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Rainbow connection number of corona product of graphs

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Abstract

In an edge-colored graph (where adjacent edges may have the same color), a rainbow path is a path whose edge colors are all distinct. The coloring is called a rainbow coloring if any two vertices can be connected by a rainbow path. The rainbow connection number $rc(G)$ is the smallest number of colors in a rainbow coloring of G. The corona product $G \circ H$ of two graphs G and H is constructed from one copy of G and $n = |V(G)|$ disjoint copies of H such that the i-th vertex of G is joined to all vertices in the i-th copy of H, for each $i \in \{1, \ldots, n\}$. Several resuls on the rainbow connection number of corona product have been published, but there are inaccuracies. In this paper, we close the gaps and add new results. The strong variant of rainbow connection number is also discussed.

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1. Introduction

For the sake of completeness, we summarize some definition and ideas of graph theory that will be used throughout the paper. We mainly follow standard terminology and notation, such as in [1]. All graphs considered will be finite, undirected, and simple. Formally, a graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$ such that every edge e is an unordered pair of vertices $e = xy = yx$, with $x, y \in V(G), x \neq y$, as its endpoints. Two vertices x and y are called neighbours or adjacent if there is an edge xy in the graph. Two edges are called adjacent if they

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share exactly one common endpoint. The degree $\deg x$ or $\deg_G x$ is the number of neighbours of the vertex x, or equivalently the number of edges having x as an endpoint. A vertex is called isolated if its degree is 0, or a pendant vertex if its degree is 1. A walk of length $r-1$ is a sequence of vertices $W : x_1 - x_2 - \cdots - x_r$ such that every pair of consecutive vertices are adjacent; the vertices $x_2, \ldots, x_{r-1}$ are called the internal vertices of the walk, and we say that the walk connects x_1 to x_r. If $W : x_1 - \cdots - x_r$ is a walk, the notation $x_i W x_j$ refers to the part of W from x_i to x_j. A path is a walk whose vertices are all distinct. A cycle is a walk $x_1 - \cdots - x_r - x_1$ with $x_1, \ldots, x_r$ all distinct. A graph is called a connected graph if every pair of vertices can be connected by a walk (equivalently, by a path). The distance $d(x, y)$ or $d_G(x, y)$ is the smallest length of a walk between the vertices x and y. If there is no walk between two vertices, their distance is defined to be ∞. The diameter of a graph is the maximum distance between two vertices in the graph, $diam(G) = \max\{d(x, y) \mid x, y \in V(G)\}$. A complete graph K_n consists of n pairwise adjacent vertices. A path graph P_n consists of a path with length $n - 1$ and no other edge. A cycle graph C_n consists of a cycle with length n and no other edge. A tree is a connected graph without any cycle; a tree with n vertices is often denoted by T_n. Two graphs G and H are isomorphic, $G \cong H$, if there is a bijection $f : V(G) \to V(H)$ such that $xy \in V(G) \iff f(x)f(y) \in E(H)$.

In the wider graph theory literature, the word "coloring" usually means proper coloring. A proper edge-coloring is any map $c : E(G) \to \{1, 2, \ldots, k\}$ such that adjacent edges have distinct colors. However, in the rainbow connection literature, the word "coloring" is more flexible: adjacent edges may have the same color. The concept of rainbow coloring can be motivated by the desire to design a secure communication network between government agencies; the reader is referred to Section 1.2 in [15] for a more detailed account. We shall explain the basic concepts. Following Chartrand et al. [5], a coloring of G is any map $\gamma : E(G) \to \{1, 2, \ldots, k\}$, where adjacent edges may have the same color. We call $\gamma(xy) = i$ the color of the edge xy, and we write $x \overset{i}{-} y$. A path is called rainbow if its edge colors are all distinct. The coloring is called rainbow if every pair of distinct vertices can be connected by a rainbow path. The rainbow connection number $rc(G)$ is the smallest number of colors in a rainbow coloring of G. Chartrand et al. also studied a stronger variant of rainbow coloring. A geodesic between two vertices x and y is any path between them with length $d(x, y)$. A strong rainbow coloring is a map $\gamma : E(G) \to \{1, 2, \ldots, k\}$ such that every pair of distinct vertices can be connected by a rainbow geodesic. The strong rainbow connection number $src(G)$ is the smallest number of colors in a strong rainbow coloring of G. If a graph has a (strong) rainbow coloring, it must be connected. Conversely, on any connected graph we can put a (strong) rainbow coloring where all edges have distinct colors. The graph must be non-trivial (i.e. have more than one vertex), otherwise $E(G) = \emptyset$. Therefore, $rc(G)$ and $src(G)$ are defined if and only if G is a non-trivial connected graph.

Since their introduction in 2008, rainbow connection numbers have been a fairly popular research topic, with several variants and generalization. For example, there is a vertex version [13]; total version [24]; directed version [7]; hypergraph version [2]; connectivity version [6], [26]; and local version [23]. Computing rainbow connection numbers in general is hard [3]. Therefore, many studies are focused on specific classes of graph e.g. complete graphs, trees, cycles, wheels, complete multipartite graphs [5]; circulant graphs [25]; line graphs [14]; comb product [10]; graph join [21]; sequential join [22], [20]; graphs arising from algebraic structures [8], [28]; etc. The reader is refered to Li and Sun's book [15] and dynamic survey [16] for more detailed expositions. Below, we collect several results that will be referenced later in the paper.

Theorem 1.1 ([5]).

1. *If G is a non-trivial connected graph, then $diam(G) \leq rc(G) \leq src(G) \leq |E(G)|$. Each inequality in this chain is tight.*
2. *The equality $rc(G) = 1$ holds if and only if G is a complete graph K_n with $n \geq 2$.*
3. *If T_n is a tree with $n \geq 2$ vertices, then $rc(T_n) = src(T_n) = |E(T_n)| = n - 1$.*
4. *If C_n is a cycle with $n \geq 3$ vertices, then $rc(C_n) = src(C_n) = \lceil n/2 \rceil$.*
5. *If H is a connected subgraph that spans G (meaning $V(H) = V(G)$) then $rc(G) \leq rc(H)$.*

Remark 1.1. From the third and fifth statements we get that if $H = T_n$ is any spanning tree of G, then $rc(G) \leq rc(T_n) = n - 1$.

