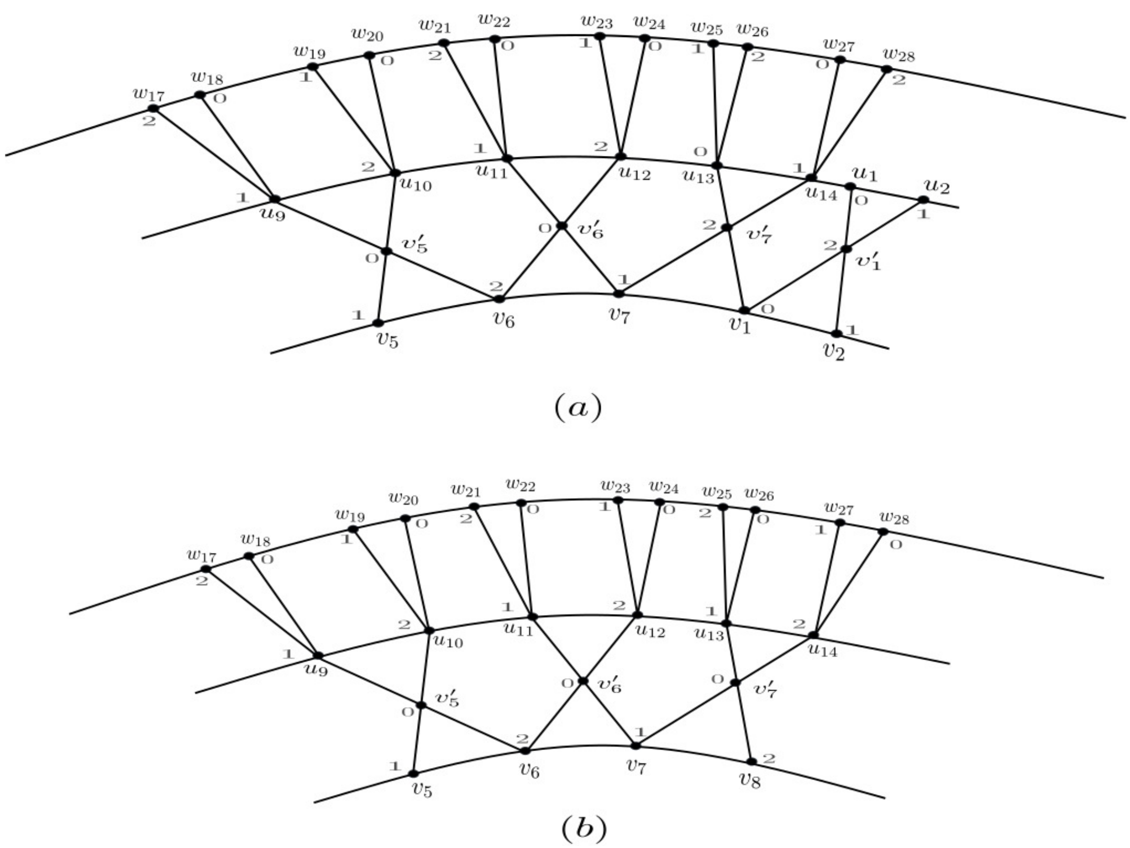


Figure 7. (a) Local points coloring of $R^{II}_{Q_7}$ and (b) for $R^{II}_{Q_8}$.

Corollary 2.1. *Cupolarotundas $R^{I}_{Q_n}$ and $R^{II}_{Q_n}$ are chromatic-AT choosable, where $n \geq 3$.*

3. The Proof of the Theorem 2

Assume $V(G_{R_n}) = \{u_1, u_2, \ldots, u_n, u'_1, u'_2, \ldots, u'_n, v_1, v_2, \ldots, v_{2n}, v'_1, v'_2, \ldots, v'_{2n}\}$. The top cycle $u_1u_2 \cdots u_nu_1$ is called C_1, the middle cycle $v_1v_2 \cdots v_{2n}v_1$ is called C_2 and the bottom cycle $v'_1v'_2 \cdots v'_{2n}v'_1$ is called C_3; the vertices $u'_1, u'_2, \ldots, u'_{2n}$ are points located between C_1 and C_2. For $i = 1, \ldots, n$, the vertices $u^{'}_iu_iu_{i+1}$ form a triangle, $u^{'}_iv_{2i}v_{2i-1}$ form a triangle and $u^{'}_iu_{i+1}u^{'}_{i+1}v_{2i+1}v_{2i}$ form a pentagon. For $j = 1, \ldots, 2n$, $v_jv_{j+1}v'_{j+1}$ form a triangle and $v_jv'_jv'_{j+1}$ form a triangle (see Figure 8).