In the following, n_i denotes the number of vertices of degree i in the given graph.

Theorem 1.2 ([19]). *If G is a non-trivial connected graph, then $rc(G) \geq n_1(G)$.*

In the present study, the author is interested in the rainbow connection number of corona product $G \circ H$ of two graphs G and H. The corona product is obtained from one copy of G and $n = |V(G)|$ copies of H such that the i-th vertex of G is joined by an edge to all vertices in the i-th copy of H, for each $i \in \{1, \dots, n\}$. Generally $G \circ H$ is not isomorphic to $H \circ G$ (except in some cases e.g. when $G \cong H$). Corona product was introduced by Frucht and Harary in 1970 [11] as an example of a graph product whose automorphism group is the wreath product of the groups of its factors. Since then, corona product has been studied in various contexts of graph labelings [12]. Note that $G \circ H$ is connected if and only if G is connected, so $rc(G \circ H)$ is defined if and only if G is connected. Several results on the rainbow connection number of corona product have been published, for example $C_n \circ K_1$ [27]; $C_n \circ P_m$ and $C_n \circ C_m$ [18]; $K_n \circ K_m$ [17]; and $G \circ H$ where $|V(G)| \geq 3$ and $|V(H)| \geq 2$ [9]; but some are inaccurate or have incomplete/unclear proofs. In the next section we present our proofs and some new results.

2. Results

2.1. General Bounds

Recall that n_i denotes the number of vertices with degree i in the given graph. Note that $K_1 \circ H = K_1 + H$ is the usual graph join (also denoted by $K_1 \vee H$). Since the rainbow connection number of graph join have been studied in [21], the following results on the (strong) rainbow connection number of $G \circ H$ assume that G is non-trivial, so that $rc(G)$ and $src(G)$ exist.

Theorem 2.1. *If G is a connected graph with $n \geq 2$ vertices, and H is any graph, then*

$$rc(G \circ H) \geq \max\left\{diam(G) + 2,\ n \cdot n_0(H),\ rc(G)\right\} \tag{1}$$

$$src(G \circ H) \geq \max\left\{src(G),\ src(K_1 \circ H)\right\} \tag{2}$$

Proof. First, from Theorem 1.1 we have $rc(G \circ H) \geq diam(G \circ H) = diam(G) + 2$ and from Theorem 1.2 we have $rc(G \circ H) \geq n_1(G \circ H) = n \cdot n_0(H)$.

Note that if a path in $G \circ H$ has both its endpoints in G, then that path must lie entirely in G (otherwise it will pass throught the entry/exit point to G more than once). This implies that any (strong) rainbow coloring of $G \circ H$ restricts to a (strong) rainbow coloring of G. Therefore, $rc(G \circ H) \geq rc(G)$ and $src(G \circ H) \geq src(G)$.

Similarly, if a path in $G \circ H$ has both its endpoints in the same subgraph $\{g_i\} \circ H_i$, then that path cannot leave the subgraph so any (strong) rainbow coloring of $G \circ H$ restricts to a (strong) rainbow coloring of $\{g_i\} \circ H_i$. This gives $rc(G \circ H) \geq rc(K_1 \circ H)$ and $src(G \circ H) \geq src(K_1 \circ H)$. □

Remark 2.1. The bound $rc(G \circ H) \geq rc(K_1 \circ H)$ was ignored because it is weaker than $n \cdot n_0(H)$. In fact $rc(K_1 \circ H) \leq \max\{3, n_0(H)\}$ (see Theorem 2.1 in [21]).

Remark 2.2. The first and second bounds in (1) are tight, e.g. $rc(K_3 \circ K_m) = diam(K_3) + 2$ (see Theorem 2.6) and $rc(C_n \circ K_1) = n \cdot n_0(K_1)$ when n is odd (see Theorem 2.8). We could not find examples of equality in the other bounds. These bounds imply that $G \circ H$ can have arbitrarily large rc and src compared to G: if H has many isolated vertices, significantly more than $rc(G)$, then (1) implies $rc(G \circ H) \geq n \cdot n_0(H) \gg rc(G)$. Similarly, if $n_0(H) \gg src(G)$ then (2) and Theorem 1.2 imply $src(G \circ H) \geq src(K_1 \circ H) \geq rc(K_1 \circ H) \geq n_1(K_1 \circ H) = n_0(H) \gg src(G)$.

Next, we prove an upper bound.

Theorem 2.2. *If G is a non-trivial connected graph and H is a graph with no isolated vertex, then $rc(G \circ H) \leq rc(G) + 3$.*

Proof. Let $V(G) = \{g_1, \ldots, g_n\}$ and let H_i be the copy of H that is attached to g_i, for each $i \in \{1, \ldots, n\}$. We will construct a rainbow coloring on $G \circ H$ with $q+3$ colors, where $q = rc(G)$. First put a rainbow coloring on G with q colors. We will put 3 new colors on the remaining edges as follows. For each $i \in \{1, \ldots, n\}$, let T_i be a spanning tree for H_i. Since any tree is bipartite, we can write $V(T_i) = A_i \cup B_i$ with $A_i \cap B_i = \emptyset$ and no edge of T_i has its endpoints both in A_i nor both in B_i. Put the color $q+1$ on every edge from g_i to A_i, put the color $q+2$ on every edge from g_i to B_i, and put the color $q+3$ on the other edges. We show that this is indeed a rainbow coloring. We started with a rainbow coloring on G, so it is enough to consider the following two cases.

- If $x \in A_i$ and $y \in B_j$ for some $i, j \in \{1, \ldots, n\}$, a rainbow path between them can be found as follows. Choose a rainbow path P in G from g_i to g_j. Then $x \overset{q+1}{-} g_i P g_j \overset{q+2}{-} y$ is a rainbow path, because P only uses colors in $\{1, \ldots, q\}$.

- If $x \in A_i$ and $y \in A_j$ for some $i, j \in \{1, \ldots, n\}$, a rainbow path between them can be found as follows. Choose a neighbour $z \in V(T_i)$ of x. Then $z \in B_i$ because A_i, B_i is a bipartition of T_i. Choose a rainbow path P from g_i to g_j. Then $x \overset{q+3}{-} z \overset{q+2}{-} g_i P g_j \overset{q+1}{-} y$ is a rainbow path, because P only uses colors in $\{1, \ldots, q\}$. Similarly if $x \in B_i$ and $y \in B_j$.

This completes the proof of the theorem. □

Remark 2.3. From Theorem 2.1 and Theorem 2.2 we get

$$n_0(H) = 0 \implies \max\{rc(G),\ diam(G) + 2\} \leq rc(G \circ H) \leq rc(G) + 3 \tag{3}$$

In [9] it was stated that $rc(G \circ H) = rc(G) + 3$ for any connected graphs G, H with $|V(G)| \geq 3$ and $|V(H)| \geq 2$. This is incorrect: it is possible to have $rc(G \circ H) < rc(G) + 3$, e.g. $rc(K_3 \circ K_m) = rc(K_3) + 2$ (see Theorem 2.6). However, there are many examples with $rc(G \circ H) = rc(G) + 3$, such as $rc(P_n \circ P_2) = rc(P_n) + 3$ (see Theorem 2.4) and $rc(K_n \circ H) = rc(K_n) + 3$ when $n \geq 4$ (see Theorem 2.5).

Remark 2.4. We could not find a similar upper bound for src. The rainbow coloring constructed in the proof of Theorem 2.2 is probably not a strong rainbow coloring. Following the notation in the proof, the rainbow path $x \overset{q+3}{-} z \overset{q+2}{-} g_i P g_j \overset{q+1}{-} y$ is not a geodesic (a shorter path can be obtained by going directly from x to g_i) so there may not be any geodesic rainbow from x to y.

Finally, we consider the corona product $G \circ H$ when $H \cong K_1$.

Theorem 2.3. *If G is a connected graph with $|V(G)| = n \geq 2$, then*

$$n \leq rc(G \circ K_1) \leq n + rc(G). \tag{4}$$

Proof. The lower bound follows from Theorem 2.1. For the upper bound, we construct a rainbow coloring on $G \circ K_1$ with $n + rc(G)$ colors as follows: first put a rainbow coloring on G using $rc(G)$ colors, then give every pendant edge its own new color. This coloring is clearly rainbow. □

Remark 2.5. These bounds are tight, for example $rc(C_n \circ K_1) = n$ when n is odd (see Theorem 2.8) and $rc(T_n \circ K_1) = n + rc(T_n)$ where T_n is any tree with n vertices (see Theorem 2.7). It is possible to have $n < rc(G \circ K_1) < n + rc(G)$, e.g. $rc(C_n \circ K_1) = n + 1$ when n is even (see Theorem 2.9).

2.2. *Exact Values*

Here we find $rc(G \circ H)$ for some particular graphs G and H, and sometimes we get $src(G \circ H)$ too. These were used in the previous section as tight examples.

Theorem 2.4. *If $n \geq 3$, then $rc(P_n \circ P_2) = n + 2$.*

Proof. Let $P_n : g_1 - g_2 - \cdots - g_n$ and for each $i \in \{1, \ldots, n\}$ let H_i be the i-th copy of $H \cong P_2$ attached to g_i with $V(H_i) = \{h_i^1, h_i^2\}$. From Theorem 1.1 and Theorem 2.2 we get $rc(P_n \circ P_2) \leq rc(P_n) + 3 = n - 1 + 3 = n + 2$. We will prove $rc(P_n \circ P_2) \geq n + 2$.

Suppose otherwise that $rc(P_n \circ P_2) \leq n + 1$. Then there is a rainbow coloring γ on $P_n \circ P_2$ with $n + 1$ colors. Under that coloring, there is a rainbow path from h_1^1 to h_n^1. The length of this path is at least $1 + (n - 1) + 1 = n + 1$. Since there are only $n + 1$ colors, the length is exactly $n + 1$. By relabeling the colors if necessary, we may assume that the colors are as follows

$$h_1^1 \overset{1}{-} g_1 \overset{2}{-} g_2 \overset{3}{-} \cdots \overset{n-1}{-} g_{n-1} \overset{n}{-} g_n \overset{n+1}{-} h_n^1$$

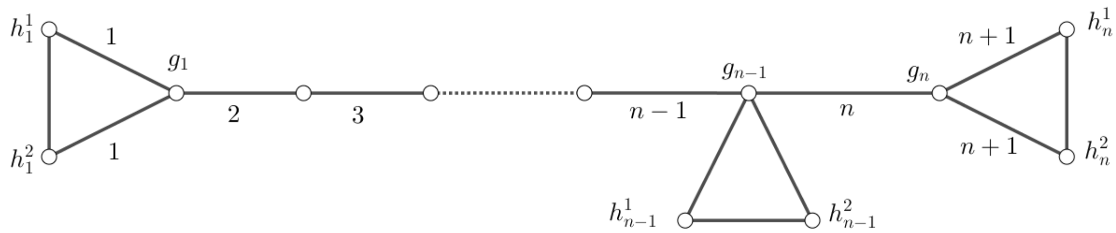

Figure 1. Considering the colors of H_{n-1} in $P_n \circ P_2$.

Similarly, by considering rainbow paths between h_1^2 and h_n^1, as well as between h_1^1 and h_n^2, we get $\gamma(h_1^2 g_1) = 1$ and $\gamma(g_n h_n^2) = n+1$. Next we consider the colors of H_{n-1}.
There are three paths from h_1^1 to h_{n-1}^1 with length at most $n+1$, namely

$$h_1^1 \overset{1}{-} g_1 \overset{2}{-} g_2 \overset{3}{-} \cdots \overset{n-1}{-} g_{n-1} - h_{n-1}^1 \tag{5}$$

$$h_1^1 - h_1^2 \overset{1}{-} g_1 \overset{2}{-} g_2 \overset{3}{-} \cdots \overset{n-1}{-} g_{n-1} - h_{n-1}^1 \tag{6}$$

$$h_1^1 \overset{1}{-} g_1 \overset{2}{-} g_2 \overset{3}{-} \cdots \overset{n-1}{-} g_{n-1} - h_{n-1}^2 - h_{n-1}^1 \tag{7}$$

and there are four paths from h_n^1 to h_{n-1}^1,

$$h_n^1 \overset{n+1}{-} g_n \overset{n}{-} g_{n-1} - h_{n-1}^1 \tag{8}$$

$$h_n^1 - h_n^2 \overset{n+1}{-} g_n \overset{n}{-} g_{n-1} - h_{n-1}^1 \tag{9}$$

$$h_n^1 \overset{n+1}{-} g_n \overset{n}{-} g_{n-1} - h_{n-1}^2 - h_{n-1}^1 \tag{10}$$

$$h_n^1 - h_n^2 \overset{n+1}{-} g_n \overset{n}{-} g_{n-1} - h_{n-1}^2 - h_{n-1}^1 \tag{11}$$