The proof of Theorem 2 will be completed by the following lemmas.

Lemma 3.1. *For a gyroelongated rotunda G_{R_n},*

$$\chi(G_{R_n}) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3; \\ 4, & \text{otherwise.} \end{cases}$$

Proof. It is easy to see that G_{R_n} contains a triangle as its subgraph, hence $\chi(G_{R_n}) \geq 3$. By the Four-Color Theorem, $\chi(G_{R_n}) \leq 4$.

Let ϕ: $V(G_{R_n}) \to \{0, 1, 2\}$. Without loss of generality, let $\phi(v'_1) = 0$ and $\phi(v'_2) = 1$, it is a simple matter to obtain the colors of $v_1, \ldots, v_{2n}, v'_3, \ldots, v'_{2n}$ in a unique way. For $q = 0, 1, \ldots, \lfloor \frac{2n}{3} \rfloor$, we have $\phi(v'_{3q+1}) = 0$, $\phi(v'_{3q+2}) = 1$, $\phi(v'_{3q+3}) = 2$ and $\phi(v_{3q+1}) = 2$, $\phi(v_{3q+2}) = 0$, $\phi(v_{3q+3}) = 1$.

Case 1. $n \equiv 1 \pmod 3$.

When $n \equiv 1 \pmod 3$, $2n \equiv 2 \pmod 3$. By the above rule, we have $\phi(v'_{2n}) = 1$,$\phi(v_{2n}) = 0$. It is easy to see that v_{2n} is the neighbor of v'_1, but $\phi(v'_1) = 0$, that a contradiction. Hence it is not 3-colorable (see Figure 8 (a) for $n = 4$).

Case 2. $n \equiv 2 \pmod 3$.

When $n \equiv 2 \pmod 3$, $2n \equiv 1 \pmod 3$. By the above rule, we have $\phi(v'_{2n}) = 0$,$\phi(v_{2n}) = 2$. Note that v'_{2n} is adjacent to v'_1, but $\phi(v'_1) = 0$, that a contradiction. Hence it is not 3-colorable (see Figure 8 (b) for $n = 5$).

Case 3. $n \equiv 0 \pmod 3$.

When $n \equiv 0 \pmod 3$, $2n \equiv 0 \pmod 3$, we can give a 3-coloring as follows:

For $q = 0, 1, \ldots, \lfloor \frac{2n}{3} \rfloor$, let $\phi(v'_{3q+1}) = 0$, $\phi(v'_{3q+2}) = 1$, $\phi(v'_{3q+3}) = 2$ and let $\phi(v_{3q+1}) = 2$, $\phi(v_{3q+2}) = 0$, $\phi(v_{3q+3}) = 1$.

For $p = 0, 1, \ldots, \lfloor \frac{n}{3} \rfloor$, let $\phi(u'_{3p+1}) = 1$, $\phi(u'_{3p+2}) = 0$, $\phi(u'_{3p+3}) = 2$ and let $\phi(u_{3p+1}) = 0$, $\phi(u_{3p+2}) = 2$, $\phi(u_{3p+3}) = 1$. It is a proper 3-coloring (see Figure 9 for $n = 3$).

□

Lemma 3.2. *For a gyroelongated rotunda G_{R_n},*

$$AT(G_{R_n}) = 4.$$

Proof. The G_{R_n} has $6n$ vertices and $13n$ edges. Since $\sum_{x \in V(D)} d^+_D(x) = |A(D)|$, by the Pigeonhole Principle, there exists some vertices have outdegree at least 3 for any orientation D of G_{R_n}. Hence $AT(G_{R_n}) \geq 4$. What is left is to show that $AT(G_{R_n}) \leq 4$.