One of the paths (5), (6), (7) must be rainbow, and one of the paths (8), (9), (10), (11) must be rainbow. Consider the following cases:

a. The path (5) or (6) is rainbow. Then $\gamma(g_{n-1}h_{n-1}^1) \in \{n, n+1\}$.
b. The path (7) is rainbow. Then $\{\gamma(g_{n-1}h_{n-1}^2), \gamma(h_{n-1}^2 h_{n-1}^1)\} = \{n, n+1\}$.
c. The path (8) or (9) is rainbow. Then $\gamma(g_{n-1}h_{n-1}^1) \notin \{n, n+1\}$.
d. The path (10) or (11) is rainbow. Then $\{\gamma(g_{n-1}h_{n-1}^2), \gamma(h_{n-1}^2 h_{n-1}^1)\} \cap \{n, n+1\} = \emptyset$.

One of a,b is true, and one of c,d is true. Cases a,c are incompatible, and so are b,d. Therefore, either we have a,d or b,c. There are three paths from h_1^1 to h_{n-1}^2 with length at most $n+1$,

$$h_1^1 \overset{1}{-} g_1 \overset{2}{-} g_2 \overset{3}{-} \cdots \overset{n-1}{-} g_{n-1} - h_{n-1}^2$$

$$h_1^1 - h_1^2 \overset{1}{-} g_1 \overset{2}{-} g_2 \overset{3}{-} \cdots \overset{n-1}{-} g_{n-1} - h_{n-1}^2$$

$$h_1^1 \overset{1}{-} g_1 \overset{2}{-} g_2 \overset{3}{-} \cdots \overset{n-1}{-} g_{n-1} - h_{n-1}^1 - h_{n-1}^2$$

If Case d is true then the colors of $g_{n-1}h_{n-1}^2$ and $h_{n-1}^1 h_{n-1}^2$ are in the set $\{1, 2, \ldots, n-1\}$, so none of the paths above is rainbow. Therefore, Cases b,c must be true and we have

$$\{\gamma(g_{n-1}h_{n-1}^2), \gamma(h_{n-1}^2 h_{n-1}^1)\} = \{n, n+1\} \quad \text{and} \quad \gamma(g_{n-1}h_{n-1}^1) \notin \{n, n+1\} \tag{12}$$

Now, by the symmetry of $H_{n-1} \cong P_2$ we can repeat the same argument (starting from the paragraph when we first considered the colors of H_{n-1}) but with h_{n-1}^1 replaced by h_{n-1}^2, to obtain

$$\{\gamma(g_{n-1}h_{n-1}^1), \gamma(h_{n-1}^1 h_{n-1}^2)\} = \{n, n+1\} \quad \text{and} \quad \gamma(g_{n-1}h_{n-1}^2) \notin \{n, n+1\} \tag{13}$$

We have a contradiction, e.g. $\gamma(g_{n-1}h_{n-1}^1) \notin \{n, n+1\}$ in (12) and $\{\gamma(g_{n-1}h_{n-1}^1), \gamma(h_{n-1}^1 h_{n-1}^2)\} = \{n, n+1\}$ in (13). This completes the proof. □

Remark 2.6. The final step above relies on the symmetry of P_2. If $m \geq 3$, then P_m is no longer symmetric and the argument does not generalize readily. But we do have a narrow range

$$n + 1 \leq rc(P_n \circ H) \leq n + 2 \tag{14}$$

if $n_0(H) = 0$, from (3) and $rc(P_n) = diam(P_n) = n - 1$ (Theorem 1.1).

Liu and Wang [17] stated that $rc(K_n \circ K_m) = 4$ if $n \geq 4$. This is generalized below and extended to src in Theorem 2.6.

Theorem 2.5. *If $n \geq 4$ and H is any graph with no isolated vertex, then $rc(K_n \circ H) = 4$.*

Proof. From (3) and $rc(K_n) = 1$ we have $3 \leq rc(K_n \circ H) \leq 4$. We prove $rc(K_n \circ H) \geq 4$ by contradiction. Suppose $rc(K_n \circ H) \leq 3$. Then there is a rainbow coloring of $K_n \circ H$ with 3 colors. Let $V(K_n) = \{g_1, \ldots, g_n\}$, and for each $i \in \{1, \ldots, n\}$ let H_i be the copy of H that is attached to g_i, with $V(H_i) = \{h_i^1, \ldots, h_i^m\}$. Consider a rainbow path from h_i^1 to h_j^1, with $i, \in \{1, \ldots, n\}$ and $i \neq j$. Since $d(h_i^1, h_j^1) = 3$, the length of the rainbow path is at least 3. But there are only 3 colors, so the length of the rainbow path is exactly 3 (it is a geodesic) and there is only one such path, namely $h_i^1 - g_i - g_j - h_j^1$. Therefore $\gamma(h_i^1 g_i) \neq \gamma(h_j^1 g_i)$. Since this is true for all i, j, we conclude that the n edges $h_1^1 g_1, \ldots, h_n^1 g_n$ all have distinct colors, contradicting $n \geq 4$. □

Theorem 2.6. *Let $n, m \in \mathbb{N}$.*

1. *If $n \in \{2, 3\}$, then $rc(K_n \circ K_m) = src(K_n \circ K_m) = 3$.*
2. *If $n \geq 4$, then $rc(K_n \circ K_1) = src(K_n \circ K_1) = n$.*
3. *If $n \geq 4$ and $m \geq 2$, then $rc(K_n \circ K_m) = 4$ and $src(K_n \circ K_m) = n$.*

Proof. If $n = 2$, then $K_2 \circ K_m$ consists of an edge xy together with complete graphs $\{x\} + K_m$ and $\{y\} + K_m$. We have $rc(K_2 \circ K_m) \geq diam(K_2) + 2 = 3$ by Theorem 2.1. The upper bound $rc(K_2 \circ K_m) \leq 3$ is proved by constructing a rainbow coloring as follows: put the color 1 on the complete graph $\{x\} + K_m$, color 2 on xy, and color 3 on the complete graph $\{y\} + K_m$.

Next, assume $n \geq 3$. First we prove an upper bound for src that will be used in all cases: $src(K_n \circ K_m) \leq n$. Define $\gamma : E(K_n \circ K_m) \to \{1, \ldots, n\}$ by

$$\gamma(e) = \begin{cases} i, & \text{if } e \in E(\{g_i\} \circ H_i) \text{ for some } i \in \{1, \ldots, n\}, \\ \min\Big(\{1, \ldots, n\} - \{i, j\}\Big), & \text{if } e = g_i g_j \text{ for some } i, j \in \{1, \ldots, n\}, i \neq j. \end{cases}$$

See Figure 2 for an illustration. We check that this is strong rainbow. It is enough to find a rainbow geodesic between $x \in H_i$ and $y \in H_j$ with $i, j \in \{1, \ldots, n\}, i \neq j$ (rainbow geodesic between any other pair is a subpath of this one). Let $x = h_i^a$ and $y = h_j^b$, for some $a, b \in \{1, \ldots, m\}$. Then $h_i^a \overset{i}{-} g_i \overset{k}{-} g_j \overset{j}{-} h_j^b$ is a rainbow geodesic because $k = \min(\{1, \ldots, n\} - \{i, j\}) \neq i, j$.

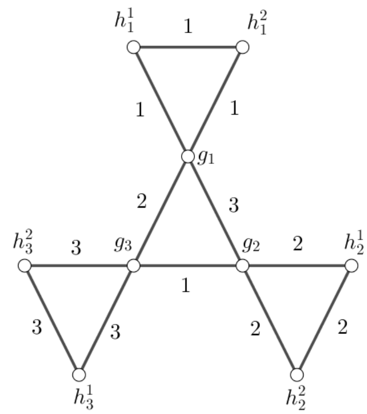

Figure 2. A strong rainbow coloring on $K_3 \circ K_2$

If $n = 3$, Theorem 2.1 and the upper bound give $3 \leq rc(K_3 \circ K_m) \leq src(K_3 \circ K_m) \leq 3$, proving the first statement. If $n \geq 4$ and $m = 1$, then Theorem 2.3 and the upper bound give $n \leq rc(K_n \circ K_1) \leq src(K_n \circ K_1) \leq n$, proving the second statement.

To prove the third statement, we assume $n \geq 4$ and $m \geq 2$. Now K_m has no isolated vertex, so by Theorem 2.5 we have $rc(K_n \circ K_m) = 4$. There is a unique geodesic from h_i^1 to h_j^1 for all $i \neq j$, namely $h_i^1 - g_i - g_j - h_j^1$, so in any strong rainbow coloring the edges $h_i^1 g_i, \ldots, h_n^1 g_n$ must use distinct colors. This gives $src(K_n \circ K_m) \geq n$. Together with the upper bound, we get equality. □

Remark 2.7. In [18] it was claimed that $C_n \circ C_m$ and $C_n \circ P_m$ have the same rainbow connection number which is 4 if $n = 3$, or $\lceil n/2 \rceil + 3$ if $n \geq 4$. Theorem 2.6 with $n = 3$ is a counter-example, since $rc(C_3 \circ C_3) = rc(K_3 \circ K_3) = 3$ and $rc(C_3 \circ P_2) = rc(K_3 \circ K_2) = 3$. Their proof for $n \geq 4$ is unclear, but we could not find a counter-example. We do have a narrow range

$$\lfloor n/2 \rfloor + 2 \leq rc(C_n \circ H) \leq \lceil n/2 \rceil + 3 \tag{15}$$

if $n_0(H) = 0$, from (3), $diam(C_n) = \lfloor n/2 \rfloor$, and $rc(C_n) = \lceil n/2 \rceil$ (Theorem 1.1).

Next we consider some corona product $G \circ H$ with $H \cong K_1$. In [9] Estetikasari and Sy proved that $rc(T_n \circ K_1) = 2n - 1$. Here we state the result again with a small addition of src.

Theorem 2.7. *If T_n is a tree with $n \geq 2$ vertices, then $rc(T_n \circ K_1) = src(T_n \circ K_1) = 2n - 1$.*

Proof. Note that $T_n \circ K_1$ is also a tree, and it has $2n$ vertices and $2n - 1$ edges, so by the third statement in Theorem 1.1 we have $rc(T_n \circ K_1) = src(T_n \circ K_1) = |E(T_n \circ K_1)| = 2n - 1$. □

The corona product $C_n \circ K_1$ of a cycle graph with the trivial graph is known as a sunlet graph or sun graph, sometimes denoted by S_n. The name comes from the shape of the graph, which is a cycle with a pendant at every vertex. In [27] it was stated that $rc(S_n) = src(S_n) = \lfloor n/2 \rfloor + n$. Unfortunately this is incorrect: this is only the upper bound in (4) and not efficient. The exact value is actually close to the lower bound in (4).

Theorem 2.8 (Odd Sunlet)**.** *If $n = 2q + 1$ with $q \geq 1$, then $rc(C_n \circ K_1) = src(C_n \circ K_1) = n$.*

Proof. Write $S_n = C_n \circ K_1$. Let the cycle be $C_n : g_1 - g_2 - \cdots - g_n - g_1$ in the clockwise direction. For each $i \in \{1, \dots, n\}$ let H_i be the i-th copy of $H \cong K_1$ joined to g_i, and $V(H_i) = \{h_i\}$. All indices will be understood modulo n, thus e.g. $g_{n+1} = g_1$.

From Theorem 2.3 we get $rc(S_n) \geq n$. We show $src(S_n) \leq n$ by constructing a strong rainbow coloring γ on S_n with n colors. For each $i \in \{1, \dots, n\}$, put the color $i \pmod n$ on the pendant edge joined to g_i, and also on the cycle-edge that is directly opposite from that pendant edge (there is such an edge precisely because n is odd). Formally, $\gamma(h_i g_i) = i \pmod n$ and $\gamma(g_i g_{i+1}) = q + 1 + i \pmod n$ for each $i \in \{1, \dots, n\}$. See Figure 3 for an illustration.