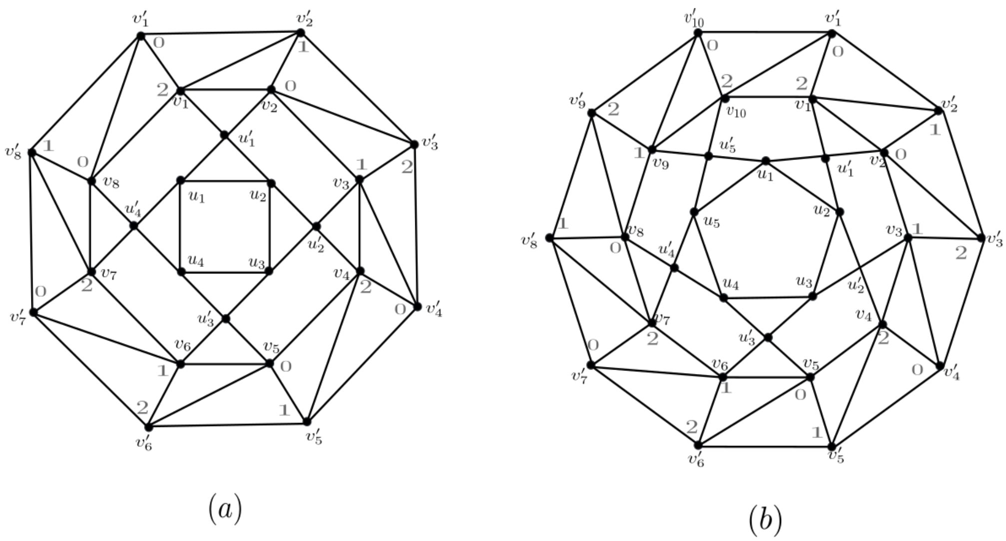


Figure 8. (a) An improper 3-coloring of G_{R_4} and (b) for G_{R_5}.

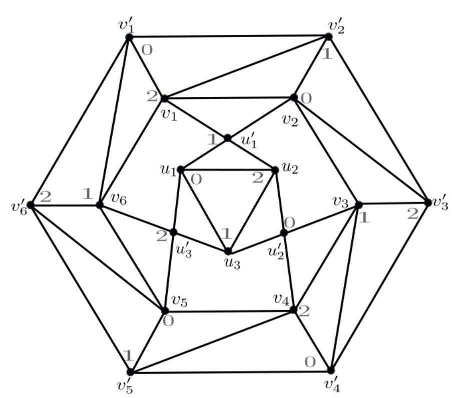


Figure 9. A proper 3-colouring of G_{R_3}.

We give G_{R_n} an orientation D. The rules of orientation D are given as following:

R1: For the cycles C_2, C_3 are clockwise. For the cycle C_1, u_1u_2 is oriented from u_2 to u_1, u_iu_{i+1} is oriented from u_i to u_{i+1} $(i = 2, \ldots, n)$.

R2: For $i = 1, 2, \ldots, n$, let the edges $u'_iu_i,\ u'_iu_{i+1}$ are oriented from u'_i to u_i, u_{i+1} and let u'_iv_{2i}, u'_iv_{2i-1} are oriented from v_{2i}, v_{2i-1} to u'_i.

R3: For $j = 1, 2, \ldots, 2n$, let the edges v'_jv_j is oriented from v'_j to v_j and let $v'_{j+1}v_j$ is oriented from v'_{j+1} to v_j.

It is easy to see the orientation D has no odd directed cycle, by Lemma 1.1, D is an AT-orientation, note that every vertex $x \in V(G_{R_n})$ has outdegree at most 3, so $AT(G_{R_n}) \leq 4$ (see Figure 10 (a) for $n = 4$,(b) for $n = 5$).