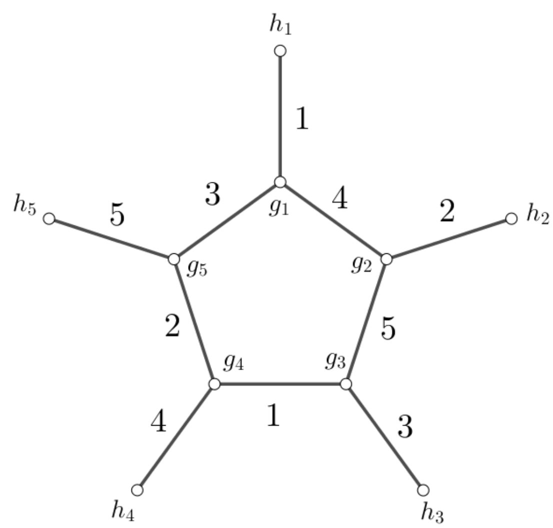

Figure 3. A strong rainbow coloring on S_5.

We check that this is a strong rainbow coloring. Let x, y be non-adjacent vertices in S_n. Since all edges in the cycle have different colors, any path in the cycle is rainbow. Any geodesic between two pendant vertices h_i and h_j must be of the form $h_i - g_i - \cdots - g_j - h_j$ so it contains a geodesic from h_i to g_j. Therefore, it is enough to find a rainbow geodesic between any two pendant vertices. Suppose that x, y are both pendant vertices. By rotational symmetry we may assume $x = h_1$ and $y = h_j$ for some $j \in \{2, \dots, n\}$. We consider two cases depending on whether g_1 is nearer to g_j in the clockwise or counterclockwise direction.

- Let $j \leq q + 1$. Then $d(x, y) = 1 + (j - 1) + 1 = j + 1$ and there is a clockwise geodesic

$$x = h_1 \overset{1}{-} g_1 \overset{q+2}{-} g_2 \overset{q+3}{-} \cdots \overset{q+j-1}{-} g_{j-1} \overset{q+j}{-} g_j \overset{j}{-} h_j = y$$

The colors are distinct mod n because $1 < j < q+2 < \cdots < q+j \leq 2q+1 = n$.

- Let $j \geq q+2$. Then $d(x,y) = 1+n-(j-1)+1 = n-j+3$ and there is a counterclockwise geodesic

$$x = h_1 \overset{1}{-} g_1 \overset{q+1+n}{-} g_n \overset{q+n}{-} \cdots \overset{q+2+j}{-} g_{j+1} \overset{q+1+j}{-} g_j \overset{j}{-} h_j = y$$

Modulo n, the colors are congruent to

$$x = h_1 \overset{1}{-} g_1 \overset{q+1}{-} g_n \overset{q}{-} \cdots \overset{j-q+1}{-} g_{j+1} \overset{j-q}{-} g_j \overset{j}{-} h_j = y$$

The colors are distinct mod n because $n \geq j > q+1 > \cdots > j-q+1 > j-q > 1$.

This completes the proof of the theorem. □

Theorem 2.9 (Even Sunlet). *If* $n = 2q$ *with* $q \geq 2$*, then* $rc(C_n \circ K_1) = src(C_n \circ K_1) = n+1$.

Proof. Write $S_n = C_n \circ K_1$. Let the cycle be $C_n : g_1 - g_2 - \cdots - g_n - g_1$ in the clockwise direction. For each $i \in \{1,\ldots,n\}$ let H_i be the i-th copy of $H \cong K_1$ joined to g_i, and $V(H_i) = \{h_i\}$. All indices will be understood modulo n, e.g. $g_{n+1} = g_1$. We will prove that

$$n+1 \leq rc(S_n) \leq src(S_n) \leq n+1.$$

Proving the lower bound $rc(S_n) \geq n+1$.

Suppose $rc(S_n) \leq n$, so S_n has a rainbow coloring γ with n colors. All colors will be understood modulo n. All pendant edges have different colors; by relabeling the colors we may assume $\gamma(g_ih_i) = i$ for every $i \in \{1,\ldots,n\}$. First we prove two claims.

Claim A: The edge colors of C_n are a permutation of $1,2,\ldots,n$.

Proof of Claim A: Since there are n edges in the cycle and only n colors, it is enough to show that all edges on the cycle have different colors. Suppose that some two edges on the cycle have the same color. This repeated color must also appear on a pendant edge. By rotating the coloring if necessary, we may assume that the repeated color is 1. So $1 = \gamma(g_1h_1) = \gamma(g_ag_{a+1}) = \gamma(g_bg_{b+1})$ for some $a,b \in \{1,\ldots,n\}$ with $a < b$. There must be a rainbow path from the pendant vertex h_1 to g_{a+1}. But there are only two paths between them, namely the clockwise path $h_1 \overset{1}{-} g_1 - \cdots - g_a \overset{1}{-} g_{a+1}$ and the counterclockwise path $h_1 \overset{1}{-} g_1 - g_m - \cdots - g_{b+1} \overset{1}{-} g_b - \cdots - g_{a+1}$ and both are not rainbow, a contradiction. This completes the proof of Claim A.

We try to contradict Claim A by showing that some color $i \in \{1,\ldots,n\}$ cannot be used on C_n. First we eliminate the cycle-edges next to g_i.

Claim B: For every $i \in \{1,\ldots,n\}$, we have $\gamma(g_ig_{i+1}) \neq i$ and $\gamma(g_{i-1}g_i) \neq i$.

Proof of Claim B: Suppose $\gamma(g_ig_{i+1}) = i$. There is a rainbow path from h_{i+1} to h_i. The counterclockwise path $h_{i+1} \overset{i+1}{-} g_{i+1} \overset{i}{-} g_i \overset{i}{-} h_i$ is not rainbow, so the clockwise path

$$h_{i+1} \overset{i+1}{-} g_{i+1} - g_{i+2} - \cdots - g_i \overset{i}{-} h_i$$

must be rainbow. By Claim A the color $i+1$ must appear on the cycle. Since $\gamma(g_i g_{i+1}) = i \neq i+1$, the color $i+1$ must appear on the remaining $n-1$ edges of the cycle which is $g_{i+1} - g_{i+2} - \cdots - g_i$, so the path above is actually not rainbow, a contradiction. The case $\gamma(g_{i-1}g_i) = i$ can be eliminated in a similar manner by "mirroring" the above argument on the line $g_i g_{i+q}$ which is a line of symmetry because $n = 2q$ is even. This completes the proof of Claim B.