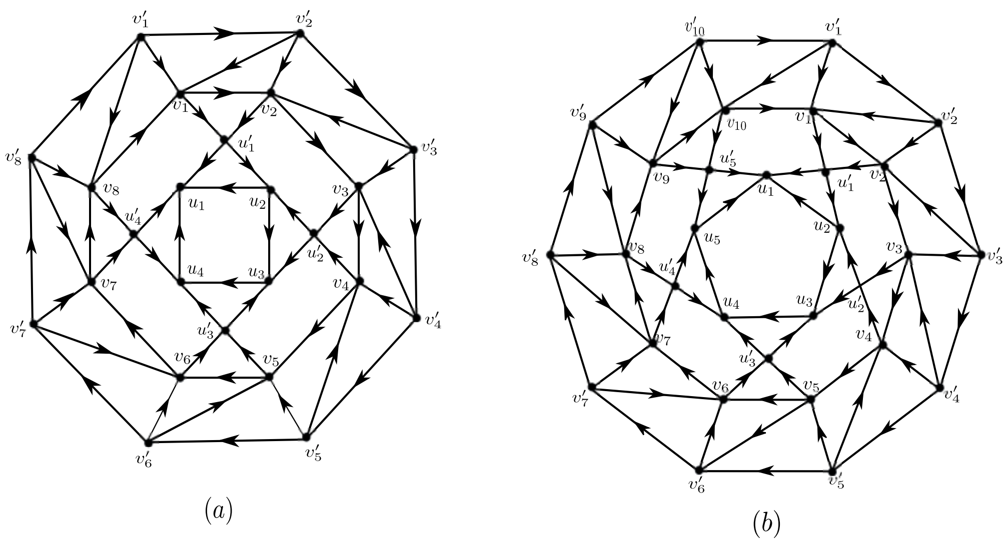


Figure 10. (a) An orientation of G_{R_4} and (b) for G_{R_5}.

□

Corollary 3.1. *The gyroelongated rotunda G_{R_n} is not $chromatic$-AT $choosable$, where $n \geq 3$.*

Corollary 3.2. *For a gyroelongated rotunda G_{R_n},*

$$ch(G_{R_n}) = 4.$$

Proof. Since $\chi(G_{R_n}) \leq ch(G_{R_n}) \leq AT(G_{R_n})$, it can be conclude that $ch(G_{R_n}) = 4$ when $n \equiv 1$ or $2 \pmod 3$.

When $n \equiv 0 \pmod 3$, we can give a 3-list assignment L of G_{R_n} using colors 0, 1, 2 and 3 as follows. Let $L(v_j) = L(v'_j) = L(u'_i) = \{0, 1, 2\}$ for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, 2n$; and $L(u_1) = \{1, 2, 3\},\ L(u_k) = \{0, 1, 3\}$ for $k = 2, \ldots, n$. Without loss of generality, let $\phi(v'_1) = 0$, $\phi(v'_2) = 1$, by Lemma 3.1, $\phi(u'_1) = 1,\ \phi(u'_2) = 0$ and $\phi(u'_n) = 2$. Since u_1 is adjacent to u'_1, u'_n, and u_2 is adjacent to u'_1, u'_2, we have $\phi(u_1) = \phi(u_2) = 3$, that a contradiction. It is an improper L-colouring of G_{R_n}, so $ch(G_{R_n}) = 4$ when $n \equiv 0 \pmod 3$ (see Figure 11 for $n = 3$). □

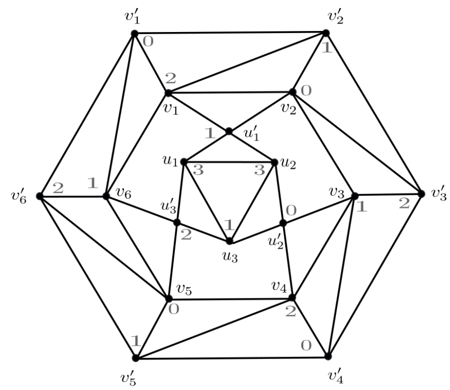


Figure 11. An improper 3-list colouring of G_{R_3}.

4. Conclusion

In this article, we obtain the exact value of the Alon-Tarsi number of cupolarotundas $R^I_{Q_n}$,$R^{II}_{Q_n}$, and gyroelongated rotunda G_{R_n} by using the AT-orientation skill. Additionally, cupolarotundas $R^I_{Q_n}$ and $R^{II}_{Q_n}$ are $chromatic$-$AT\ choosable$, but the gyroelongated rotunda G_{R_n} is not $chromatic$-$AT\ choosable$.

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