A similar argument can be used to show that the color $i \in \{1, \ldots, n\}$ also does not appear in the four edges around g_i, namely $g_{i-2}g_{i-1}$, $g_{i-1}g_i$, $g_i g_{i+1}$, and $g_{i+1}g_{i+2}$. Instead, we show more generally that the color i never appears on the $2j$ edges around g_i, for every $j \in \{1, \ldots, q\}$.

Claim C: For every $i \in \{1, \ldots, n = 2q\}$ and $j \in \{1, \ldots, q\}$, the color i is absent from the $2j$ edges around g_i (namely, j edges to the left and j edges to the right of g_i).

Proof of Claim C: We prove this by induction on j. The basis $j = 1$ is Claim B. For the induction step, let $i \in \{1, \ldots, n\}$ and $j \in \{2, \ldots, q\}$ be such that the color i appears among the $2j$ edges around g_i. Suppose that the color i appears in the clockwise direction from g_i, namely $\gamma(g_{i+a-1}g_{i+a}) = i$ for some $a \in \{1, \ldots, j\}$ (the case when the color i appears in the counterclockwise direction can be handled similarly by mirror symmetry). Consider the set of colors of the edges between g_i and g_{i+a-1}, namely

$$S = \{\gamma(g_i g_{i+1}), \ldots, \gamma(g_{i+a-2}g_{i+a-1})\}$$

We will show that

$$i+a, i+a+1, i+a+2, \ldots, i+2a-1 \in S \tag{16}$$

To prove this, take any color $i+a+k$ with $k \in \{0, 1 \ldots, a-1\}$. Note that $h_{i+a+k} \neq h_i$ because $i < i+a+k \leq i+2a-1 \leq i+2j-1 \leq i+2q-1 < i+n$ (recall that $n = 2q$) so there is a rainbow path from h_{i+a+k} to h_i. Because the counterclockwise path from g_{i+a+k} to h_i repeats the color i (namely $\gamma(g_{i+a}g_{i+a-1}) = \gamma(h_i g_i) = i$), the rainbow path must be the clockwise path

$$h_{i+a+k} \overset{i+a+k}{-} g_{i+a+k} - g_{i+a+k+1} - \cdots - g_n - g_1 - \cdots g_i \overset{i}{-} h_i$$

By Claim A, the color $i+a+k$ must appear on the cycle. Since the path above is rainbow, $\gamma(g_{i+a+k}g_{i+a+k+1}), \ldots, \gamma(g_{i-1}g_i) \neq i+a+k$. Therefore, the color $i+a+k$ must appear among the remaining edges of the cycle, i.e.

$$g_i - g_{i+1} - \cdots - g_{i+a-2} - g_{i+a-1} \overset{i}{-} g_{i+a} - g_{i+a+1} - \cdots - g_{i+a+k-1} - g_{i+a+k}$$

We use the inductive hypotesis to conclude that the color $i+a+k$ is absent from the $2(j-1)$ edges around g_{i+a+k}. Since $k \leq a-1 \leq j-1$, these $2(j-1)$ edges include the $2k$ edges around g_{i+a+k}, including the k edges $g_{i+a} - g_{i+a+1} - \cdots - g_{i+a+k-1} - g_{i+a+k}$. Therefore, the color $i+a+k$ can only occur on the edges $g_i - g_{i+1} - \cdots - g_{i+a-2} - g_{i+a-1}$ so $i+a+k \in S$. This proves (16).

Now, the consecutive numbers $i+a, i+a+1, \ldots, i+2a-1$ are all distinct modulo n, so (16) shows that S has at least a distinct members. But from the definition of S it is clear that $|S| \leq a-1$, so we get a contradiction. This completes the proof of Claim C.

Finally, using Claim C with $i = 1$ and $j = q$, we conclude that the color 1 is absent from the $2q = n$ edges around g_1. But there are only n edges in the cycle, so the color 1 does not occur anywhere in the cycle. This contradicts Claim A, and completes the proof of the lower bound.

Proving the upper bound $src(S_n) \leq n + 1$.

Color the edges as in Figure 4. More formally, the coloring is given by $\gamma(h_i g_i) = i$ for every $i \in \{1, \dots, 2q\}$ and

$$\gamma(g_i g_{i+1}) = \begin{cases} q + i, & i \in \{1, 2, \dots, q - 1\} \\ 2q + 1, & i \in \{q, 2q\} \\ i - q + 1, & i \in \{q + 1, q + 2, \dots, 2q - 1\} \end{cases}$$

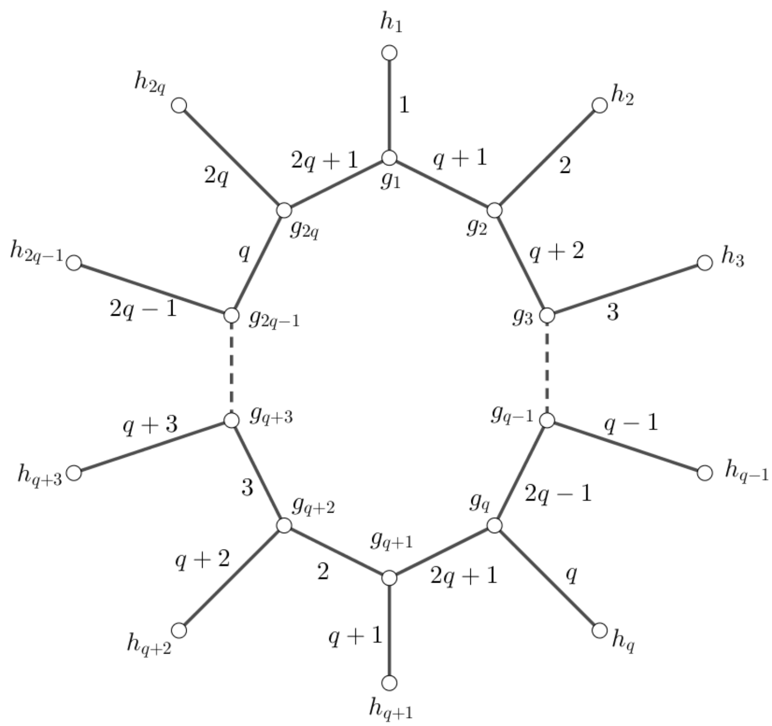

Figure 4. A strong rainbow coloring on S_{2q+1}.

We verify that this is strong rainbow. Let x, y be non-adjacent vertices in S_n. Since all edges in the cycle have different colors, any path within the cycle is rainbow. The remaining cases are as follows. Split the edge-colored graph into two subgraphs as in Figure 5.

- Let x, y be in the same subgraph. In each subgraph, all the edges have different colors so any path within the subgraph is rainbow. Moreover, in each subgraph, distance between any two vertices is always equal to distance in the whole graph. So there is always a rainbow geodesic between x, y.

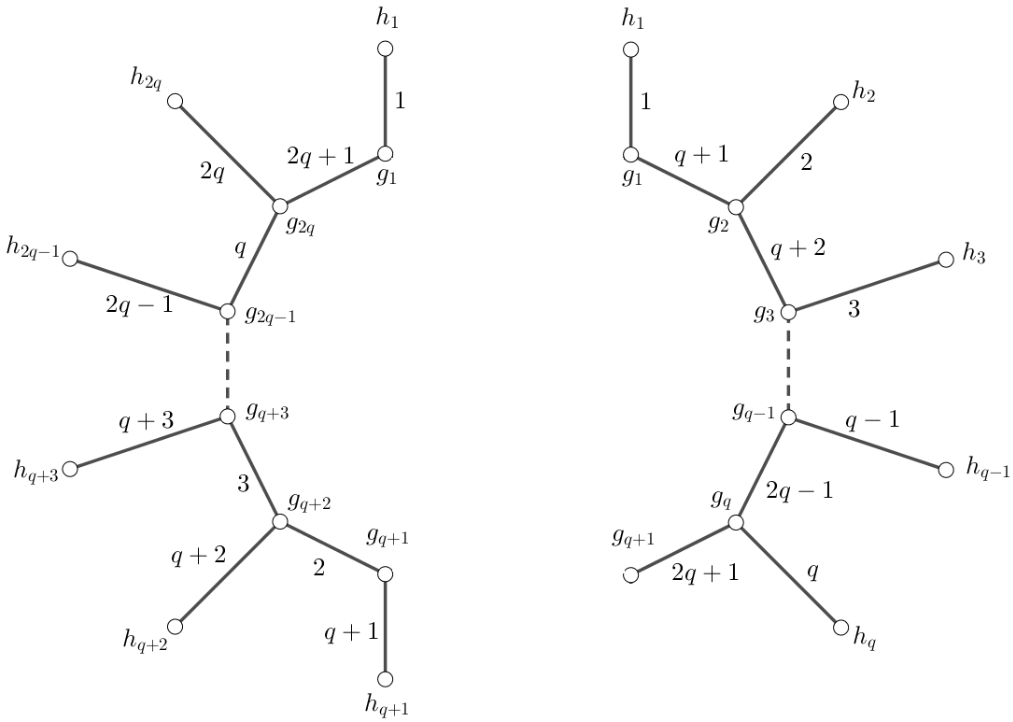

Figure 5. Two subgraphs of S_{2q+1}.

- Let x, y be in different subgraphs, say x is in the subgraph on the left of Figure 5 and y is in the subgraph on the right. Since the two subgraphs intersect on $\{h_1, g_1, g_{q+1}\}$, we may assume that x and y are none of these vertices.
  - (i) If $x = h_{q+1}$, then the counterclockwise path to $y \in \{g_{q+1}, g_q, h_q, \ldots, g_2, h_2\}$ is always a rainbow geodesic, because the right subgraph only uses the color $q+1$ on g_2g_1.
  - (ii) Let $x = h_i$ and $y = h_j$ with $2 \leq j \leq q < q+1 < i \leq 2q$.
    If $i - j < q$, then $d(x, y) = i - j$ and the counterclockwise path below is a rainbow geodesic

    $$x = h_i \overset{i}{-} g_i \overset{i-q}{-} g_{i-1} \overset{i-q-1}{-} \cdots \overset{2}{-} g_{q+1} \overset{2q+1}{-} g_q \overset{2q-1}{-} g_{q-1} \overset{2q-2}{-} \cdots \overset{q+j}{-} g_j \overset{j}{-} h_j = y$$

    because $2 < \cdots < i - q < j < i < q + j < \cdots < 2q - 1 < 2q + 1$.
    If $i - j \geq q$, then the clockwise path below is a rainbow geodesic

    $$x = h_i \overset{i}{-} g_i \overset{i-q+1}{-} g_{i+1} \overset{i-q+2}{-} \cdots \overset{q}{-} g_{2q} \overset{2q+1}{-} g_1 \overset{q+1}{-} g_2 \overset{q+2}{-} \cdots \overset{q+j-1}{-} g_j \overset{j}{-} h_j = y$$

    because $j < i - q + 1 < \cdots < q < q + 1 < \cdots < q + j - 1 < i < 2q + 1$.
  - (iii) If $(x, y) = (h_i, g_j)$ or (g_i, h_j) with $2 \leq j \leq q < q + 1 \leq i \leq 2q$, then we can use the rainbow geodesic from h_i to h_j in Case (ii) and cut the last or first vertex of the path respectively.

This completes the proof of the upper bound $rc(S_n) \leq n + 1$, hence the theorem. □

3. Concluding Remarks

In this paper we have investigated the rainbow connection number of corona product of two graphs $G \circ H$. We have obtained a general lower bound (Theorem 2.1), an upper bound when H has no isolated vertex (Theorem 2.2), a lower bound and an upper bound when $H \cong K_1$ (Theorem 2.3), and some exact values (Theorems 2.4, 2.5, 2.6, 2.7, 2.8, 2.9). Tightness of the bounds were also discussed. There are some open problems, for example:

1. Examples of $rc(G \circ H) = rc(G)$ or $src(G \circ H) = \max\{src(G), src(K_1 \circ H)\}$.
2. Exact values for $P_n \circ H$ and $C_n \circ H$ in the narrow ranges (14) and (15), cf. Theorem 2.5.
3. Better bounds for $src(G \circ H)$.

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